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AVERAGING OPERATOR IN THE L_p SPACES FOR $0 < p < 1$

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The role of the Sobolev averaging operators in mathematical analysis is well known. These operators are defined for any measurable set $E \subset \mathbf{R}^n$, for any function f, integrable over the intersection of E with any ball from $\mathbf{R}^n$, and for any $\delta > 0$ by the equality

$$\forall x \in \mathbf{R}^n \quad (A_\delta f)(x) = (\omega_\delta * \overset{0}{f})(x) = \frac{1}{\delta^n} \int_{E \cap B(x,\delta)} \omega\left(\frac{x-y}{\delta}\right) f(y)\,dy. \tag{1}$$

Here ω is the kernel of the averaging, having the form

$$\omega(x) = c \begin{cases} e^{1/(|x|^2-1)}, & |x| < 1, \\ 0, & |x| \geqslant 1, \end{cases} \tag{2}$$

where the constant c is selected so that

$$\int_{\mathbf{R}^n} \omega(x)\,dx = 1, \tag{3}$$

$\omega_\delta(x) = 1/\delta^n \omega(x/\delta)$, $\overset{0}{f}$ is the extension of the function f beyond the domain of definition by zero [if $x \in E$, then $\overset{0}{f}(x) = f(x)$, if $x \notin E$, then $\overset{0}{f}(x) = 0$], $B(x, \delta)$ is the open ball with center at the point x and radius δ.

These operators possess the following properties: If $f \in L_p(E)$, $1 \leq p \leq \infty$, then

$$A_\delta f \in C^\infty(\mathbf{R}^n), \tag{4}$$

i.e., the function $A_\delta f$ has continuous derivatives of every order,

$$\| A_\delta f \|_{L_p(\mathbf{R}^n)} \leqslant \| f \|_{L_p(E)} \quad (1 \leqslant p \leqslant \infty), \tag{5}$$

and

$$\lim_{\delta \to +0} \| A_\delta f - f \|_{L_p(E)} = 0 \quad (1 \leqslant p < \infty), \tag{6}$$

(see, for example, [1]).

Other averaging kernels are also used, satisfying condition (3), but in the sequel it is essential that the kernel ω should have the special form (2).

If $f \in L_p(E)$ and $0 < p < 1$, then there may exist a point $x_0 \in \bar{E}$ ($\bar{E}$ is the closure of the set E) such that $\forall \delta > 0 \; f \notin L_1(E \cap B(x_0, \delta))$ – in this case definition (1) loses senses for each $\delta > 0$ and points $x \in \mathbf{R}^n$ such that $x \in B(x_0, \delta)$.

The purpose of the present paper is the construction of operator $B_\delta \equiv B_{p,\delta}$, which for the functions $f \in L_p(E)$ with $0 < p < 1$ should have the same properties (4)-(6) as the Sobolev averaging operator.

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We denote $f_+(x) = \max\{f(x), 0\}$, $f_-(x) = \max\{-f(x), 0\}$ (we have $f = f_+ - f_-$, $|f| = f_+ + f_-$). For $0 < p < 1$, for any measurable set E, any function f belonging to L_p on the intersection of E with any ball from $\mathbf{R}^n$, and for any $\delta > 0$ we set

$$\forall x \in \mathbf{R}^n \quad (B_\delta f)(x) = (A_\delta f_+^p)^{1/p}(x) - (A_\delta f_-^p)^{1/p}(x). \tag{7}$$

<u>THEOREM</u>. Let $0 < p < 1$, let E be a measurable set in $\mathbf{R}^n$, and let $f \in L_p(E)$ (i.e., the function f is measurable on E and $\|f\|_{L_p(E)} = \left(\int_E |f|^p \, dx\right)^{1/p} < \infty$). Then

$$B_\delta f \in C^\infty(\mathbf{R}^n), \tag{8}$$

$$\|B_\delta f\|_{L_p(\mathbf{R}^n)} \leqslant 2^{1/p-1} \|f\|_{L_p(E)} \tag{9}$$

and

$$\lim_{\delta \to +0} \|B_\delta f - f\|_{L_p(E)} = 0. \tag{10}$$

For the proof of this theorem we need the following lemmas.

