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Journal of Functional Analysis 262 (2012) 4948–4971

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

The Brezis–Browder Theorem in a general Banach space

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Received 26 October 2011; accepted 30 March 2012

Available online 10 April 2012

Communicated by H. Brezis

Abstract

During the 1970s Brezis and Browder presented a now classical characterization of maximal monotonicity of monotone linear relations in reflexive spaces. In this paper, we extend (and refine) their result to a general Banach space. We also provide an affirmative answer to a problem posed by Phelps and Simons.

Keywords: Adjoint; Brezis–Browder Theorem; Linear relation; Monotone operator; Operator of type (D); Operator of type (FP); Operator of type (NI); Set-valued operator; Skew operator; Symmetric operator

1. Introduction

Throughout this paper, we assume that

$$X \text{ is a real Banach space with norm } \|\cdot\|,$$

that X^* is the continuous dual of X, and that X and X^* are paired by $\langle\cdot,\cdot\rangle$. The *closed unit ball* in X is denoted by $B_X = \{x \in X \mid \|x\| \leqslant 1\}$, and $\mathbb{N} = \{1, 2, 3, \ldots\}$.

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0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2012.03.023

We identify X with its canonical image in the bidual space X^{**}. As always, $X \times X^*$ and $(X \times X^*)^* = X^* \times X^{**}$ are paired via

$$\langle (x, x^*), (y^*, y^{**}) \rangle = \langle x, y^* \rangle + \langle x^*, y^{**} \rangle,$$

where $(x, x^*) \in X \times X^*$ and $(y^*, y^{**}) \in X^* \times X^{**}$. The norm on $X \times X^*$, written as $\|\cdot\|_1$, is defined by $\|(x, x^*)\|_1 = \|x\| + \|x^*\|$ for every $(x, x^*) \in X \times X^*$.

Let $A : X \rightrightarrows X^*$ be a *set-valued operator* (also known as multifunction) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A = \{(x, x^*) \in X \times X^* \mid x^* \in Ax\}$ be the *graph* of A. The *domain* of A, written as $\operatorname{dom} A$, is $\operatorname{dom} A = \{x \in X \mid Ax \neq \varnothing\}$ and $\operatorname{ran} A = A(X)$ is the *range* of A. We say A is a *linear relation* if $\operatorname{gra} A$ is a linear subspace. By saying $A : X \rightrightarrows X^*$ is *at most single-valued*, we mean that for every $x \in X$, Ax is either a singleton or empty. In this case, we follow a slight but common abuse of notation and write $A : \operatorname{dom} A \to X^*$. Conversely, if $T : D \to X^*$, we may identify T with $A : X \rightrightarrows X^*$, where A is at most single-valued with $\operatorname{dom} A = D$.

Now let $U \times V \subseteq X \times X^*$. We say that A is *monotone* with respect to $U \times V$, if for every $(x, x^*) \in (\operatorname{gra} A) \cap (U \times V)$ and $(y, y^*) \in (\operatorname{gra} A) \cap (U \times V)$, we have

$$\langle x - y, x^* - y^* \rangle \geqslant 0. \tag{1}$$

Of course, by (classical) monotonicity we mean monotonicity with respect to $X \times X^*$. Furthermore, we say that A is *maximally monotone* with respect to $U \times V$ if A is monotone with respect to $U \times V$ and for every operator $B : X \rightrightarrows X^*$ that is monotone with respect to $U \times V$ and such that $(\operatorname{gra} A) \cap (U \times V) \subseteq (\operatorname{gra} B) \cap (U \times V)$, we necessarily have $(\operatorname{gra} A) \cap (U \times V) = (\operatorname{gra} B) \cap (U \times V)$. Thus, (classical) maximal monotonicity corresponds to maximal monotonicity with respect to $X \times X^*$. This slightly unusual presentation is required to state our main results; moreover, it yields a more concise formulation of monotone operators of type (FP).

Now let $A : X \rightrightarrows X^*$ be monotone and $(x, x^*) \in X \times X^*$. We say (x, x^*) is *monotonically related to* $\operatorname{gra} A$ if

$$\langle x - y, x^* - y^* \rangle \geqslant 0, \quad \forall (y, y^*) \in \operatorname{gra} A.$$

If Z is a real Banach space with continuous dual Z^* and a subset S of Z, we define $S^\perp$ by $S^\perp = \{z^* \in Z^* \mid \langle z^*, s \rangle = 0,\ \forall s \in S\}$. Given a subset D of Z^*, we define $D_\perp$ by $D_\perp = \{z \in Z \mid \langle z, d^* \rangle = 0,\ \forall d^* \in D\} = D^\perp \cap Z$.

The operator *adjoint* of A, written as A^*, is defined by

$$\operatorname{gra} A^* = \{(x^{**}, x^*) \in X^{**} \times X^* \mid (x^*, -x^{**}) \in (\operatorname{gra} A)^\perp\}.$$

Note that the adjoint is always a linear relation with $\operatorname{gra} A^* \subseteq X^{**} \times X^* \subseteq X^{**} \times X^{***}$. These inclusions make it possible to consider monotonicity properties of A^*; however, care is required: as a linear relation, $\operatorname{gra} A^* \subseteq X^{**} \times X^*$ while as a potential monotone operator we are led to consider $\operatorname{gra} A^* \subseteq X^{**} \times X^{***}$. Now let $A : X \rightrightarrows X^*$ be a linear relation. We say that A is *skew* if $\operatorname{gra} A \subseteq \operatorname{gra}(-A^*)$; equivalently, if $\langle x, x^* \rangle = 0$, $\forall (x, x^*) \in \operatorname{gra} A$. Furthermore, A is *symmetric* if $\operatorname{gra} A \subseteq \operatorname{gra} A^*$; equivalently, if $\langle x, y^* \rangle = \langle y, x^* \rangle$, $\forall (x, x^*), (y, y^*) \in \operatorname{gra} A$.

We now recall three fundamental subclasses of maximally monotone operators.

Definition 1.1. Let $A : X \rightrightarrows X^*$ be maximally monotone. The three key types of monotone operators are defined as follows.

(i) A is *of type* (D) (1976, [24]; see also [29] and [40, Theorem 9.5]) if for every $(x^{**}, x^*) \in X^{**} \times X^*$ with

$$\inf_{(a,a^*)\in \operatorname{gra} A} \langle a - x^{**}, a^* - x^* \rangle \geqslant 0,$$

there exists a bounded net $(a_\alpha, a^*_\alpha)_{\alpha\in\Gamma}$ in $\operatorname{gra} A$ such that $(a_\alpha, a^*_\alpha)_{\alpha\in\Gamma}$ weak* $\times$ strong converges to (x^{**}, x^*).

(ii) A is *of type negative infimum* (NI) (1996, [35]) if

$$\sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle a, x^* \rangle + \langle a^*, x^{**} \rangle - \langle a, a^* \rangle\big) \geqslant \langle x^{**}, x^* \rangle, \quad \forall (x^{**}, x^*) \in X^{**} \times X^*.$$

(iii) A is *of type Fitzpatrick–Phelps* (FP) (1992, [22]) if whenever V is an open convex subset of X^* such that $V \cap \operatorname{ran} A \neq \varnothing$, it must follow that A is maximally monotone with respect to $X \times V$.

As we see in the following result, it is a consequence of recent work that the three classes of Definition 1.1 coincide.

Fact 1.2. (See [5, Corollary 3.2] and [37,35,26].) Let $A : X \rightrightarrows X^*$ be maximally monotone. Then the following are equivalent.

(i) A is of type (D).
(ii) A is of type (NI).
(iii) A is of type (FP).

This is a powerful result because it is often easier to establish (ii) or (iii) than (i).

Fact 1.3. (See [36,38,16].) The following are maximally monotone of types (D), (NI), and (FP).

(i) ∂f, where $f : X \to \left]-\infty, +\infty\right]$ is convex, lower semicontinuous, and proper;
(ii) $A : X \rightrightarrows X^*$, where A is maximally monotone and X is reflexive.

These and other relationships known amongst these and other monotonicity notions are described in [16, Chapter 9]. Monotone operators have proven to be a key class of objects in both modern Optimization and Analysis; see, e.g., [13–15], the books [7,16,20,30,36,38,33,44–46] and the references therein.

Let us now precisely state the aforementioned Brezis–Browder Theorem:

Theorem 1.4 *(Brezis–Browder in reflexive Banach space). (See [18,19].) Suppose that X is reflexive. Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that* $\operatorname{gra} A$ *is closed. Then A is maximally monotone if and only if the adjoint A^* is monotone.*

In this paper, we generalize the Brezis–Browder Theorem to an arbitrary Banach space. (See also [39] and [41] for Simons' recent extensions in the context of symmetrically self-dual Banach (SSDB) spaces as defined in [38, §21] and of Banach SNL spaces.)

Our main result is the following.

Theorem 1.5 *(Brezis–Browder in general Banach space). Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that* gra A *is closed. Then A is maximally monotone of type* (D) *if and only if A^* is monotone.*

This result will also give an affirmative answer to a question posed by Phelps and Simons in [31, Section 9, item 2]:

> *Let $A : \operatorname{dom} A \to X^*$ be linear and maximally monotone. Assume that A^* is monotone. Is A necessarily of type* (D)?

The paper is organized as follows. In Section 2, we collect auxiliary results for future reference and for the reader's convenience. In Section 3, we provide the key technical step enabling us to show that when A^* is monotone then A is of type (D). Our central result, the generalized Brezis–Browder Theorem (Theorem 1.5), is then proved in Section 4. Finally, in Section 5 with the necessary proviso that the domain be closed, we establish further results such as Theorem 5.5 relating to the skew part of the operator. This was motivated by and extends [2, Theorem 4.1] which studied the case of a bounded linear operator.

Finally, let us mention that we adopt standard convex analysis notation. Given a subset C of X, $\operatorname{int} C$ is the *interior* of C, $\overline{C}$ is the *norm closure* of C. For the set $D \subseteq X^*$, $\overline{D}^{\mathrm{w}*}$ is the weak* closure of D. If $E \subseteq X^{**}$, $\overline{E}^{\mathrm{w}*}$ is the weak* closure of E in X^{**} with the topology induced by X^*. The *indicator function* of C, written as ι_C, is defined at $x \in X$ by

$$\iota_C(x) = \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise.} \end{cases} \tag{2}$$

For every $x \in X$, the *normal cone* operator of C at x is defined by $N_C(x) = \{x^* \in X^* \mid \sup_{c \in C} \langle c - x, x^* \rangle \leqslant 0\}$, if $x \in C$; and $N_C(x) = \varnothing$, if $x \notin C$.

