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Integral representations of some series involving $\binom{2k}{k}^{-1} k^{-n}$ and some related series

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Abstract

In this paper we analysed some series involving

$$\binom{2k}{k}^{-1} k^{-n} \quad \text{and} \quad \binom{2k}{k}^{-2} k^{-n}$$

and some other related series and derived the integral representations of those series considered by using some elementary properties of polylogarithms. The results we obtained show that all the integral representations involve so called log-sine terms. On using their representations we made some generalizations and closed form evaluations. We also obtained two new acceleration formulas for $\zeta(3)$ and Catalan's constant G.

Keywords: Integral representations; Polylogarithms; Log-sine integrals; Catalan constant; Special functions; Riemann zeta function

1. Introduction

After Apéry [1] proved the irrationality of $\zeta(3)$, ζ being the Riemann zeta function, starting from the identity

$$\zeta(3) = \sum_{k=1}^{\infty} \left[(-1)^{k+1} \Big/ \binom{2k}{k} k^3 \right]$$

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many authors have considered these type of series. Among them Zucker [7], Bradley [4], Borwein (with Bradley) [3] and long before Ramanujan [2, vol. I, Chapter 9] are noticable.

The aim of this note is to present some integral representations of some series involving

$$\binom{2k}{k}^{-1} k^{-n} \quad \text{and} \quad \binom{2k}{k}^{-2} k^{-n}$$

and related series. The results found here show that the explicit evaluations of such series depend on the evaluations of the integrals involving log-sine terms. This emphasize the importance of these integrals. For some low values of n, we can make closed form evaluations but for higher values this seems difficult to do. Our main results given in Section 2 enable also us to make some generalizations by making use of some elementary properties of the polylogarithms, especially identity (1.3).

Throughout this paper, we will use the following standard definitions and identities involving the Euler gamma function Γ, beta function β, polylogarithms $\mathrm{Li}_n(z)$ and polyinverse tangent series $\mathrm{Ti}_n(z)$:

$$\beta(s,t) := \int_0^1 u^{s-1}(1-u)^{t-1}\,\mathrm{d}u = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \quad \text{for } s>0,\ t>0 \tag{1.1}$$

(see [6, Theorem 7.69]),

$$\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n} = \frac{(-1)^{n-1}}{(n-1)!}\int_0^1 \frac{z\log^{n-1}\phi\,\mathrm{d}\phi}{1-z\phi} \quad \text{for } |z| \leqslant 1, \tag{1.2}$$

$$\mathrm{Li}_n(z^m) = m^{n-1}\sum_{k=1}^{m} \mathrm{Li}_n(\omega^k z), \tag{1.3}$$

where m is a positive integer and $\omega = \mathrm{e}^{2\pi i/m}$, mth root of unity, is called the factorization formula for polylogarithm series. These and further properties of polylogarithms and related functions can be found in [5]

$$\mathrm{Ti}_n(z) := \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)^n} = \mathrm{Li}_n(z) - 2^{-n}\mathrm{Li}_n(z^2). \tag{1.4}$$

2. The main results

We present a collection of some integral representations of some series involving

$$\binom{2k}{k}^{-1} k^{-n} \quad \text{and} \quad \binom{2k}{k}^{-2} k^{-n}.$$

Almost all results given here were obtained by extensive usage of the identities (1.1) and (1.2).

Theorem 2.1. *$|x| \leqslant 2$ and $n = 2, 3, \dots$ we have*

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}(k!)^2}{(2k+1)^n (2k)!} = \frac{(-1)^{n-2}}{(n-2)!} \int_0^{\arcsin(x/2)} \frac{\phi}{\sin\phi} \log^{n-2}[2\sin\phi/x]\, \mathrm{d}\phi.$$

Proof. We start by using the Eq. (1.1).