<u>LEMMA 1</u>. Assume that the function $f \in L_1(\mathbf{R}^n)$ is nonnegative. Then for any multiindex α ($\alpha = (\alpha_1, \ldots, \alpha_n)$), where $\forall j \in \{1, \ldots, n\}$ α_j are nonnegative integers) and for any $\varepsilon \in [0, 1)$ there exists a positive number $c_{\alpha,\varepsilon}$ such that

$$\forall \delta > 0 \quad \forall x \in \mathbf{R}^n \quad |D^\alpha (A_\delta f)(x)| \leqslant \frac{c_{\alpha,\varepsilon}}{\delta^{|\alpha| + n(1-\varepsilon)}} \|f\|_{L_1(\mathbf{R}^n)}^{1-\varepsilon} (A_\delta f)^\varepsilon (x) \tag{11}$$

(for $\varepsilon = 0$ we assume that $0^0 = 1$).

<u>Proof</u>. Since for every multiindex α there exists a polynomial P_α such that for $|x| < 1$ we have

$$(D^\alpha \omega)(x) = \frac{P_\alpha(x)}{(1 - |x|^2)^{2|\alpha|}} \omega(x),$$

and for any $\lambda > 0$ there exists a positive number c_λ such that for $|x| < 1$ we have

$$(1 - |x|^2)^{-1} \leqslant c_\lambda \omega^{-\lambda}(x),$$

it follows that for any multiindex α and any $\varepsilon \in [0, 1)$ there exists a positive number $c_{\alpha,\varepsilon}$ such that

$$\forall x \in \mathbf{R}^n \quad |(D^\alpha \omega)(x)| \leqslant c_{\alpha,\varepsilon} \omega^\varepsilon (x).$$

From here there follows that

$$\forall x \in \mathbf{R}^n \quad |(D^\alpha \omega_\delta)(x)| \leqslant \frac{c_{\alpha,\varepsilon}}{\delta^{|\alpha| + n(1-\varepsilon)}} \omega^\varepsilon (x).$$

Further, for $0 < \varepsilon < 1$, taking into account the nonnegativity of the function f, we have

$$|D^\alpha (A_\delta f)(x)| = |D^\alpha (\omega_\delta * f)(x)| = |(D^\alpha \omega_\delta * f)(x)| \leqslant (|D^\alpha \omega_\delta| * f)(x) \leqslant \frac{c_{\alpha,\varepsilon}}{\delta^{|\alpha| + n(1-\varepsilon)}} (\omega_\delta^\varepsilon * f)(x).$$

Applying Hölder's inequality, we obtain that

$$(\omega_\delta^\varepsilon * f)(x) = \int_{\mathbf{R}^n} \omega_\delta^\varepsilon (x - y) f^\varepsilon (y) f^{1-\varepsilon}(y) \, dy \leqslant \left(\int_{\mathbf{R}^n} \omega_\delta (x - y) f(y) \, dy \right)^\varepsilon \left(\int_{\mathbf{R}^n} f(y) \, dy \right)^{1-\varepsilon} = \|f\|_{L_1(\mathbf{R}^n)}^{1-\varepsilon} (A_\delta f)^\varepsilon (x),$$

whence there follows (11) for $0 < \varepsilon < 1$.

For $\varepsilon = 0$ we have

$$(|D^\alpha \omega_\delta| * f)(x) = \int_{B(x,\delta)} |(D^\alpha \omega_\delta)(x - y)| f(y) \, dy \leqslant \frac{c_{\alpha,0}}{\delta^{|\alpha|+n}} \|f\|_{L_1(\mathbf{R}^n)},$$

from where there follows (11) for $\varepsilon = 0$.

We mention the following consequence of this lemma, which will not be used in the sequel but may be of independent interest.

COROLLARY 1 (complement to the lemma on "cap" type functions). For any set $E \subset \mathbf{R}^n$ and any $\delta > 0$ there exists a function $\eta \in C^\infty(\mathbf{R}^n)$ such that $\forall x \in \mathbf{R}^n \quad 0 \leqslant \eta(x) \leqslant 1$, $\forall x \in E$ $\eta(x) = 1$ $\forall x \in E^\delta$ (E^δ is the δ-neighborhood of the set E) $\eta(x) = 0$ and for any multiindex α and any $\varepsilon \in [0, 1)$ there exists a positive number $c_{\alpha,\varepsilon,\delta}$ such that

$$\forall x \in \mathbf{R}^n \quad |(D^\alpha \eta)(x)| \leqslant c_{\alpha,\varepsilon,\delta}\eta^\varepsilon(x). \tag{12}$$

Proof. It is sufficient to set $\eta = A_{\delta/2}\chi_{E^{\delta/2}}$ ($\chi_{E^{\delta/2}}$ is the characteristic function of the set $E^{\delta/2}$) and to make use of Lemma 1.