Let $f : X \to \left]-\infty, +\infty\right]$. Then $\operatorname{dom} f = f^{-1}(\mathbb{R})$ is the *domain* of f, and $f^* : X^* \to [-\infty, +\infty] : x^* \mapsto \sup_{x \in X}(\langle x, x^* \rangle - f(x))$ is the *Fenchel conjugate* of f. The *epigraph* of f is $\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} \mid f(x) \leqslant r\}$. The *lower semicontinuous hull* of f, denoted by $\overline{f}$, is the function defined at every $x \in X$ by $\overline{f}(x) = \inf\{t \in \mathbb{R} \mid (x, t) \in \overline{\operatorname{epi} f}\}$. We say f is proper if $\operatorname{dom} f \neq \varnothing$. Let f be proper. The *subdifferential* of f is defined by

$$\partial f : X \rightrightarrows X^* : x \mapsto \{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^* \rangle + f(x) \leqslant f(y)\}.$$

For $\varepsilon \geqslant 0$, the *ε-subdifferential* of f is defined by $\partial_\varepsilon f : X \rightrightarrows X^* : x \mapsto \{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^* \rangle + f(x) \leqslant f(y) + \varepsilon\}$. Note that $\partial f = \partial_0 f$.

Let $F : X \times X^* \to \left]-\infty, +\infty\right]$. We say F is a *representative* of the monotone operator $A : X \rightrightarrows X^*$ if F is proper, lower semicontinuous and convex with $F \geqslant \langle \cdot, \cdot \rangle$ on $X \times X^*$ and

$$\operatorname{gra} A = \{(x, x^*) \in X \times X^* \mid F(x, x^*) = \langle x, x^* \rangle\}.$$

Let $(z,z^*)\in X\times X^*$ and $F:X\times X^*\to\,]-\infty,+\infty]$. Then $F_{(z,z^*)}:X\times X^*\to\,]-\infty,+\infty]$ [27,38] is defined by

$$\begin{aligned}F_{(z,z^*)}(x,x^*)&=F(z+x,z^*+x^*)-\big(\langle x,z^*\rangle+\langle z,x^*\rangle+\langle z,z^*\rangle\big)\\&=F(z+x,z^*+x^*)-\langle z+x,z^*+x^*\rangle+\langle x,x^*\rangle,\quad\forall(x,x^*)\in X\times X^*.\end{aligned}\tag{3}$$

Let now Y be another real Banach space. We set $P_X:X\times Y\to X:(x,y)\mapsto x$. Let $F_1,F_2:X\times Y\to\,]-\infty,+\infty]$. Then the *partial inf-convolution* $F_1\,\Box_2\,F_2$ is the function defined on $X\times Y$ by

$$F_1\,\Box_2\,F_2:(x,y)\mapsto\inf_{v\in Y}\big[F_1(x,y-v)+F_2(x,v)\big].$$

2. Prerequisite results

In this section, we gather some facts and auxiliary results used in the sequel.

Fact 2.1. (See [28, Proposition 2.6.6(c)] or [34, Theorem 4.7 and Theorem 3.12].) Let C be a subspace of X, and D be a subspace of X^*. Then

$$\big(C^\perp\big)_\perp=\overline{C}\quad\text{and}\quad(D_\perp)^\perp=\overline{D}^{\mathrm{w}*}.$$

Fact 2.2 *(Rockafellar).* (See [32, Theorem 3], [38, Corollary 10.3 and Theorem 18.1] or [44, Theorem 2.8.7(iii)].) Let $f,g:X\to\,]-\infty,+\infty]$ be proper convex functions. Assume that there exists a point $x_0\in\operatorname{dom}f\cap\operatorname{dom}g$ such that g is continuous at x_0. Then

$$(f+g)^*(x^*)=\min_{y^*\in X^*}\big[f^*(y^*)+g^*(x^*-y^*)\big],\quad\forall x^*\in X^*,$$
$$\partial(f+g)=\partial f+\partial g.$$

Fact 2.3 *(Borwein).* (See [44, Theorem 3.1.1] which is based on [11, Theorem 1].) Let $f:X\to\,]-\infty,+\infty]$ be a proper lower semicontinuous and convex function. Let $\varepsilon>0$ and $\beta\geqslant 0$ (where $\frac{1}{0}=\infty$). Assume that $x_0\in\operatorname{dom}f$ and $x_0^*\in\partial_\varepsilon f(x_0)$. There exist $x_\varepsilon\in X$, $x_\varepsilon^*\in X^*$ such that

$$\|x_\varepsilon-x_0\|+\beta\big|\langle x_\varepsilon-x_0,x_0^*\rangle\big|\leqslant\sqrt{\varepsilon},\quad x_\varepsilon^*\in\partial f(x_\varepsilon),$$
$$\big\|x_\varepsilon^*-x_0^*\big\|\leqslant\sqrt{\varepsilon}\big(1+\beta\big\|x_0^*\big\|\big),\qquad\big|\langle x_\varepsilon-x_0,x_\varepsilon^*\rangle\big|\leqslant\varepsilon+\frac{\sqrt{\varepsilon}}{\beta}.$$

Fact 2.4 *(Attouch–Brezis).* (See [1, Theorem 1.1] or [38, Remark 15.2].) Let $f,g:X\to\,]-\infty,+\infty]$ be proper lower semicontinuous and convex. Assume that $\bigcup_{\lambda>0}\lambda[\operatorname{dom}f-\operatorname{dom}g]$ is a closed subspace of X. Then

$$(f+g)^*(z^*)=\min_{y^*\in X^*}\big[f^*(y^*)+g^*(z^*-y^*)\big],\quad\forall z^*\in X^*.$$

Fact 2.5 *(Simons and Zălinescu).* (See [42, Theorem 4.2] or [38, Theorem 16.4(a)].) Let Y be a real Banach space and $F_1, F_2 : X \times Y \to \left]-\infty, +\infty\right]$ be proper, lower semicontinuous, and convex. Assume that for every $(x, y) \in X \times Y$,

$$(F_1 \Box_2 F_2)(x, y) > -\infty$$

and that $\bigcup_{\lambda>0} \lambda[P_X \operatorname{dom} F_1 - P_X \operatorname{dom} F_2]$ is a closed subspace of X. Then for every $(x^*, y^*) \in X^* \times Y^*$,

$$(F_1 \Box_2 F_2)^*\big(x^*, y^*\big) = \min_{u^* \in X^*} \big[F_1^*\big(x^* - u^*, y^*\big) + F_2^*\big(u^*, y^*\big)\big].$$

The following result was first established in [12, Theorem 7.4]. We next provide a new proof.

Fact 2.6 *(Borwein).* Let $A, B : X \rightrightarrows X^*$ be linear relations such that $\operatorname{gra} A$ and $\operatorname{gra} B$ are closed. Assume that $\operatorname{dom} A - \operatorname{dom} B$ is closed. Then

$$(A + B)^* = A^* + B^*.$$

Proof. We have

$$\iota_{\operatorname{gra}(A+B)} = \iota_{\operatorname{gra} A} \Box_2 \iota_{\operatorname{gra} B}. \tag{4}$$

Let $(x^{**}, x^*) \in X^{**} \times X^*$. Since $\operatorname{gra} A$ and $\operatorname{gra} B$ are closed convex, $\iota_{\operatorname{gra} A}$ and $\iota_{\operatorname{gra} B}$ are proper lower semicontinuous and convex. Then by Fact 2.5 and (4),

$$\begin{aligned}
\iota_{\operatorname{gra}(A+B)^*}\big(x^{**}, x^*\big) &= \iota_{(\operatorname{gra}(A+B))^\perp}\big(-x^*, x^{**}\big) \\
&= \iota^*_{\operatorname{gra}(A+B)}\big(-x^*, x^{**}\big) \quad \big(\text{since } \operatorname{gra}(A+B) \text{ is a subspace}\big) \\
&= \min_{y^* \in X^*} \big[\iota^*_{\operatorname{gra} A}\big(y^*, x^{**}\big) + \iota^*_{\operatorname{gra} B}\big(-x^* - y^*, x^{**}\big)\big] \\
&= \min_{y^* \in X^*} \big[\iota_{(\operatorname{gra} A)^\perp}\big(y^*, x^{**}\big) + \iota_{(\operatorname{gra} B)^\perp}\big(-x^* - y^*, x^{**}\big)\big] \\
&= \min_{y^* \in X^*} \big[\iota_{\operatorname{gra} A^*}\big(x^{**}, -y^*\big) + \iota_{\operatorname{gra} B^*}\big(x^{**}, x^* + y^*\big)\big] \\
&= \iota_{\operatorname{gra}(A^*+B^*)}\big(x^{**}, x^*\big).
\end{aligned}$$

It follows that $\operatorname{gra}(A + B)^* = \operatorname{gra}(A^* + B^*)$, i.e., $(A + B)^* = A^* + B^*$. $\Box$

Fact 2.7 *(Simons).* (See [38, Lemma 19.7 and Section 22].) Let $A : X \rightrightarrows X^*$ be a monotone operator such that $\operatorname{gra} A$ is convex with $\operatorname{gra} A \neq \varnothing$. Then the function

$$g : X \times X^* \to \left]-\infty, +\infty\right] : \big(x, x^*\big) \mapsto \big\langle x, x^*\big\rangle + \iota_{\operatorname{gra} A}\big(x, x^*\big) \tag{5}$$

is proper and convex.

Fact 2.8 *(Marques Alves and Svaiter).* (See [26, Theorem 4.4].) Let $A : X \rightrightarrows X^*$ be maximally monotone, and let $F : X \to]-\infty, +\infty]$ be a representative of A. Then A is of type (D) if and only if for every $(x_0, x_0^*) \in X \times X^*$,

$$\inf_{(x,x^*)\in X\times X^*}\left[F_{(x_0,x_0^*)}\big(x,x^*\big)+\frac{1}{2}\|x\|^2+\frac{1}{2}\left\|x^*\right\|^2\right]=0.$$

We also recall the following somewhat more precise version of Theorem 1.4.

Fact 2.9 *(Brezis and Browder).* (See [19, Theorem 2], or [17,18,39,43].) Suppose that X is reflexive. Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that $\operatorname{gra} A$ is closed. Then the following are equivalent.

