Comment. Math. Helvetici 57 (1982) 221–225 0010–2571/82/002221–05$01.50+0.20/0

Hölder continuity of conformal mappings and non-quasiconformal Jordan curves

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Let $\Gamma\subset\mathbf{C}$ be a quasiconformal curve (quasicircle), that is the image of the unit circle under a quasiconformal mapping of the plane. Let f and f^* denote conformal mappings of the unit disk $\mathbf{D}$ onto int Γ and ext Γ respectively.

It is well-known that f and f^* have quasiconformal extensions onto the plane [3], p. 98. Because of the Hölder continuity of quasiconformal mappings ([3], p. 71), f and f^* as well as their inverse mappings f^{-1} and f^{*-1} satisfy Hölder conditions:

$$|f(z_1)-f(z_2)|\le K\,|z_2-z_1|^\alpha \qquad (z_1, z_2\in\mathbf{D}), \tag{1}$$
$$|f^{-1}(w_1)-f^{-1}(w_2)|\le L\,|w_1-w_2|^\beta \qquad (w_1, w_2\in \operatorname{int}\Gamma)$$

and similarly for f^* where one has to use the spherical metric in ext Γ. The Hölder exponents α $(0<\alpha\le 1)$ and β $(1\le\beta<2)$ depend only on Γ [4], [5], [8], p. 287, 289, 347.

By means of a simple geometrical characterization of (1) (Theorem 1), we construct a non-quasiconformal Jordan curve (Theorem 2) such that nevertheless f, f^*, and also f^{-1}, f^{*-1} remain Hölder continuous. This shows that Hölder continuity of all these conformal mappings is only a necessary, but not a sufficient condition for quasicircles.

Let $G\subsetneqq\mathbf{C}$ be a simply connected domain and let $h(w_1, w_2)$ denote the hyperbolic distance of the points w_1, $w_2\in G$ defined by

$$h(w_1, w_2)=\log\frac{|1-\bar z_1 z_2|+|z_2-z_1|}{|1-\bar z_1 z_2|-|z_2-z_1|} \qquad (w_i=f(z_i),\ i=1,2) \tag{2}$$

where f is a conformal mapping of $\mathbf{D}$ onto G. Let $\delta(w)=\operatorname{dist}(w,\partial G)$ denote the (Euclidean) boundary distance.

THEOREM 1. *If G is a bounded simply connected domain, $w_0\in G$, and f a conformal mapping of $\mathbf{D}$ onto G, then* (1) *is satisfied if and only if*

$$\limsup_{\delta(w)\to 0}\,[h(w_0, w)+\frac{1}{\alpha}\log\delta(w)]<+\infty. \tag{3}$$

Proof. It is well-known that (1) is equivalent to

$$|f'(z)| \leq M(1-|z|^2)^{\alpha-1} \qquad (|z|<1) \tag{4}$$

(see for instance [2], p. 361–363). Introducing the non-Euclidean length element

$$\rho(w)\,|dw| = \frac{2\,|dz|}{1-|z|^2} \qquad (w = f(z))$$

corresponding to (2) we see that (4) means

$$h(f(0), f(z)) \leq \frac{1}{\alpha} \log \left[\frac{M}{2}(1+|z|)^{2\alpha} \rho(f(z))\right].$$

Since G is bounded we have $\delta(f(z)) \to 0$ if and only if $|z| \to 1$. Thus we obtain as an equivalent condition

$$\limsup_{\delta(w)\to 0} \left[h(f(0), w) - \frac{1}{\alpha} \log \rho(w)\right] < +\infty. \tag{5}$$

Because of

$$\tfrac{1}{2} \leq \rho(w)\delta(w) \leq 2$$

[8], p. 22 it follows by the triangle inequality for $h(w_1, w_2)$ that (5) is the same as (3) which proves Theorem 1.

The following corollary gives a convenient sufficient condition for Hölder continuity.

CORROLLARY. *Let f be a bounded univalent function defined in* $\mathbf{D}$, $G = f(\mathbf{D})$, *then f satisfies* (1) *if there are positive numbers M, δ_0, δ_1 with the property that, for every $w \in G$ with $\delta(w) < \delta_0$, there exists $w_1 \in G$ with $\delta(w_1) \geq \delta_1$ and a connecting arc $C \subset G$, such that*

$$\int_C \frac{|d\omega|}{\delta(\omega)} \leq M + \frac{1}{2\alpha} \log \frac{1}{\delta(w)}. \tag{6}$$

Proof. We choose $w_0 = f(0)$. Then we have

$$h(w_0, w) \leq h(w_0, w_1) + \int_C \rho(\omega)\,|d\omega|$$
$$\leq \log \frac{1+|f^{-1}(w_1)|}{1-|f^{-1}(w_1)|} + 2\int_C \frac{|d\omega|}{\delta(\omega)}.$$

From (6) and $\delta(w_1) \geq \delta_1 > 0$ it follows that (3) is satisfied which implies (1) by Theorem 1.

We are now ready to construct a Jordan curve with the desired properties.