The "usual" lemma on "cap" type functions corresponds to the case $\varepsilon = 0$. The necessity in "cap" type functions, satisfying condition (12) with $\varepsilon = 1/2$, arises at the derivation of a priori type inequalities in L_2 for generalized solutions u of elliptic or more general types of equations, when the "test" function v is chosen to be equal to ηu.

LEMMA 2. Assume that the function $\psi \in C^\infty(\mathbf{R}^n)$ is nonnegative and for any multiindex α and any $\varepsilon \in (0, 1)$ there exists a positive number $c_{\alpha,\varepsilon}$ such that

$$\forall x \in \mathbf{R}^n \quad |(D^\alpha \psi)(x)| \leqslant c_{\alpha,\varepsilon}\psi^\varepsilon(x). \tag{13}$$

Then for any $\gamma > 0$ we have $\psi^\gamma \in C^\infty(\mathbf{R}^n)$.

Proof. If $x \in \mathbf{R}^n$ and $\psi(x) > 0$, then for any multiindex α there exists a derivative $D^\alpha(\psi^\gamma)(x)$, computed by the formula

$$D^\alpha(\psi^\gamma)(x) = \sum_{m=1}^{|\alpha|} \psi^{\gamma-m}(x) \sum_{\beta_1+\ldots+\beta_m=\alpha} b^{(\alpha)}_{\beta_1,\ldots,\beta_m} (D^{\beta_1}\psi)(x) \ldots (D^{\beta_m}\psi)(x) \tag{14}$$

(here $|\alpha| = \alpha_1 + \ldots + \alpha_m$, $\beta_1, \ldots, \beta_m$ are multiindices, $b_{\beta_1,\ldots,\beta_m}{}^{(\alpha)}$ are some real numbers, independent of ψ and x). We consider now points x_0 in $\mathbf{R}^n$ such that $\psi(x_0) = 0$ and in every neighborhood of the point x_0 there is a point x for which $\psi(x) > 0$. According to (13), for the multiindex α we have $(D^\alpha\psi)(x_0) = 0$ and from Taylor's formula there follows that

$$\forall k \in \mathbf{N} \quad \psi(x) = o(|x - x_0|^k), \quad x \to x_0. \tag{15}$$

Since in (13) the number can be taken arbitrarily close to 1, from (14) there follows that there exist positive numbers A and μ such that $\forall x \in \mathbf{R}^n$ for which $\psi(x) > 0$ we have

$$|D^\alpha(\psi^\gamma)(x)| \leqslant A\psi^\mu(x)$$

and, consequently, for the indicated x we have

$$\forall k \in \mathbf{N}\ D^\alpha(\psi^\gamma)(x) = o(|x - x_0|^k),\ x \to x_0. \tag{16}$$

From the definition of the derivative and from (15) there follows that for all considered points x_0 we have $x_0 \frac{\partial_i(\psi^\gamma)}{\partial x_j}(x_0) = 0$. Further, by induction, making use of (16), we obtain that for any multiindex α we have $D^\alpha(\psi^\gamma((x) = 0$. Thus, at each point $x \in \mathbf{R}^n$ the function ψ^γ has derivatives of every order and, according to (14) and (16) they are continuous functions on $\mathbf{R}^n$, i.e., $\psi^\gamma \in C^\infty(\mathbf{R}^n)$.

Proof of the Theorem. 1. Since the functions $\overset{0}{f}{}^p_\pm \in L_1(\mathbf{R}^n)$ and are nonnegative, it follows that the functions $A_\delta f^p_\pm = A_\delta \overset{0}{f}{}^p_\pm \in C^\infty(\mathbf{R}^n)$ are nonnegative, and, according to Lemma 1, satisfy the assumptions of Lemma 2; thus,

$$(A_\delta f^p_\pm)^{1/p} \in C^\infty(\mathbf{R}^n) \text{ and } B_\delta f \in C^\infty(\mathbf{R}^n).$$