(i) A is maximally monotone.
(ii) A^* is maximally monotone.
(iii) A^* is monotone.

Now let us cite some basic properties of linear relations.

The following result appeared in Cross' book [21]. We give new proofs of items (iv)–(vi). The proof of item (vi) below was adapted from [10, Remark 2.2].

Fact 2.10. Let $A : X \rightrightarrows X^*$ be a linear relation. Then the following hold.

(i) $Ax = x^* + A0$, $\forall x^* \in Ax$.
(ii) $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$, $\forall(\alpha, \beta) \in \mathbb{R}^2 \smallsetminus \{(0,0)\}$, $\forall x, y \in \operatorname{dom} A$.
(iii) $\langle A^*x, y\rangle = \langle x, Ay\rangle$ is a singleton, $\forall x \in \operatorname{dom} A^*$, $\forall y \in \operatorname{dom} A$.
(iv) $(\operatorname{dom} A)^\perp = A^*0$ is (weak*) closed and $\overline{\operatorname{dom} A} = (A^*0)_\perp$.
(v) If $\operatorname{gra} A$ is closed, then $(\operatorname{dom} A^*)_\perp = A0$ and $\overline{\operatorname{dom} A^*}^{\mathrm{w}*} = (A0)^\perp$.
(vi) If $\operatorname{dom} A$ is closed, then $\operatorname{dom} A^* = (\bar{A}0)^\perp$ and thus $\operatorname{dom} A^*$ is (weak*) closed, where $\bar{A}$ is the linear relation whose graph is the closure of the graph of A.

Proof. (i): See [21, Proposition I.2.8(a)]. (ii): See [21, Corollary I.2.5]. (iii): See [21, Proposition III.1.2].

(iv): We have

$$x^* \in A^*0 \quad\Leftrightarrow\quad \big(x^*, 0\big) \in (\operatorname{gra} A)^\perp \quad\Leftrightarrow\quad x^* \in (\operatorname{dom} A)^\perp.$$

Hence $(\operatorname{dom} A)^\perp = A^*0$ and thus A^*0 is weak* closed. By Fact 2.1, $\overline{\operatorname{dom} A} = (A^*0)_\perp$.

(v): Using Fact 2.1,

$$x^* \in A0 \quad\Leftrightarrow\quad \big(0, x^*\big) \in \operatorname{gra} A = \big[(\operatorname{gra} A)^\perp\big]_\perp = \big[\operatorname{gra} -\big(A^*\big)^{-1}\big]_\perp \quad\Leftrightarrow\quad x^* \in \big(\operatorname{dom} A^*\big)_\perp.$$

Hence $(\operatorname{dom} A^*)_\perp = A0$ and thus, by Fact 2.1, $\overline{\operatorname{dom} A^*}^{\mathrm{w}*} = (A0)^\perp$.

(vi): Let $\bar{A}$ be the linear relation whose graph is the closure of the graph of A. Then $\operatorname{dom} A = \operatorname{dom} \bar{A}$ and $A^* = \bar{A}^*$. Then by the Attouch–Brezis Theorem (Fact 2.4),

$$\iota_{X^*\times(\bar{A}0)^\perp} = \iota^*_{\{0\}\times\bar{A}0} = (\iota_{\operatorname{gra}\bar{A}} + \iota_{\{0\}\times X^*})^* = \iota_{\operatorname{gra}(-\bar{A}^*)^{-1}} \,\Box\, \iota_{X^*\times\{0\}} = \iota_{X^*\times\operatorname{dom}\bar{A}^*}.$$

It is clear that $\operatorname{dom} A^* = \operatorname{dom} \bar{A}^* = (\bar{A}0)^\perp$ is weak* closed, hence closed. $\square$

3. A key result

The proof of Proposition 3.1 was in part inspired by that of [46, Theorem 32.L] and by that of [25, Theorem 2.1].

Proposition 3.1. *Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that* gra A *is closed and A^* is monotone. Define*

$$F : X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \iota_{\operatorname{gra} A}(x, x^*) + \langle x, x^* \rangle.$$

Then F is a representative of A, and

$$\inf_{(x,x^*)\in X\times X^*}\left[F_{(v_0,v_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2\right] = 0, \quad \forall (v_0, v_0^*) \in X \times X^*.$$

Proof. Since A is monotone and gra A is closed, Fact 2.7 implies that F is proper lower semicontinuous and convex, and a representative of A. Let $(v_0, v_0^*) \in X \times X^*$. Recalling (3), note that

$$F_{(v_0,v_0^*)} : (x, x^*) \mapsto \iota_{\operatorname{gra} A}(v_0 + x, v_0^* + x^*) + \langle x, x^* \rangle \tag{6}$$

is proper lower semicontinuous and convex. By Fact 2.2, there exists $(y^{**}, y^*) \in X^{**} \times X^*$ such that

$$\begin{aligned} K &:= \inf_{(x,x^*)\in X\times X^*}\left[F_{(v_0,v_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2\right] \\ &= -\left(F_{(v_0,v_0^*)} + \frac{1}{2}\|\cdot\|^2 + \frac{1}{2}\|\cdot\|^2\right)^*(0,0) \\ &= -F^*_{(v_0,v_0^*)}(y^*, y^{**}) - \frac{1}{2}\|y^{**}\|^2 - \frac{1}{2}\|y^*\|^2. \end{aligned} \tag{7}$$

Since $(x, x^*) \mapsto F_{(v_0,v_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2$ is coercive, there exist $M > 0$ and a sequence $(a_n, a_n^*)_{n\in\mathbb{N}}$ in $X \times X^*$ such that

$$\|a_n\| + \|a_n^*\| \leqslant M \tag{8}$$

and

$$\begin{aligned} &F_{(v_0,v_0^*)}(a_n, a_n^*) + \frac{1}{2}\|a_n\|^2 + \frac{1}{2}\|a_n^*\|^2 \\ &\quad < K + \frac{1}{n^2} = -F^*_{(v_0,v_0^*)}(y^*, y^{**}) - \frac{1}{2}\|y^{**}\|^2 - \frac{1}{2}\|y^*\|^2 + \frac{1}{n^2} \quad \text{(by (7))} \\ &\quad \Rightarrow \quad F_{(v_0,v_0^*)}(a_n, a_n^*) + \frac{1}{2}\|a_n\|^2 + \frac{1}{2}\|a_n^*\|^2 + F^*_{(v_0,v_0^*)}(y^*, y^{**}) \\ &\qquad\quad + \frac{1}{2}\|y^{**}\|^2 + \frac{1}{2}\|y^*\|^2 < \frac{1}{n^2} \end{aligned} \tag{9}$$

$$\Rightarrow \quad F_{(v_0,v_0^*)}\big(a_n,a_n^*\big)+F^*_{(v_0,v_0^*)}\big(y^*,y^{**}\big)+\langle a_n,-y^*\rangle+\langle a_n^*,-y^{**}\rangle<\frac{1}{n^2} \tag{10}$$

$$\Rightarrow \quad \big(y^*,y^{**}\big)\in\partial_{\frac{1}{n^2}}F_{(v_0,v_0^*)}\big(a_n,a_n^*\big) \quad \big(\text{by [44, Theorem 2.4.2(ii)]}\big). \tag{11}$$

Set $\beta=\frac{1}{\max\{\|y^*\|,\|y^{**}\|\}+1}$. Then by Fact 2.3, there exist sequences $(\widetilde{a_n},\widetilde{a_n^*})_{n\in\mathbb{N}}$ in $X\times X^*$ and $(y_n^*,y_n^{**})_{n\in\mathbb{N}}$ in $X^*\times X^{**}$ such that

$$\|a_n-\widetilde{a_n}\|+\big\|a_n^*-\widetilde{a_n^*}\big\|+\beta\big|\langle\widetilde{a_n}-a_n,y^*\rangle+\langle\widetilde{a_n^*}-a_n^*,y^{**}\rangle\big|\leqslant\frac{1}{n}, \tag{12}$$

$$\max\big\{\big\|y_n^*-y^*\big\|,\big\|y_n^{**}-y^{**}\big\|\big\}\leqslant\frac{2}{n}, \tag{13}$$

$$\big|\langle\widetilde{a_n}-a_n,y_n^*\rangle+\langle\widetilde{a_n^*}-a_n^*,y_n^{**}\rangle\big|\leqslant\frac{1}{n^2}+\frac{1}{n\beta}, \tag{14}$$

$$\big(y_n^*,y_n^{**}\big)\in\partial F_{(v_0,v_0^*)}\big(\widetilde{a_n},\widetilde{a_n^*}\big), \quad \forall n\in\mathbb{N}. \tag{15}$$

Then we have

$$\begin{aligned}
&\langle\widetilde{a_n},y_n^*\rangle+\langle\widetilde{a_n^*},y_n^{**}\rangle-\langle a_n,y^*\rangle-\langle a_n^*,y^{**}\rangle\\
&\quad=\langle\widetilde{a_n}-a_n,y_n^*\rangle+\langle a_n,y_n^*-y^*\rangle+\langle\widetilde{a_n^*}-a_n^*,y_n^{**}\rangle+\langle a_n^*,y_n^{**}-y^{**}\rangle\\
&\quad\leqslant\big|\langle\widetilde{a_n}-a_n,y_n^*\rangle+\langle\widetilde{a_n^*}-a_n^*,y_n^{**}\rangle\big|+\big|\langle a_n,y_n^*-y^*\rangle\big|+\big|\langle a_n^*,y_n^{**}-y^{**}\rangle\big|\\
&\quad\leqslant\frac{1}{n^2}+\frac{1}{n\beta}+\|a_n\|\cdot\big\|y_n^*-y^*\big\|+\big\|a_n^*\big\|\cdot\big\|y_n^{**}-y^{**}\big\| \quad \big(\text{by (14)}\big)\\
&\quad\leqslant\frac{1}{n^2}+\frac{1}{n\beta}+\big(\|a_n\|+\big\|a_n^*\big\|\big)\cdot\max\big\{\big\|y_n^*-y^*\big\|,\big\|y_n^{**}-y^{**}\big\|\big\}\\
&\quad\leqslant\frac{1}{n^2}+\frac{1}{n\beta}+\frac{2}{n}M \quad \big(\text{by (8) and (13)}\big),\ \forall n\in\mathbb{N}.
\end{aligned} \tag{16}$$