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{x^{2k+1}(k!)^2}{(2k+1)^n (2k)!} &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^{n-1}} \frac{\Gamma(k+1)^2}{\Gamma(2k+2)} \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^{n-1}} \beta(k+1, k+1) \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^{n-1}} \int_0^1 t^k (1-t)^k\, \mathrm{d}t.\end{aligned}$$

Interchanging the order of integration and summation and then using (1.4) we get

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{x^{2k+1}(k!)^2}{(2k+1)^n (2k)!} &= \int_0^1 \frac{\mathrm{Ti}_{n-1}[x(t(1-t))^{1/2}]}{[t(1-t)]^{1/2}}\, \mathrm{d}t \\ &= \int_0^1 \frac{\mathrm{Li}_{n-1}[x(t(1-t))^{1/2}] - 2^{1-n}\mathrm{Li}_{n-1}[x^2 t(1-t)]}{[t(1-t)]^{1/2}}\, \mathrm{d}t \\ &= S_1 + S_2, \end{aligned} \tag{2.1}$$

where

$$S_1 := \int_0^1 \frac{\mathrm{Li}_{n-1}[x(t(1-t))^{1/2}]}{[t(1-t)]^{1/2}}\, \mathrm{d}t \quad \text{and} \quad S_2 := 2^{1-n} \int_0^1 \frac{\mathrm{Li}_{n-1}[x^2 t(1-t)]}{[t(1-t)]^{1/2}}\, \mathrm{d}t.$$

We continue to simplify these two expressions by means of the integral representation (1.2) of $\mathrm{Li}_n(x)$. On using the identity (1.2) we obtain

$$S_1 = \frac{(-1)^{n-2}}{(n-2)!} \int_0^1 \int_0^1 \frac{x \log^{n-2}\phi}{1 - x[t(1-t)]^{1/2}\phi}\, \mathrm{d}\phi. \tag{2.2}$$

Inverting the order of integration, this is

$$S_1 = \frac{(-1)^{n-2}}{(n-2)!} \int_0^1 \frac{\log^{n-2}\phi}{\phi} \int_0^1 \frac{\mathrm{d}t}{(x\phi)^{-1} - [t(1-t)]^{1/2}}\, \mathrm{d}\phi. \tag{2.3}$$

We will justify the inversion of the order of these two integrals at the end of the proof. Now it is easily verified that

$$\int_0^1 \frac{\mathrm{d}t}{(x\phi)^{-1} - [t(1-t)]^{1/2}} = -\pi + \frac{8}{[4 - x^2\phi^2]^{1/2}} \operatorname{arctg}\left[\frac{2 + x\phi}{2 - x\phi}\right]^{1/2}.$$

Substituting this into (2.3), we get

$$S_1 = \frac{(-1)^{n-2}}{(n-2)!} \int_0^1 \frac{\log^{n-2}\phi}{\phi} \left[-\pi + \frac{8}{[4 - x^2\phi^2]^{1/2}} \operatorname{arctg}\left[\frac{2 + x\phi}{2 - x\phi}\right]^{1/2}\right] \mathrm{d}\phi. \tag{2.4}$$

Following the same procedure given for getting S_1, we arrive at

$$S_2 = 2^{-n+1}\frac{(-1)^{n-2}}{(n-2)!} \int_0^1 \frac{\log^{n-2} u}{u} \left[-\pi + \frac{2\pi}{[4 - x^2 u]^{1/2}}\right] \mathrm{d}u.$$

Setting $u = \phi^2$, we find

$$S_2 = \frac{(-1)^{n-2}}{(n-2)!} \int_0^1 \frac{\log^{n-2}\phi}{\phi} \left[-\pi + \frac{2\pi}{[4 - x^2\phi^2]^{1/2}}\right] \mathrm{d}\phi. \tag{2.5}$$

Substituting the values of S_1 and S_2 given in (2.4) and (2.5) respectively into (2.1) and then combining the two integrals into a single integral, we obtain

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}(k!)^2}{(2k+1)^n(2k)!} = \frac{(-1)^{n-2}}{(n-2)!} \times \int_0^1 \frac{\log^{n-2}\phi}{\phi} \left[\frac{8}{[4 - x^2\phi^2]^{1/2}} \arctan\left[\frac{2 + x\phi}{2 - x\phi}\right]^{1/2} - \frac{2\pi}{[4 - x^2\phi^2]^{1/2}}\right] \mathrm{d}\phi.$$