2. Since for $0 < p < 1$, for f, $g \in L_p(E)$ we have

$$\|f + g\|_{L_p(E)} \leqslant 2^{1/p-1}(\|f\|_{L_p(E)} + \|g\|_{L_p(E)}), \tag{17}$$

making use of (5) and the numerical inequality $x^\alpha + y^\alpha \leq (x + y)^\alpha$, valid for x, $y \geq 0$, $\alpha \geq 1$, we have

$$\|B_\delta f\|_{L_p(\mathbf{R}^n)} = \|(A_\delta f^p_+)^{1/p} - (A_\delta f^p_-)^{1/p}\|_{L_p(\mathbf{R}^n)} \leqslant 2^{1/p-1}(\|A_\delta f^p_+\|^{1/p}_{L_1(\mathbf{R}^n)} + \|A_\delta f^p_-\|^{1/p}_{L_1(\mathbf{R}^n)}) \leqslant$$

$$\leqslant 2^{1/p-1}(\|f^p_+\|^{1/p}_{L_1(E)} + \|f^p_-\|^{1/p}_{L_1(E)} \leqslant 2^{1/p-1}(\|f^p_+\|_{L_1(E)} + \|f^p\|_{L_1(E)})^{1/p} = 2^{1/p-1}\|f\|_{L_p(E)},$$

and we obtain (9).

3. From (17) there follows that

$$\| B_\delta f - f \|_{L_p(E)} = \| (A_\delta f_+^p)^{1/p} - (A_\delta f_-^p)^{1/p} - f_+ + f_- \|_{L_p(E)} \leqslant$$
$$\leqslant 2^{1/p-1} (\| (A_\delta f_+^p)^{1/p} - f_+ \|_{L_p(E)} + \| (A_\delta f_-^p)^{1/p} - f_- \|_{L_p(E)}) \tag{18}$$

Further, from the numerical inequality

$$| x^\alpha - y^\alpha | \leqslant \alpha | x - y | (| x |^{\alpha-1} + | y |^{\alpha-1}),$$

valid for x, y, $\alpha \geq 0$ (for $0 \leq \alpha < 1$, x, y > 0), and from (17) there follows that

$$\| (A_\delta f_+^p)^{1/p} - f_+ \|_{L_p(E)} = \| (A_\delta f_+^p)^{1/p} - (f_+^p)^{1/p} \|_{L_p(E)} \leqslant \frac{1}{p} \| | A_\delta f_+^p - f_+^p | ((A_\delta f_+^p)^{1/p-1} + (f_+^p)^{1/p-1}) \|_{L_p(E)} =$$
$$= \frac{1}{p} \| | A_\delta f_+^p - f_+^p | (A_\delta f^p)^{1-p/p} + | A_\delta f_+^p - f_+^p | f_+^{1-p} \|_{L_p(E)} \leqslant$$
$$\leqslant c_p (\| | A_\delta f_+^p - f_+^p | (A_\delta f_+^p)^{(1-p)/p} \|_{L_p(E)} + \| | A_\delta f_+^p - f_+^p | f_+^{1-p} \|_{L_p(E)}),$$

where

$$c_p = \frac{1}{p} \cdot 2^{1/p-1}.$$

Applying now the inequality

$$\| gh \|_{L_p(E)} \leqslant \| g \|_{L_1(E)} \| h \|_{L_{p/(1-p)}(E)} \quad (0 < p < 1)$$

(a consequence of Hölder's inequality) and the inequality (5), we obtain that

$$\| (A_\delta f_+^p)^{1/p} - f_+ \|_{L_p(E)} \leqslant c_p \| A_\delta f_+^p - f_+^p \|_{L_1(E)} (\| A_\delta f_+^p \|_{L_1(E)}^{1-p/p} + \| f_+^p \|_{L_1(E)}^{1-p/p}) \leqslant 2c_p \| f_+ \|_{L_p(E)}^{1-p} \| A_\delta f_+^p - f_+^p \|_{L_1(E)} \to 0$$

for $\delta \to +0$ according to (6) since $f_+^p \in L_1(E)$.

In a similar manner one proves that the second term in (18) tends to zero when $\delta \to +0$, and we obtain (10).

COROLLARY 2. Let $0 < p < 1$ and let E be a measurable set in $\mathbf{R}^n$. Then the set $C^\infty(\mathbf{R}^n)$ is dense in $L_p(E)$. If Ω is an open set in $\mathbf{R}^n$, then the set $C_0^\infty(\Omega)$ is dense in $L_p(\Omega)$.

Proof. The first assertion follows from (8) and (10), the second is proved in a standard manner with the aid of a multiplication by a truncating function.

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