By (12), we have

$$\big|\|a_n\|-\|\widetilde{a_n}\|\big|+\big|\big\|a_n^*\big\|-\big\|\widetilde{a_n^*}\big\|\big|\leqslant\frac{1}{n}. \tag{17}$$

Thus by (8), we have

$$\begin{aligned}
&\big|\|a_n\|^2-\|\widetilde{a_n}\|^2\big|+\big|\big\|a_n^*\big\|^2-\big\|\widetilde{a_n^*}\big\|^2\big|\\
&\quad=\big|\|a_n\|-\|\widetilde{a_n}\|\big|\big(\|a_n\|+\|\widetilde{a_n}\|\big)+\big|\big\|a_n^*\big\|-\big\|\widetilde{a_n^*}\big\|\big|\big(\big\|a_n^*\big\|+\big\|\widetilde{a_n^*}\big\|\big)\\
&\quad\leqslant\frac{1}{n}\left(2\|a_n\|+\frac{1}{n}\right)+\frac{1}{n}\left(2\big\|a_n^*\big\|+\frac{1}{n}\right) \quad \big(\text{by (17)}\big)\\
&\quad\leqslant\frac{1}{n}\left(2M+\frac{2}{n}\right)=\frac{2}{n}M+\frac{2}{n^2}, \quad \forall n\in\mathbb{N}.
\end{aligned} \tag{18}$$

Similarly, by (13), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \big|\|y_n^*\|^2 - \|y^*\|^2\big| &\leqslant \frac{4}{n}\|y^*\| + \frac{4}{n^2} \leqslant \frac{4}{n\beta} + \frac{4}{n^2}, \\ \big|\|y_n^{**}\|^2 - \|y^{**}\|^2\big| &\leqslant \frac{4}{n}\|y^{**}\| + \frac{4}{n^2} \leqslant \frac{4}{n\beta} + \frac{4}{n^2}. \end{aligned} \tag{19}$$

Thus

$$\begin{aligned} &F_{(v_0,v_0^*)}\big(\widetilde{a_n}, \widetilde{a_n^*}\big) + F^*_{(v_0,v_0^*)}\big(y_n^*, y_n^{**}\big) + \frac{1}{2}\|\widetilde{a_n}\|^2 + \frac{1}{2}\big\|\widetilde{a_n^*}\big\|^2 + \frac{1}{2}\big\|y_n^*\big\|^2 + \frac{1}{2}\big\|y_n^{**}\big\|^2 \\ &= \left[F_{(v_0,v_0^*)}\big(\widetilde{a_n}, \widetilde{a_n^*}\big) + F^*_{(v_0,v_0^*)}\big(y_n^*, y_n^{**}\big) + \frac{1}{2}\|\widetilde{a_n}\|^2 + \frac{1}{2}\big\|\widetilde{a_n^*}\big\|^2 + \frac{1}{2}\big\|y_n^*\big\|^2 + \frac{1}{2}\big\|y_n^{**}\big\|^2\right] \\ &\quad - \left[F_{(v_0,v_0^*)}\big(a_n, a_n^*\big) + \frac{1}{2}\|a_n\|^2 + \frac{1}{2}\big\|a_n^*\big\|^2 + F^*_{(v_0,v_0^*)}\big(y^*, y^{**}\big) + \frac{1}{2}\big\|y^{**}\big\|^2 + \frac{1}{2}\big\|y^*\big\|^2\right] \\ &\quad + \left[F_{(v_0,v_0^*)}\big(a_n, a_n^*\big) + \frac{1}{2}\|a_n\|^2 + \frac{1}{2}\big\|a_n^*\big\|^2 + F^*_{(v_0,v_0^*)}\big(y^*, y^{**}\big) + \frac{1}{2}\big\|y^{**}\big\|^2 + \frac{1}{2}\big\|y^*\big\|^2\right] \\ &< \big[F_{(v_0,v_0^*)}\big(\widetilde{a_n}, \widetilde{a_n^*}\big) + F^*_{(v_0,v_0^*)}\big(y_n^*, y_n^{**}\big) - F_{(v_0,v_0^*)}\big(a_n, a_n^*\big) - F^*_{(v_0,v_0^*)}\big(y^*, y^{**}\big)\big] \\ &\quad + \frac{1}{2}\big[\|\widetilde{a_n}\|^2 + \big\|\widetilde{a_n^*}\big\|^2 - \|a_n\|^2 - \big\|a_n^*\big\|^2\big] \\ &\quad + \frac{1}{2}\big[\big\|y_n^*\big\|^2 + \big\|y_n^{**}\big\|^2 - \big\|y^{**}\big\|^2 - \big\|y^*\big\|^2\big] + \frac{1}{n^2} \quad \big(\text{by } (9)\big) \\ &\leqslant \big[\langle\widetilde{a_n}, y_n^*\rangle + \langle\widetilde{a_n^*}, y_n^{**}\rangle - \langle a_n, y^*\rangle - \langle a_n^*, y^{**}\rangle\big] \quad \big(\text{by } (15)\big) \\ &\quad + \frac{1}{2}\big(\big|\|\widetilde{a_n}\|^2 - \|a_n\|^2\big| + \big|\big\|\widetilde{a_n^*}\big\|^2 - \big\|a_n^*\big\|^2\big|\big) \\ &\quad + \frac{1}{2}\big(\big|\big\|y_n^*\big\|^2 - \big\|y^*\big\|^2\big| + \big|\big\|y_n^{**}\big\|^2 - \big\|y^{**}\big\|^2\big|\big) + \frac{1}{n^2} \\ &\leqslant \frac{1}{n^2} + \frac{1}{n\beta} + \frac{2}{n}M + \frac{1}{n}M + \frac{1}{n^2} + \frac{4}{n\beta} + \frac{4}{n^2} + \frac{1}{n^2} \quad \big(\text{by } (16), (18) \text{ and } (19)\big) \\ &= \frac{7}{n^2} + \frac{5}{n\beta} + \frac{3}{n}M, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{20}$$

By (15), (6), and [44, Theorem 3.2.4(vi)&(ii)], there exists a sequence $(z_n^*, z_n^{**})_{n\in\mathbb{N}}$ in $(\operatorname{gra} A)^{\perp}$ such that

$$\big(y_n^*, y_n^{**}\big) = \big(\widetilde{a_n^*}, \widetilde{a_n}\big) + \big(z_n^*, z_n^{**}\big), \quad \forall n \in \mathbb{N}. \tag{21}$$

Since A^* is monotone and $(z_n^{**}, z_n^*) \in \operatorname{gra}(-A^*)$, it follows from (21) that

$$\begin{aligned} &\big\langle y_n^*, y_n^{**}\big\rangle - \big\langle y_n^*, \widetilde{a_n}\big\rangle - \big\langle y_n^{**}, \widetilde{a_n^*}\big\rangle + \big\langle\widetilde{a_n^*}, \widetilde{a_n}\big\rangle = \big\langle y_n^* - \widetilde{a_n^*}, y_n^{**} - \widetilde{a_n}\big\rangle = \big\langle z_n^*, z_n^{**}\big\rangle \leqslant 0 \\ &\quad\Rightarrow\quad \big\langle y_n^*, y_n^{**}\big\rangle \leqslant \big\langle y_n^*, \widetilde{a_n}\big\rangle + \big\langle y_n^{**}, \widetilde{a_n^*}\big\rangle - \big\langle\widetilde{a_n^*}, \widetilde{a_n}\big\rangle, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Then by (6) and (15), we have $\langle \widetilde{a_n^*}, \widetilde{a_n} \rangle = F_{(v_0, v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*})$ and

$$\langle y_n^*, y_n^{**} \rangle \leqslant \langle y_n^*, \widetilde{a_n} \rangle + \langle y_n^{**}, \widetilde{a_n^*} \rangle - F_{(v_0, v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}) = F^*_{(v_0, v_0^*)}(y_n^*, y_n^{**}), \quad \forall n \in \mathbb{N}. \tag{22}$$

By (20) and (22), we have

$$F_{(v_0, v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}) + \langle y_n^*, y_n^{**} \rangle + \frac{1}{2}\|\widetilde{a_n}\|^2 + \frac{1}{2}\|\widetilde{a_n^*}\|^2 + \frac{1}{2}\|y_n^*\|^2 + \frac{1}{2}\|y_n^{**}\|^2 < \frac{7}{n^2} + \frac{5}{n\beta} + \frac{3}{n}M$$

$$\Rightarrow \quad F_{(v_0, v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}) + \frac{1}{2}\|\widetilde{a_n}\|^2 + \frac{1}{2}\|\widetilde{a_n^*}\|^2 < \frac{7}{n^2} + \frac{5}{n\beta} + \frac{3}{n}M, \quad \forall n \in \mathbb{N}. \tag{23}$$

Thus by (23),

$$\inf_{(x,x^*) \in X \times X^*} \left[F_{(v_0, v_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \right] \leqslant 0. \tag{24}$$

By (6),

$$\inf_{(x,x^*) \in X \times X^*} \left[F_{(v_0, v_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \right] \geqslant 0. \tag{25}$$

Combining (24) with (25), we obtain

$$\inf_{(x,x^*) \in X \times X^*} \left[F_{(v_0, v_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \right] = 0. \qquad \square \tag{26}$$

Proposition 3.2. *Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that* gra A *is closed and A^* is monotone. Then A is maximally monotone of type* (D).