Now make the change of variable

$$y = \arctan\left[\frac{2 + x\phi}{2 - x\phi}\right]^{1/2}$$

to yield

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}(k!)^2}{(2k+1)^n(2k)!} = -2\frac{(-1)^{n-2}}{(n-2)!} \times \int_0^{\arcsin(x/2)/2+\pi/4} \frac{4y - \pi}{\cos 2y} \log^{n-2}(-2\cos 2y/x)\, \mathrm{d}y, \tag{2.6}$$

since

$$\arctan\left[\frac{2+x}{2-x}\right]^{1/2}=\frac{1}{2}\arcsin(x/2)+\frac{\pi}{4}.$$

Substituting $\phi=(4y-\pi)/2$ in (2.6), we get

$$\sum_{k=0}^{\infty}\frac{x^{2k+1}(k!)^2}{(2k+1)^n(2k)!}=\frac{(-1)^{n-2}}{(n-2)!}\int_0^{\arcsin(x/2)}\frac{\phi}{\sin\phi}\log^{n-2}(2\sin\phi/x)\,\mathrm{d}\phi.$$

To complete the proof we need to justify the inversion of the order of the two integrals in (2.2). If we make the change of variables $\phi=1/y$ in the inner integral in (2.2), it takes the form

$$S_1=\frac{1}{(n-2)!}\int_0^1\int_1^{\infty}\frac{x\log^{n-2}y}{y^2-x[t(1-t)]^{1/2}y}\,\mathrm{d}y\,\mathrm{d}t.$$

Since for every $0\leqslant t\leqslant 1$ and $-2\leqslant x\leqslant 2$

$$\frac{\log^{n-2}y}{y^2-x[t(1-t)]^{1/2}y}\leqslant\frac{\log^{n-2}y}{y^2-y}$$

and the improper integral

$$\int_1^{\infty}\frac{\log^{n-2}y}{y^2-y}\,\mathrm{d}y$$

converges,

$$\int_1^{\infty}\frac{x\log^{n-2}y}{y^2-x[t(1-t)]^{1/2}y}\,\mathrm{d}y$$

converges uniformly. Thus, the inversion of the orders of the integrals in (2.2) is justified. This completes the proof of Theorem 2.1. □

Theorem 2.2. *For* $|x|\leqslant 4$ *and* $n=3,4,\ldots$ *we have*

$$\sum_{k=1}^{\infty}\frac{x^{2k}(k!)^4}{(2k)^n[(2k)!]^2}=\frac{(-1)^{n-3}}{(n-3)!}\int_0^{\pi/2}\frac{1}{\sin y}\int_0^{\arcsin(x\sin y/4)}\phi\times\log^{n-3}[4\sin\phi/x\sin y]\,\mathrm{d}\phi\,\mathrm{d}y.$$

Proof. We start by making use of the identity (1.1).

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{x^{2k}(k!)^4}{k^n[(2k)!]^2}&=\sum_{k=1}^{\infty}\frac{x^{2k}}{k^n}\frac{\Gamma(k+1)^2\Gamma(k+1)^2}{\Gamma(2k+1)^2}\\&=\sum_{k=1}^{\infty}\frac{x^{2k}}{k^{n-2}}\frac{\Gamma(k)^2\Gamma(k+1)^2}{\Gamma(2k+1)^2}=\sum_{k=1}^{\infty}\frac{x^{2k}}{k^{n-2}}B(k,k+1)^2\\&=\sum_{k=1}^{\infty}\frac{x^{2k}}{k^{n-2}}\left(\int_0^1 u^{k-1}(1-u)^k\,\mathrm{d}u\right)\left(\int_0^1 y^{k-1}(1-y)^k\,\mathrm{d}y\right)\\&=\sum_{k=1}^{\infty}\frac{x^{2k}}{k^{n-2}}\int_0^1\int_0^1 u^{k-1}y^{k-1}(1-u)^k(1-y)^k\,\mathrm{d}u\,\mathrm{d}y\\&=\int_0^1\int_0^1\sum_{k=1}^{k=\infty}\frac{x^{2k}}{k^{n-2}}u^{k-1}y^{k-1}(1-u)^k(1-y)^k\,\mathrm{d}u\,\mathrm{d}y.\end{aligned}$$

Here we interchanged the order of summation and integration. Thus, we have, by the definition (1.2),