THEOREM 2. *There is a non-quasiconformal Jordan curve Γ such that the conformal mappings f and f^* of $\mathbf{D}$ onto* int Γ *and* ext Γ *respectively, and also their inverse mappings are Hölder continuous.*

Proof. We set $Q = \{w = u + iv : |u| < 1, |v| < 1\}$ and $R_n = \{w = u + iv : 0 \leq u - 1 < a_n, |v - v_n| < \varepsilon_n\}$ where we choose $v_n = 1/n$, $\varepsilon_n = 2^{-n}$, $a_n = -\varepsilon_n \log \varepsilon_n$, $n = 2, 3, \ldots$. We consider the domain $G = Q \cup \bigcup_{n=2}^{\infty} R_n$. Since $\Gamma = \partial G$ is a locally connected continuum without cut points it is a Jordan curve [8], p. 281. But it is not a quasicircle, because the Ahlfors criterion [1] is not satisfied. For we have

$$\frac{\operatorname{diam} R_n}{2\varepsilon_n} \geq \frac{a_n}{2\varepsilon_n} \to +\infty (n \to \infty).$$

On the other hand, we shall show below that, for every $w \in G$, $\delta(w) < \frac{1}{2}$, there exists $w_1 \in G$, $\delta(w_1) \geq \frac{1}{2}$, and a connecting arc C such that

$$\int_C \frac{|d\omega|}{\delta(\omega)} \leq 2 \log \frac{1}{\delta(w)}. \tag{7}$$

Thus (6) is satisfied with $\alpha = \frac{1}{4}$ which shows that f is α-Hölder-continuous for some $\alpha \geq \frac{1}{4}$.

The Hölder continuity of f^{-1}, f^*, f^{*-1} follows easily from results by Näkki and Palka [6], [7]:

Let $d(w_1, w_2) = \inf_C \int_C |dw|$ denote the inner distance of $w_1, w_2 \in G$ where the infimum is taken over all connecting arcs $C \subset G$. Then, obviously, there exists a constant A such that

$$d(w_1, w_2) \leq A\,|w_2 - w_1|\,(w_1, w_2 \in G).$$

It follows from [6] that f^* and f^{-1} are Hölder continuous.

Considering the inner distance $d^*(w_1, w_2)$ with regard to $G^* = \operatorname{ext} \Gamma$ one easily sees that, for every β $(0<\beta<1)$, and $R>0$, there exists a constant B such that

$$d^*(w_1, w_2) \leq B\,|w_2 - w_1|^\beta \quad (w_i \in G^*, |w_i| \leq R, i = 1, 2)$$

This implies, by another result of Näkki and Palka [6], [7], that f^{*-1} is also Hölder continuous.

Hence it remains to prove (7). We have to consider the following cases.

(a) $w = u + iv \in Q$, and, if $u \geq v \geq 0$, $|v - v_n| \geq \varepsilon_n$, $n = 2, 3, \ldots$. Hence $\delta(w) = \min(1-|u|, 1-|v|)$. Choose $w_1 = 0$ and $C = [w_1, w_2]$ (segment with end points w_1, w_2). Then

$$\int_C \frac{|d\omega|}{\delta(\omega)} \leq \int_{\delta(w)}^1 \frac{\sqrt{2}\,dt}{t} = \sqrt{2} \log \frac{1}{\delta(w)}.$$

(b) $w = u + iv \in Q$, $u \geq v \geq 0$, $|v - v_n| < \varepsilon_n$ for some n (hence $\delta(w) > 1 - u$). Choose, for $\delta(w) < \frac{1}{2}$, $w_1 = \frac{1}{2} + iv$, $C = [w_1, w]$. Because of

$$\delta(w) = \sqrt{((1 - x^2) + (\varepsilon_n - |v - v_n|)^2)}, \qquad \omega = x + iv \in C,$$

we obtain

$$\int_C \frac{|d\omega|}{\delta(\omega)} = \log \frac{\frac{1}{2} + \sqrt{(\frac{1}{4} + (\varepsilon_n - |v - v_n|)^2)}}{1 - u + \sqrt{((1-u)^2 + (\varepsilon_n - |v - v_n|)^2)}}$$
$$\leq \log \frac{1 + \sqrt{(1 + 4\varepsilon_2^2)}}{2\delta(w)} < 2 \log \frac{1}{\delta(w)}.$$

(c) $w = u + iv \in R_n$ for some n, $0 \leq u - 1 \leq a_n - \varepsilon_n$ hence $\delta(w) = \varepsilon_n - |v - v_n|$. Choose $w_1 = \frac{1}{2} + iv_n$ and $C = [w_1, u + iv_n] \cup [u + iv_n, w]$. Then

$$\int_C \frac{|d\omega|}{\delta(\omega)} = \int_{1/2}^1 \frac{dx}{\sqrt{((1-x)^2 + \varepsilon_n^2)}} + \int_1^u \frac{dx}{\varepsilon_n} + \int_0^{|v - v_n|} \frac{dy}{\varepsilon_n - y}$$
$$< \frac{a_n}{\varepsilon_n} - \log(\varepsilon_n - |v - v_n|) \leq 2 \log \frac{1}{\delta(w)}.$$

(d) $w = u + iv \in R_n$ for some n, $a_n - \varepsilon_n < u - 1$ hence $\delta(w) = \min(\varepsilon_n - |v - v_n|, 1 + a_n - u)$. Choose $w_1 = \frac{1}{2} + iv_n$, $C = [w_1, 1 + a_n - \varepsilon_n + iv_n] \cup [1 + a_n - \varepsilon_n + iv_n, w]$.

Then

$$\int_C \frac{|d\omega|}{\delta(\omega)} \leq \int_{1/2}^{1} \frac{dx}{\sqrt{(1-x)^2+\varepsilon_n^2}} + \int_0^{a_n-\varepsilon_n} \frac{dx}{\varepsilon_n} + \int_{\delta(w)}^{\varepsilon_n} \frac{\sqrt{2}\, dt}{t}$$

$$< -\log \varepsilon_n + \frac{a_n}{\varepsilon_n} + \sqrt{2} \log \frac{\varepsilon_n}{\delta(w)} \leq 2 \log \frac{1}{\delta(w)}.$$

Thus we have proved that (7) holds in each case which completes the proof of Theorem 2.

Remark. The Hölder continuity of f^* could have also been shown by this method.

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Received January 21, 1982