Proof. By Fact 2.8 and Proposition 3.1, it suffices to show that A is maximally monotone. Let $(z, z^*) \in X \times X^*$. Assume that

$$(z, z^*) \text{ is monotonically related to } \operatorname{gra} A. \tag{27}$$

Define

$$F : X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \iota_{\operatorname{gra} A}(x, x^*) + \langle x, x^* \rangle.$$

We have

$$F_{(z,z^*)} : (x, x^*) \mapsto \iota_{\operatorname{gra} A}(z + x, z^* + x^*) + \langle x, x^* \rangle. \tag{28}$$

Proposition 3.1 implies that there exists a sequence $(x_n, x_n^*)_{n \in \mathbb{N}}$ in dom $F_{(z,z^*)}$ such that

$$F_{(z,z^*)}(x_n, x_n^*) + \frac{1}{2}\|x_n\|^2 + \frac{1}{2}\|x_n^*\|^2 \to 0. \tag{29}$$

Set $(a_n, a_n^*) := (z + x_n, z^* + x_n^*)$, $\forall n \in \mathbb{N}$. Then by (28), we have

$$F_{(z,z^*)}\big(x_n, x_n^*\big) = \iota_{\operatorname{gra} A}\big(z + x_n, z^* + x_n^*\big) + \big\langle x_n, x_n^*\big\rangle \tag{30}$$

$$= \iota_{\operatorname{gra} A}\big(a_n, a_n^*\big) + \big\langle a_n - z, a_n^* - z^*\big\rangle. \tag{31}$$

By (29) and (31),

$$\big(a_n, a_n^*\big) \in \operatorname{gra} A, \quad \forall n \in \mathbb{N}. \tag{32}$$

Then by (32) and (27), we have

$$\big\langle x_n, x_n^*\big\rangle = \big\langle a_n - z, a_n^* - z^*\big\rangle \geqslant 0, \quad \forall n \in \mathbb{N}. \tag{33}$$

Combining (31) and (33),

$$F_{(z,z^*)}\big(x_n, x_n^*\big) \geqslant 0, \quad \forall n \in \mathbb{N}. \tag{34}$$

In view of (34) and (29),

$$\|x_n\|^2 + \big\|x_n^*\big\|^2 \to 0. \tag{35}$$

Thus $(x_n, x_n^*) \to (0,0)$ and hence $(a_n, a_n^*) \to (z, z^*)$. Finally, by (32) and since $\operatorname{gra} A$ is closed, we see $(z, z^*) \in \operatorname{gra} A$. Therefore, A is maximally monotone. □

Remark 3.3. Proposition 3.2 provides an affirmative answer to a problem posed by Phelps and Simons in [31, Section 9, item 2] on the converse of [31, Theorem 6.7(c) ⇒ (f)]. In [4, Proposition 3.1], we present an alternative proof for Proposition 3.2.

Example 3.4. Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that $\operatorname{gra} A$ is closed. We note that we cannot guarantee the maximal monotonicity of A even if A is at most single-valued and densely defined. To see this, suppose that $X = \ell^2$, and that $A : \ell^2 \rightrightarrows \ell^2$ is given by

$$Ax := \frac{(\sum_{i<n} x_i - \sum_{i>n} x_i)_{n\in\mathbb{N}}}{2} = \left(\sum_{i<n} x_i + \frac{1}{2}x_n\right)_{n\in\mathbb{N}}, \quad \forall x = (x_n)_{n\in\mathbb{N}} \in \operatorname{dom} A, \tag{36}$$

where $\operatorname{dom} A := \{x := (x_n)_{n\in\mathbb{N}} \in \ell^2 \mid \sum_{i\geqslant 1} x_i = 0,\ (\sum_{i\leqslant n} x_i)_{n\in\mathbb{N}} \in \ell^2\}$. Then A is an at most single-valued linear relation. Now [9, Propositions 3.6] states that

$$A^*x = \left(\frac{1}{2}x_n + \sum_{i>n} x_i\right)_{n\in\mathbb{N}}, \tag{37}$$

where

$$x = (x_n)_{n\in\mathbb{N}} \in \operatorname{dom} A^* = \left\{x = (x_n)_{n\in\mathbb{N}} \in \ell^2 \;\middle|\; \left(\sum_{i>n} x_i\right)_{n\in\mathbb{N}} \in \ell^2\right\}.$$

Moreover, [9, Propositions 3.2, 3.5, 3.6 and 3.8], [31, Theorem 2.5] and Fact 2.9 show that:

(i) A is maximally monotone and skew;
(ii) $\operatorname{dom} A$ is dense and $\operatorname{dom} A \subsetneqq \operatorname{dom} A^*$;
(iii) A^* is maximally monotone, but not skew;
(iv) $-A$ is not maximally monotone and $A^*|_{\operatorname{dom} A} = -A|_{\operatorname{dom} A}$.

Hence, $-A$ is monotone with dense domain and $\operatorname{gra}(-A)$ is closed, but nonetheless $-A$ is not maximally monotone.

4. The general Brezis–Browder Theorem

We may now pack everything together to establish our central result. For the reader's convenience we repeat Theorem 1.5:

Theorem 4.1 *(Brezis–Browder in general Banach space). Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that* $\operatorname{gra} A$ *is closed. Then A is maximally monotone of type* (D) *if and only if A^* is monotone.*

Proof. " ⇒ ": By Fact 1.2, A is of type (NI). Suppose to the contrary that there exists $(a_0^{**}, a_0^*) \in \operatorname{gra} A^*$ such that $\langle a_0^{**}, a_0^*\rangle < 0$. Then we have

$$\sup_{(a,a^*)\in \operatorname{gra} A} \left(\langle a, -a_0^*\rangle + \langle a_0^{**}, a^*\rangle - \langle a, a^*\rangle\right) = \sup_{(a,a^*)\in \operatorname{gra} A} \left\{-\langle a, a^*\rangle\right\} = 0 < \langle -a_0^{**}, a_0^*\rangle,$$

which contradicts that A is type of (NI). Hence A^* is monotone.
"⇐": Apply Proposition 3.2 directly. □

Remark 4.2. The proof of the necessary part in Theorem 4.1 follows closely that of [19, Theorem 2].

The original Brezis and Browder result follows.

Corollary 4.3 *(Brezis and Browder). Suppose that X is reflexive. Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that* $\operatorname{gra} A$ *is closed. Then the following are equivalent.*

(i) *A is maximally monotone.*
(ii) *A^* is maximally monotone.*
(iii) *A^* is monotone.*

Proof. "(i) ⇔ (iii)": Apply Theorem 4.1 and Fact 1.3 directly.
"(ii) ⇒ (iii)": Clear.
"(iii) ⇒ (ii)": Since $\operatorname{gra} A$ is closed, $(A^*)^* = A$. Apply Theorem 4.1 to A^*. □

In the case of a skew operator we can add maximality of the adjoint and so we prefigure results of the next section:

Corollary 4.4 *(The skew case). Let $A : X \rightrightarrows X^*$ be a skew operator such that* $\operatorname{gra} A$ *is closed. Then the following are equivalent.*

(i) *A is maximally monotone of type* (D).
(ii) *A^* is monotone.*
(iii) *A^* is maximally monotone with respect to $X^{**} \times X^*$.*

Proof. By Theorem 4.1, it only remains to show:

"(ii) ⇒ (iii)": Let $(z^{**}, z^*) \in X^{**} \times X^*$ be monotonically related to $\operatorname{gra} A^*$. Since $\operatorname{gra}(-A) \subseteq \operatorname{gra} A^*$, (z^{**}, z^*) is monotonically related to $\operatorname{gra}(-A)$. Thus $(z^*, z^{**}) \in [\operatorname{gra}(-A)]^\perp$ since $\operatorname{gra} A$ is linear. Hence $(z^{**}, z^*) \in \operatorname{gra} A^*$. Hence A^* is maximally monotone. □

Remark 4.5. We cannot say A^* is maximally monotone with respect to $X^{**} \times X^{***}$ in Corollary 4.4(iii): indeed, let A be defined by

$$\operatorname{gra} A = \{0\} \times X^*.$$

Then $\operatorname{gra} A^* = \{0\} \times X^*$. If X is not reflexive, then $X^* \subsetneqq X^{***}$ and so $\operatorname{gra} A^*$ is a proper subset of $\{0\} \times X^{***}$. Hence A^* is not maximally monotone with respect to $X^{**} \times X^{***}$ although A is maximally monotone of type (D) by Fact 1.3(i), because $A = N_{\{0\}}$ is a normal cone (and hence subdifferential) operator.

We conclude with an application of Theorem 4.1 to an operator studied previously by Phelps and Simons [31].

Example 4.6. Suppose that $X = L^1[0,1]$ so that $X^* = L^\infty[0,1]$, let

$$D = \{x \in X \mid x \text{ is absolutely continuous, } x(0) = 0,\ x' \in X^*\},$$

and set

$$A : X \rightrightarrows X^* : x \mapsto \begin{cases} \{x'\}, & \text{if } x \in D; \\ \varnothing, & \text{otherwise.} \end{cases}$$

By [31, Example 4.3], A is an at most single-valued maximally monotone linear relation with proper dense domain, and A is neither symmetric nor skew. Moreover,

$$\operatorname{dom} A^* = \{z \in X^{**} \mid z \text{ is absolutely continuous, } z(1) = 0,\ z' \in X^*\} \subseteq X$$

$A^* z = -z', \forall z \in \operatorname{dom} A^*$, and A^* is monotone. Therefore, Theorem 4.1 implies that A is of type (D).

In the next section, we turn to the question of how the skew part of the adjoint behaves.

5. Decomposition of monotone linear relations

In this section, our main result states that A is maximally monotone of type (D) if and only if its skew part $A_\circ$ is maximally monotone of type (D) and $A^*0 = A0$. When $\operatorname{dom} A$ is closed, we also obtained a refined version of the Brezis–Browder Theorem.

Let us first gather some basic properties about monotone linear relations, and conditions for them to be maximally monotone.

The next three propositions were proven in reflexive spaces in [8, Proposition 2.2]. We adjust the proofs to a general Banach space setting.

Proposition 5.1 *(Monotone linear relations). Let $A : X \rightrightarrows X^*$ be a linear relation. Then the following hold.*

(i) *Suppose A is monotone. Then* $\operatorname{dom} A \subseteq (A0)_\perp$ *and* $A0 \subseteq (\operatorname{dom} A)^\perp$; *consequently, if* $\operatorname{gra} A$ *is closed, then* $\operatorname{dom} A \subseteq \overline{\operatorname{dom} A^*}^{\mathrm{w}*} \cap X$ *and* $A0 \subseteq A^*0$.
(ii) $(\forall x \in \operatorname{dom} A)$ $(\forall z \in (A0)_\perp)$ $\langle z, Ax\rangle$ *is single-valued.*
(iii) $(\forall z \in (A0)_\perp)$ $\operatorname{dom} A \to \mathbb{R} : y \mapsto \langle z, Ay\rangle$ *is linear.*
(iv) *If A is monotone, then* $(\forall x \in \operatorname{dom} A)$ $\langle x, Ax\rangle$ *is single-valued.*
(v) *A is monotone* $\Leftrightarrow$ $(\forall x \in \operatorname{dom} A)$ $\inf\langle x, Ax\rangle \geqslant 0$.
(vi) *If A is monotone, then so is $A + A^*$.*
(vii) *If A is monotone, then so are $A - A^*$ and $A^* - A$.*
(viii) *If $(x, x^*) \in (\operatorname{dom} A) \times X^*$ is monotonically related to* $\operatorname{gra} A$ *and $x_0^* \in Ax$, then $x^* - x_0^* \in$* $(\operatorname{dom} A)^\perp$.