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{x^{2k}(k!)^4}{k^n[(2k)!]^2}&=\int_0^1\int_0^1\frac{1}{uy}\sum_{k=1}^{k=\infty}\frac{[x^2uy(1-u)(1-y)]^k}{k^{n-2}}\mathrm{d}u\,\mathrm{d}y\\&=\int_0^1\int_0^1\frac{\mathrm{Li}_{n-2}[x^2uy(1-u)(1-y)]}{uy}\mathrm{d}u\,\mathrm{d}y\\&=\frac{(-1)^{n-3}}{(n-3)!}\int_0^1\int_0^1\int_0^1\frac{x^2(1-u)(1-y)\log^{n-3}\phi}{1-x^2uy(1-u)(1-y)\phi}\mathrm{d}\phi\,\mathrm{d}u\,\mathrm{d}y,\end{aligned}\tag{2.7}$$

where in the last equality we used the integral representation (1.2) of polylogarithms. Interchanging the order of the two inner integrals in (2.7), leaving the justification of it at the end of the proof, we get

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{x^{2k}(k!)^4}{k^n[(2k)!]^2}&=\frac{(-1)^{n-3}}{(n-3)!}\\&\quad\times\int_0^1\frac{1}{y}\left[\int_0^1\frac{\log^{n-3}\phi}{\phi}\left[\int_0^1\frac{(1-u)\,\mathrm{d}u}{[x^2y(1-y)\phi]^{-1}-u+u^2}\right]\mathrm{d}\phi\right]\mathrm{d}y.\end{aligned}\tag{2.8}$$

It is very easy to verify that

$$\begin{aligned}\int_0^1\frac{(1-u)\,\mathrm{d}u}{[x^2y(1-y)\phi]^{-1}-u+u^2}&=\left[\frac{4x^2y(1-y)\phi}{4-x^2y(1-y)\phi}\right]^{1/2}\\&\quad\times\arctan\left[\frac{x^2y(1-y)\phi}{4-x^2y(1-y)\phi}\right]^{1/2}.\end{aligned}$$

Hence (2.8) can be written as

$$\sum_{k=1}^{\infty} \frac{x^{2k}(k!)^4}{k^n[(2k)!]^2} = \frac{(-1)^{n-3}}{(n-3)!} \int_0^1 \frac{1}{y} \left[\int_0^1 \frac{\log^{n-3}\phi}{\phi} \left[\frac{4x^2y(1-y)\phi}{4-x^2y(1-y)\phi} \right]^{1/2} \times \arctan \left[\frac{x^2y(1-y)\phi}{4-x^2y(1-y)\phi} \right]^{1/2} \mathrm{d}\phi \right] \mathrm{d}y. \tag{2.9}$$

Make the substitution

$$z = \arctan \left[\frac{x^2y(1-y)\phi}{4-x^2y(1-y)\phi} \right]^{1/2}$$

in the inner integral in (2.9) to get

$$\sum_{k=1}^{\infty} \frac{x^{2k}(k!)^4}{k^n[(2k)!]^2} = \frac{4(-1)^{n-3}}{(n-3)!} \int_0^1 \frac{1}{y} \left[\int_0^{\alpha(x,y)} z \log^{n-3}[4\sin^2 z/x^2y(1-y)]\,\mathrm{d}z \right] \mathrm{d}y, \tag{2.10}$$

where

$$\alpha(x,y) = \arctan \left[\frac{x^2y(1-y)}{4-x^2y(1-y)} \right]^{1/2}.$$

Making the successive substitutions $y = \frac{1}{2} + z$ and $z = \frac{1}{2}\sin t$ in (2.10) again, we get

$$\sum_{k=1}^{\infty} \frac{x^{2k}(k!)^4}{k^n[(2k)!]^2} = \frac{2^{n-1}(-1)^{n-3}}{(n-3)!} \int_{-\pi/2}^{\pi/2} \frac{\cos t}{1+\sin t} \left[\int_0^{\lambda(x,t)} u \log^{n-3}[4\sin u/x\cos t]\,\mathrm{d}u \right] \mathrm{d}t, \tag{2.11}$$

where

$$\lambda(x,t) = \arctan \left[\frac{x^2\cos^2 t}{16 - x^2\cos^2 t} \right]^{1/2}. \tag{2.12}$$