Proof. (i): Pick $x \in \operatorname{dom} A$. Then there exists $x^* \in X^*$ such that $(x, x^*) \in \operatorname{gra} A$. By monotonicity of A and since $\{0\} \times A0 \subseteq \operatorname{gra} A$, we have $\langle x, x^*\rangle \geqslant \sup\langle x, A0\rangle$. Since $A0$ is a linear subspace, we obtain $x \perp A0$. This implies $\operatorname{dom} A \subseteq (A0)_\perp$ and $A0 \subseteq (\operatorname{dom} A)^\perp$. If $\operatorname{gra} A$ is closed, then Facts 2.10(v)&(iv) yield $\operatorname{dom} A \subseteq (A0)_\perp \subseteq (A0)^\perp = \overline{\operatorname{dom} A^*}^{\mathrm{w}*}$ and $A0 \subseteq A^*0$.

(ii): Take $x \in \operatorname{dom} A$, $x^* \in Ax$, and $z \in (A0)_\perp$. By Fact 2.10(i), $\langle z, Ax\rangle = \langle z, x^* + A0\rangle = \langle z, x^*\rangle$.

(iii): Take $z \in (A0)_\perp$. By (ii), $(\forall y \in \operatorname{dom} A)$ $\langle z, Ay\rangle$ is single-valued. Now let x, y be in $\operatorname{dom} A$, and let α, β be in $\mathbb{R}$. If $(\alpha, \beta) = (0, 0)$, then $\langle z, A(\alpha x + \beta y)\rangle = \langle z, A0\rangle = 0 = \alpha\langle z, Ax\rangle + \beta\langle z, Ay\rangle$. And if $(\alpha, \beta) \neq (0, 0)$, then Fact 2.10(ii) yields $\langle z, A(\alpha x + \beta y)\rangle = \langle z, \alpha Ax + \beta Ay\rangle = \alpha\langle z, Ax\rangle + \beta\langle z, Ay\rangle$. This verifies linearity.

(iv): Apply (i)&(ii).

(v): "$\Rightarrow$": This follows from the fact that $(0, 0) \in \operatorname{gra} A$. "$\Leftarrow$": If x and y belong to $\operatorname{dom} A$, then Fact 2.10(ii) yields $\langle x - y, Ax - Ay\rangle = \langle x - y, A(x - y)\rangle \geqslant 0$.

(vi): Let A be monotone. Set $B := A + A^*$ and take $x \in \operatorname{dom} B = \operatorname{dom} A \cap \operatorname{dom} A^*$. By (v) and Fact 2.10(iii), $0 \leqslant \langle x, Ax\rangle = \langle A^*x, x\rangle$. Hence $0 \leqslant 2\langle x, Ax\rangle = \langle x, Bx\rangle$. Again by (v), B is monotone.

(vii): Set $B := A - A^*$ and note that $\operatorname{dom} B = \operatorname{dom} A \cap \operatorname{dom} A^* = \operatorname{dom}(-B)$. Take $x \in \operatorname{dom} B$. By Fact 2.10(iii), $\langle x, Bx\rangle = \langle x, Ax\rangle - \langle A^*x, x\rangle = 0$. The monotonicity of $\pm B$ thus follows from (v).

(viii): Let $(x, x^*) \in \operatorname{dom} A \times X^*$ be monotonically related to $\operatorname{gra} A$, and take $x_0^* \in Ax$. For every $(v, v^*) \in \operatorname{gra} A$, we have $x_0^* + v^* \in A(x + v)$ (by Fact 2.10(ii)); hence, $\langle x - (x + v), x^* - (x_0^* + v^*)\rangle \geqslant 0$ and thus $\langle v, v^*\rangle \geqslant \langle v, x^* - x_0^*\rangle$. Now take $\lambda > 0$ and replace (v, v^*) in the last inequality by $(\lambda v, \lambda v^*)$. Then divide by λ and let $\lambda \to 0^+$ to see that $0 \geqslant \sup\langle \operatorname{dom} A, x^* - x_0^*\rangle$. Since $\operatorname{dom} A$ is linear, it follows that $x^* - x_0^* \in (\operatorname{dom} A)^\perp$. □

We define the *symmetric part* and the *skew part* of A via

$$A_+ := \frac{1}{2}A + \frac{1}{2}A^* \quad \text{and} \quad A_\circ := \frac{1}{2}A - \frac{1}{2}A^*, \tag{38}$$

respectively. It is easy to check that A_+ is symmetric and that $A_\circ$ is skew.

Proposition 5.2 *(Maximally monotone linear relations). Let $A : X \rightrightarrows X^*$ be a monotone linear relation. Then the following hold.*

(i) *A_+ and $A_\circ$ are monotone.*
(ii) *If A is maximally monotone, then $(\operatorname{dom} A)^\perp = A0$ and hence $\overline{\operatorname{dom} A} = (A0)_\perp$.*
(iii) *If $\operatorname{dom} A$ is closed, then: A is maximally monotone $\Leftrightarrow$ $(\operatorname{dom} A)^\perp = A0$.*
(iv) *If A is maximally monotone, then $\overline{\operatorname{dom} A^*}^{\mathrm{w}*} \cap X = \overline{\operatorname{dom} A} = (A0)_\perp$, and $A0 = A^*0 = A_+0 = A_\circ 0 = (\operatorname{dom} A)^\perp$ is (weak*) closed.*
(v) *If A is maximally monotone and $\operatorname{dom} A$ is closed, then $\operatorname{dom} A^* \cap X = \operatorname{dom} A$.*
(vi) *If A is maximally monotone and $\operatorname{dom} A \subseteq \operatorname{dom} A^*$, then $A = A_+ + A_\circ$, $A_+ = A - A_\circ$, and $A_\circ = A - A_+$.*
(vii) *If A is maximally monotone and $\operatorname{dom} A$ is closed, then both A_+ and $A_\circ$ are maximally monotone.*
(viii) *If A is maximally monotone and $\operatorname{dom} A$ is closed, then $A^* = (A_+)^* + (A_\circ)^*$.*

Proof. (i): This follows from Proposition 5.1(vi)&(vii).

(ii): Let $x^* \in (\operatorname{dom} A)^\perp$. Then, for all $(a, a^*) \in \operatorname{gra} A$, $0 \leqslant \langle a - 0, a^* - 0\rangle = \langle a - 0, a^* - x^*\rangle$. By the maximal monotonicity of A, we have $(0, x^*) \in \operatorname{gra} A$, i.e., $x^* \in A0$. This establishes the inclusion $(\operatorname{dom} A)^\perp \subseteq A0$. (We are grateful to the referee for suggesting this simple proof.) Combining with Proposition 5.1(i), we have $(\operatorname{dom} A)^\perp = A0$. By Fact 2.1, $\overline{\operatorname{dom} A} = (A0)_\perp$.

(iii): "$\Rightarrow$": Clear from (ii). "$\Leftarrow$": The assumptions and Fact 2.1 imply that $\operatorname{dom} A = \overline{\operatorname{dom} A} = [(\operatorname{dom} A)^\perp]_\perp = (A0)_\perp$. Let (x, x^*) be monotonically related to $\operatorname{gra} A$. We have $\inf\langle x - 0, x^* - A0\rangle \geqslant 0$. Then we have $x \in (A0)_\perp$ and hence $x \in \operatorname{dom} A$. Then by Proposition 5.1(viii) and Fact 2.10(i), $x^* \in Ax$. Hence A is maximally monotone.

(iv): By (ii) and Fact 2.10(iv), $A0 = (\operatorname{dom} A)^\perp = A^*0$ is weak* closed and thus $A_+0 = A_\circ 0 = A0 = (\operatorname{dom} A)^\perp$. Then by Fact 2.10(v) and (ii), $\overline{\operatorname{dom} A^*}^{\mathrm{w}*} \cap X = (A0)_\perp = \overline{\operatorname{dom} A}$.

(v): Combine (iv) with Fact 2.10(vi).

(vi): We show only the proof of $A = A_+ + A_\circ$ as the other two proofs are analogous. Clearly, $\operatorname{dom} A_+ = \operatorname{dom} A_\circ = \operatorname{dom} A \cap \operatorname{dom} A^* = \operatorname{dom} A$. Let $x \in \operatorname{dom} A$, and $x^* \in Ax$ and $y^* \in A^*x$. We write $x^* = \frac{x^*+y^*}{2} + \frac{x^*-y^*}{2} \in (A_+ + A_\circ)x$. Then, by (iv) and Fact 2.10(i), $Ax = x^* + A0 = x^* + (A_+ + A_\circ)0 = (A_+ + A_\circ)x$. Therefore, $A = A_+ + A_\circ$.

(vii): By (v),

$$\operatorname{dom} A_+ = \operatorname{dom} A_\circ = \operatorname{dom} A \text{ is closed.} \tag{39}$$

Hence, by (iv),

$$A_\circ 0 = A_+0 = A0 = (\operatorname{dom} A)^\perp = (\operatorname{dom} A_+)^\perp = (\operatorname{dom} A_\circ)^\perp. \tag{40}$$

Since A is monotone, so are A_+ and $A_\circ$. Combining (39), (40), and (iii), we deduce that A_+ and $A_\circ$ are maximally monotone.

(viii): By (v)&(vi),

$$A = A_+ + A_\circ. \tag{41}$$

Then by (vii), (v), and Fact 2.6, $A^* = (A_+)^* + (A_\circ)^*$. □

For a monotone linear relation $A : X \rightrightarrows X^*$ it will be convenient to define – as in, e.g., [3] – a generalized quadratic form

$$(\forall x \in X) \quad q_A(x) = \begin{cases} \frac{1}{2}\langle x, Ax\rangle, & \text{if } x \in \operatorname{dom} A; \\ +\infty, & \text{otherwise.} \end{cases}$$

Proposition 5.3. *Let $A : X \rightrightarrows X^*$ be a monotone linear relation, let x and y be in* dom A*, and let $\lambda \in \mathbb{R}$. Then q_A is single-valued, $q_A \geqslant 0$ and*

$$\begin{aligned} \lambda q_A(x) + (1-\lambda)q_A(y) - q_A\big(\lambda x + (1-\lambda)y\big) &= \lambda(1-\lambda)q_A(x-y) \\ &= \frac{1}{2}\lambda(1-\lambda)\langle x-y, Ax-Ay\rangle. \end{aligned} \tag{42}$$

Consequently, q_A is convex.

Proof. Proposition 5.1(iv)&(v) show that q_A is single-valued and that $q_A \geqslant 0$. Combining with Proposition 5.1(i)&(iii), we obtain (42). Therefore, q_A is convex. □

As in the classical case, q_A allows us to connect properties of A_+ to those of A and A^*. The proof of Proposition 5.4(iv) was borrowed from [19, Theorem 2]. Results very similar to Proposition 5.4(i)&(ii) are verified in [43, Proposition 18.9]. We write $\overline{q_A}$ for the lower semicontinuous hull of q_A.

Proposition 5.4 *(Properties of the symmetric part and the adjoint). Let $A : X \rightrightarrows X^*$ be a monotone linear relation. Then the following hold.*

(i) *$\overline{q_A} + \iota_{\operatorname{dom} A_+} = q_{A_+}$ and thus q_{A_+} is convex.*
(ii) gra $A_+ \subseteq$ gra $\partial\overline{q_A}$*. If A_+ is maximally monotone, then $A_+ = \partial\overline{q_A}$.*
(iii) *If A is maximally monotone and* dom A *is closed, then $A_+ = \partial\overline{q_A}$.*
(iv) *If A is maximally monotone, then $A^*|_X$ is monotone.*
(v) *If A is maximally monotone and* dom A *is closed, then $A^*|_X$ is maximally monotone.*

Proof. Let $x \in \operatorname{dom} A_+$.

(i): By Fact 2.10(iii) and Proposition 5.1(iv), $q_{A_+} = q_A|_{\operatorname{dom} A_+}$. Then by Proposition 5.3, q_{A_+} is convex. Let $y \in \operatorname{dom} A$. By Fact 2.10(iii), $\langle A(x-y), x-y\rangle = \langle Ax - Ay, x-y\rangle = \langle Ax, x\rangle + \langle Ay, y\rangle - 2\langle A_+x, y\rangle$ and therefore all pairing terms are single-valued by Proposition 5.1(iv). It therefore follows with Fact 2.10(iii) that

$$\langle A_+x, y\rangle \text{ and } \langle Ax, x\rangle = \langle A_+x, x\rangle \text{ are single-valued.} \tag{43}$$

Hence

$$0 \leqslant \frac{1}{2}\langle Ax - Ay, x - y\rangle = \frac{1}{2}\langle Ay, y\rangle + \frac{1}{2}\langle Ax, x\rangle - \langle A_+x, y\rangle, \tag{44}$$

and thus $q_A(y) \geqslant \langle A_+x, y\rangle - q_A(x)$. Take the lower semicontinuous hull of q_A at y to deduce that $\overline{q_A}(y) \geqslant \langle A_+x, y\rangle - q_A(x)$. For $y = x$, we have $\overline{q_A}(x) \geqslant q_A(x)$. On the other hand, $\overline{q_A} \leqslant q_A$. Altogether, $\overline{q_A}(x) = q_A(x) = q_{A_+}(x)$. Thus (i) holds.

(ii): Let $y \in \operatorname{dom} A$. By (43), (44) and (i),

$$q_A(y) \geqslant q_A(x) + \langle A_+x, y - x\rangle = \overline{q_A}(x) + \langle A_+x, y - x\rangle, \tag{45}$$

where the pairing terms are single-valued. Since $\operatorname{dom}\overline{q_A} \subseteq \overline{\operatorname{dom} q_A} = \overline{\operatorname{dom} A}$, by (45), $\overline{q_A}(z) \geqslant \overline{q_A}(x) + \langle x^*, z - x\rangle$, $\forall x^* \in A_+x$, $\forall z \in \operatorname{dom}\overline{q_A}$. Hence $A_+x \subseteq \partial\overline{q_A}(x)$. If A_+ is maximally monotone, then $A_+ = \partial\overline{q_A}$. Thus (ii) holds.

(iii): Combine Proposition 5.2(vii) with (ii).

(iv): Suppose to the contrary that $A^*|_X$ is not monotone. By Proposition 5.1(v), there exists $(x_0, x_0^*) \in \operatorname{gra} A^*$ with $x_0 \in X$ such that $\langle x_0, x_0^*\rangle < 0$. Now we have

$$\begin{aligned}\bigl\langle -x_0 - y, x_0^* - y^*\bigr\rangle &= -\bigl\langle x_0, x_0^*\bigr\rangle + \bigl\langle y, y^*\bigr\rangle + \bigl\langle x_0, y^*\bigr\rangle - \bigl\langle y, x_0^*\bigr\rangle \\ &= -\bigl\langle x_0, x_0^*\bigr\rangle + \bigl\langle y, y^*\bigr\rangle > 0, \quad \forall\bigl(y, y^*\bigr) \in \operatorname{gra} A.\end{aligned} \tag{46}$$

Thus, $(-x_0, x_0^*)$ is monotonically related to $\operatorname{gra} A$. By the maximal monotonicity of A, $(-x_0, x_0^*) \in \operatorname{gra} A$. Then $\langle -x_0 - (-x_0), x_0^* - x_0^*\rangle = 0$, which contradicts (46). Hence $A^*|_X$ is monotone.

(v): By Fact 2.10(vi), $\operatorname{dom} A^*|_X = (A0)_\perp$ and thus $\operatorname{dom} A^*|_X$ is closed. By Fact 2.1 and Proposition 5.2(ii), $(\operatorname{dom} A^*|_X)^\perp = ((A0)_\perp)^\perp = \overline{A0}^{\mathrm{w}*} = A0$. Then by Proposition 5.2(iv), $(\operatorname{dom} A^*|_X)^\perp = A^*0 = A^*|_X 0$. Applying (iv) and Proposition 5.2(iii), we see that $A^*|_X$ is maximally monotone. □

The proof of the next Theorem 5.5(i) ⇒ (ii) was partially inspired by that of [2, Theorem 4.1(v) ⇒ (vi)]. In it we present our main findings relating monotonicity and adjoint properties of A and those of its skew part $A_\circ$.

Theorem 5.5 *(Monotone linear relations with closed graph and domain). Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that* $\operatorname{gra} A$ *is closed and* $\operatorname{dom} A$ *is closed. Then the following are equivalent.*

- (i) *A is maximally monotone of type* (D).
- (ii) *$A_\circ$ is maximally monotone of type* (D) *with respect to $X \times X^*$ and $A^*0 = A0$.*
- (iii) *$(A_\circ)^*$ is maximally monotone with respect to $X^{**} \times X^*$ and $A^*0 = A0$.*
- (iv) *$(A_\circ)^*$ is monotone and $A^*0 = A0$.*
- (v) *A^* is monotone.*
- (vi) *A^* is maximally monotone with respect to $X^{**} \times X^*$.*

Proof. "(i) ⇒ (ii)": By Theorem 4.1,

$$A^* \text{ is monotone.} \tag{47}$$

By Proposition 5.4(iii), A_+ is a subdifferential operator; hence, by Fact 1.3(i),

$$A_+ \text{ is maximally monotone of type (D).} \tag{48}$$

By Theorem 4.1,

$$(A_+)^* \text{ is monotone.} \tag{49}$$

Now we show that

$$(A_\circ)^* \text{ is monotone.} \tag{50}$$

Proposition 5.2(viii) implies

$$A^* = (A_+)^* + (A_\circ)^*. \tag{51}$$

Since A is maximally monotone and $\operatorname{dom} A$ is closed, Proposition 5.2(vii) implies that

$$A_\circ \text{ is maximally monotone; consequently, } \operatorname{gra}(A_\circ) \text{ is closed.} \tag{52}$$

On the other hand, again since A is maximally monotone and $\operatorname{dom} A$ is closed, Proposition 5.2(v) yields $\operatorname{dom}(A_\circ) = \operatorname{dom} A$ is closed. Altogether, and combining with Fact 2.10(vi) applied to $A_\circ$, we obtain $\operatorname{dom}(A_\circ)^* = (A_\circ 0)^\perp$. Furthermore, since $A0 = A_\circ 0$ by Proposition 5.2(iv), we have $(A0)^\perp = (A_\circ 0)^\perp$. Moreover, applying Fact 2.10(vi) to A, we deduce that $\operatorname{dom} A^* = (A0)^\perp$. Therefore,

$$\operatorname{dom}(A_\circ)^* = (A_\circ 0)^\perp = (A0)^\perp = \operatorname{dom} A^*. \tag{53}$$

Similarly, we have

$$\operatorname{dom}(A_+)^* = \operatorname{dom} A^*. \tag{54}$$

Take $(x^{**}, x^*) \in \operatorname{gra}(A_\circ)^*$. By (51), (53) and (54), there exist $a^*, b^* \in X^*$ such that

$$\left(x^{**}, a^*\right) \in \operatorname{gra} A^*, \qquad \left(x^{**}, b^*\right) \in \operatorname{gra}(A_+)^* \tag{55}$$

and

$$a^* = b^* + x^*. \tag{56}$$

Since A_+ is symmetric, $\operatorname{gra} A_+ \subseteq \operatorname{gra}(A_+)^*$. Thus, by (49), (x^{**}, b^*) is monotonically related to $\operatorname{gra} A_+$. By (48), there exists a bounded net $(a_\alpha, b_\alpha^*)_{\alpha\in\Gamma}$ in $\operatorname{gra} A_+$ such that $(a_\alpha, b_\alpha^*)_{\alpha\in\Gamma}$

weak* × strong converges to (x^{**}, b^*). Note that $(a_\alpha, b^*_\alpha) \in \operatorname{gra}(A_+)^*$, $\forall \alpha \in \Gamma$. By (51), (53) and (54), there exist $a^*_\alpha \in A^* a_\alpha, c^*_\alpha \in (A_\circ)^* a_\alpha$ such that

$$a^*_\alpha = b^*_\alpha + c^*_\alpha, \quad \forall \alpha \in \Gamma. \tag{57}$$

Thus by Fact 2.10(iii),

$$\langle a_\alpha, c^*_\alpha \rangle = \langle A_\circ a_\alpha, a_\alpha \rangle = 0, \quad \forall \alpha \in \Gamma. \tag{58}$$

Hence for every $\alpha \in \Gamma$, $(-a_\alpha, c^*_\alpha)$ is monotonically related to $\operatorname{gra} A_\circ$. By Proposition 5.2(vii),

$$(-a_\alpha, c^*_\alpha) \in \operatorname{gra} A_\circ, \quad \forall \alpha \in \Gamma. \tag{59}$$

By (47) and (55), we have

$$\begin{aligned} 0 &\leqslant \langle x^{**} - a_\alpha, a^* - a^*_\alpha \rangle = \langle x^{**} - a_\alpha, a^* - b^*_\alpha - c^*_\alpha \rangle \quad (\text{by (57)}) \\ &= \langle x^{**} - a_\alpha, a^* - b^*_\alpha \rangle - \langle x^{**}, c^*_\alpha \rangle + \langle a_\alpha, c^*_\alpha \rangle \\ &= \langle x^{**} - a_\alpha, a^* - b^*_\alpha \rangle - \langle x^{**}, c^*_\alpha \rangle \quad (\text{by (58)}) \\ &= \langle x^{**} - a_\alpha, a^* - b^*_\alpha \rangle + \langle x^*, a_\alpha \rangle \quad (\text{by (59) and } (x^{**}, x^*) \in \operatorname{gra}(A_\circ)^*). \end{aligned} \tag{60}$$

Taking the limit in (60) along with $a_\alpha \overset{\mathrm{w}*}{\rightharpoonup} x^{**}$ and $b^*_\alpha \to b^*$, we have

$$\langle x^{**}, x^* \rangle \geqslant 0.$$

Hence $(A_\circ)^*$ is monotone and thus (50) holds. Combining this, (52), and Theorem 4.1, we deduce that $A_\circ$ is maximally monotone of type (D). Finally, $A^*0 = A0$ by Proposition 5.2(iv).

"(ii) ⇒ (iii) ⇒ (iv)": Apply Corollary 4.4 to $A_\circ$, which has closed graph by maximal monotonicity.

"(iv) ⇒ (v)": By Fact 2.10(iv) and Proposition 5.2(iii), A is maximally monotone. Then by Proposition 5.2(viii) and Proposition 5.4(iii), we have

$$A^* = (A_+)^* + (A_\circ)^* \quad \text{and} \quad A_+ = \partial \overline{q_A}. \tag{61}$$

As a subdifferential operator, A_+ is of type (D) (see Fact 1.3(i)), and hence $(A_+)^*$ is monotone by Theorem 4.1. Thus, by the assumption and (61), we have A^* is monotone.

"(v) ⇒ (vi)": By Proposition 3.2, A is maximally monotone. Then by Fact 2.10(vi) and Proposition 5.2(iv),

$$\operatorname{dom} A^* = (A^*0)^\perp. \tag{62}$$

Then by Fact 2.1 and Fact 2.10(iv),

$$(\operatorname{dom} A^*)_\perp = A^*0. \tag{63}$$

Let $(x^{**}, x^*) \in X^{**} \times X^*$ be monotonically related to $\operatorname{gra} A^*$. Because $\{0\} \times A^*0 \subseteq \operatorname{gra} A^*$, we have $\inf\langle x^{**}, x^* - A^*0\rangle \geqslant 0$. Since A^*0 is a subspace, $x^{**} \in (A^*0)^{\perp}$. Then by (62),

$$x^{**} \in \operatorname{dom} A^*. \tag{64}$$

Take $(x^{**}, x_0^*) \in \operatorname{gra} A^*$ and $\lambda > 0$. For every $(a^{**}, a^*) \in \operatorname{gra} A^*$, we have $(\lambda a^{**}, \lambda a^*) \in \operatorname{gra} A^*$ and hence $(x^{**} + \lambda a^{**}, x_0^* + \lambda a^*) \in \operatorname{gra} A^*$ (since $\operatorname{gra} A^*$ is a subspace). Thus

$$\lambda\big\langle a^{**}, x_0^* + \lambda a^* - x^*\big\rangle = \big\langle x^{**} + \lambda a^{**} - x^{**}, x_0^* + \lambda a^* - x^*\big\rangle \geqslant 0.$$

Now divide by λ to obtain $\lambda\langle a^{**}, a^*\rangle \geqslant \langle a^{**}, x^* - x_0^*\rangle$. Then let $\lambda \to 0^+$ to see that $0 \geqslant \sup\langle \operatorname{dom} A^*,\ x^* - x_0^*\rangle$. Thus, $x^* - x_0^* \in (\operatorname{dom} A^*)_{\perp}$. By (63), $x^* \in x_0^* + A^*0 \subseteq A^*x^{**} + A^*0$. Then there exists $(0, z^*) \in \operatorname{gra} A^*$ such that $(x^{**}, x^* - z^*) \in \operatorname{gra} A^*$. Since $\operatorname{gra} A^*$ is a subspace, $(x^{**}, x^*) = (0, z^*) + (x^{**}, x^* - z^*) \in \operatorname{gra} A^*$. Hence A^* is maximally monotone with respect to $X^{**} \times X^*$.

"(vi) $\Rightarrow$ (i)": Apply Proposition 3.2 directly. □

The next three examples show the need for our various auxiliary hypotheses.

Example 5.6. We cannot remove the condition that $A^*0 = A0$ in Theorem 5.5(iv). For example, suppose that $X = \mathbb{R}^2$ and set $e_1 = (1, 0)$, $e_2 = (0, 1)$. We define $A : X \rightrightarrows X$ by

$$\operatorname{gra} A = \operatorname{span}\{e_1\} \times \{0\} \quad \text{so that } \operatorname{gra} A^* = X \times \operatorname{span}\{e_2\}.$$

Then A is monotone, $\operatorname{dom} A$ is closed, and $\operatorname{gra} A$ is closed. Thus

$$\operatorname{gra} A_\circ = \operatorname{span}\{e_1\} \times \operatorname{span}\{e_2\} \tag{65}$$

and so

$$\operatorname{gra}(A_\circ)^* = \operatorname{span}\{e_1\} \times \operatorname{span}\{e_2\}.$$

Hence $(A_\circ)^*$ is monotone, but A is not maximally monotone because $\operatorname{gra} A \subsetneqq \operatorname{gra} N_X$.

Example 5.7. We cannot replace "$\operatorname{dom} A$ is closed" by "$\operatorname{dom} A$ is dense" in the statement of Theorem 5.5. For example, let X, A be defined as in Example 3.4 and consider the operator A^*. Example 3.4(iii)&(ii) imply that A^* is maximally monotone with dense domain; hence, $\operatorname{gra} A^*$ is closed. Moreover, by Example 3.4(i)&(iv),

$$\big(A^*\big)_\circ = \frac{A^* - A^{**}}{2} = \frac{A^* - A}{2} = -A. \tag{66}$$

Hence

$$\big[\big(A^*\big)_\circ\big]^* = -A^*. \tag{67}$$

Thus $[(A^*)_\circ]^*$ is not monotone by Example 3.4(iii); even though A^* is a classically maximally monotone and densely defined linear operator.

Example 5.8. We cannot remove the condition that $(A_\circ)^*$ is monotone in Theorem 5.5(iv). For example, consider the Gossez operator A (see [23] and [2]). It satisfies $X = \ell^1$, $\operatorname{dom} A = X$, $A_\circ = A$, $A0 = \{0\} = A^*0$, yet A^* is not monotone.

Remark 5.9. Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation.

(i) In general, $(A^*)_\circ \neq (A_\circ)^*$. To see that, let X, A be as in Example 3.4 again. By Example 3.4(i), we have

$$\left(A^*\right)_\circ = -A \quad \text{and} \quad (A_\circ)^* = A^*.$$

Hence $(A^*)_\circ \neq (A_\circ)^*$ by Example 3.4(ii). Moreover, even when A is a linear and continuous operator, one cannot deduce that $(A^*)_\circ = (A_\circ)^*$. Indeed, assume that X and A are as in Example 5.8. Then $(A^*)_\circ$ is skew and hence monotone; however, $(A_\circ)^* = A^*$ is not monotone.

(ii) However, if X is finite-dimensional, we do have $(A^*)_\circ = (A_\circ)^*$. Indeed, by Fact 2.6,

$$(A_\circ)^* = \left(\frac{A - A^*}{2}\right)^* = \frac{A^* - A^{**}}{2} = \left(A^*\right)_\circ.$$

We do not know whether $(A^*)_\circ = (A_\circ)^*$ for all maximally monotone linear relations if and only if X is finite-dimensional.

The work in [6] suggests that in every nonreflexive Banach space there is a maximally monotone linear relation which is not of type (D).

When A is linear and continuous, Theorem 5.5 can also be deduced from [2, Theorem 4.1]. When X is reflexive and $\operatorname{dom} A$ is closed, Theorem 5.5 turns into the following refined version of Fact 2.9:

Corollary 5.10 *(Monotonicity of the adjoint in reflexive space). Suppose that X is reflexive and let $A : X \rightrightarrows X^*$ be a monotone linear relation such that* $\operatorname{gra} A$ *is closed and* $\operatorname{dom} A$ *is closed. Then the following are equivalent.*

(i) *A is maximally monotone.*
(ii) *A^* is monotone.*
(iii) *A^* is maximally monotone.*
(iv) $A0 = A^*0$.

Proof. "(i) ⇔ (ii) ⇔ (iii) ⇒ (iv)": This follows from Theorem 5.5 and the fact that every maximally monotone operator on a reflexive space is of type (D), see Fact 1.3(ii).

"(iv) ⇒ (i)": Fact 2.10(iv) implies that $(\operatorname{dom} A)^\perp = A^*0 = A0$. By Proposition 5.2(iii), A is maximally monotone. □

When X is finite-dimensional, the closure assumptions in the previous result are automatically satisfied and we thus obtain the following:

Corollary 5.11 *(Monotonicity of the adjoint in finite-dimensional space). Suppose that X is finite-dimensional. Let $A : X \rightrightarrows X^*$ be a monotone linear relation. Then the following are equivalent.*

(i) *A is maximally monotone.*
(ii) *A^* is monotone.*
(iii) *A^* is maximally monotone.*
(iv) *$A0 = A^*0$.*

Acknowledgments

We are very grateful to a referee for very careful reading and many constructive comments. Heinz Bauschke was partially supported by the Natural Sciences and Engineering Research Council of Canada and by the Canada Research Chair Program. Jonathan Borwein and Liangjin Yao were both partially supported by the Australian Research Council. Xianfu Wang was partially supported by the Natural Sciences and Engineering Research Council of Canada.

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