To simplify the expression in (2.11) we separate it into two parts as

$$\sum_{k=1}^{\infty} \frac{x^{2k}(k!)^4}{k^n[(2k)!]^2} = \frac{2^{n-1}(-1)^{n-3}}{(n-3)!} \left[\int_{-\pi/2}^{0} \frac{\sigma(x,t)\cos t}{1+\sin t}\mathrm{d}t + \int_0^{\pi/2} \frac{\sigma(x,t)\cos t}{1+\sin t}\mathrm{d}t \right], \tag{2.13}$$

where

$$\sigma(x,t) = \int_0^{\lambda(x,t)} u \log^{n-3}[4\sin u/x\cos t]\,\mathrm{d}u$$

with $\lambda(x,t)$ is defined as in (2.12). Now make the change of variable $t=-y$ in the first integral in (2.13) and then combine the two integrals into a single integral to yield

$$\sum_{k=1}^{\infty}\frac{x^{2k}(k!)^4}{(2k)^n[(2k)!]^2}=\frac{(-1)^{n-3}}{(n-3)!}\int_0^{\pi/2}\frac{1}{\sin y}\times\int_0^{\arcsin(x\sin y/4)}u\log^{n-3}[4\sin u/x\sin y]\,\mathrm{d}u\,\mathrm{d}y,$$

since

$$\arctan\left[\frac{x^2\cos^2 y}{16-x^2\cos^2 y}\right]^{1/2}=\arcsin(|x\cos y|/4).$$

Finally, in order to complete the proof we should justify the interchanging of the order of the two inner integrals given in (2.7). To fulfill it let $\phi=1/t$ in the inner integral in (2.7). Then we have

$$\int_0^1\int_0^1\frac{x^2(1-u)(1-y)\log^{n-3}\phi}{1-x^2uy(1-u)(1-y)\phi}\mathrm{d}\phi\,\mathrm{d}u$$
$$=(-1)^{n-3}\int_0^1\int_1^{\infty}\frac{x^2(1-u)(1-y)\log^{n-3}t}{t^2-x^2uy(1-u)(1-y)t}\mathrm{d}t\,\mathrm{d}u.$$

Since for all $0\leqslant u\leqslant 1$, $-4\leqslant x\leqslant 4$ and $0\leqslant y\leqslant 1$

$$\frac{x^2(1-u)(1-y)\log^{n-3}t}{t^2-x^2uy(1-u)(1-y)t}\leqslant\frac{\log^{n-3}t}{t^2-t}$$

and the integral

$$\int_1^{\infty}\frac{\log^{n-3}t}{t^2-t}\mathrm{d}t$$

converges,

$$\int_1^{\infty}\frac{x^2(1-u)(1-y)\log^{n-3}t}{t^2-x^2uy(1-u)(1-y)t}\mathrm{d}t$$

converges uniformly. This justifies the inversion of the order of the inner two integrals in (2.7). □

Theorem 2.3. *For $|x|\leqslant 4$ and $n=3,4,\dots$ We have*

$$\sum_{k=1}^{\infty}\frac{x^{2k+1}(k!)^4}{(2k+1)^n[(2k)!]^2}=\frac{4(-1)^{n-3}}{(n-3)!}\int_0^{\pi/2}\int_0^{\arcsin(x\cos y/4)}\frac{\phi}{\sin\phi}\times\log^{n-3}[4\sin\phi/x\cos y]\,\mathrm{d}\phi\,\mathrm{d}y.$$

Proof. We start with the identities given in (1.2) again.

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{x^{2k+1}(k!)^4}{(2k+1)^n[(2k)!]^2} &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^{n-2}} \frac{\Gamma(k+1)^4}{\Gamma(2k+2)^2} \\
&= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^{n-2}} \beta(k+1,k+1)^2 \\
&= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^{n-2}} \left[\int_0^1 y^k(1-y)^k\,\mathrm{d}y\right]\left[\int_0^1 z^k(1-z)^k\,\mathrm{d}z\right] \\
&= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^{n-2}} \int_0^1 \int_0^1 [yz(1-z)(1-y)]^k\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^1 \int_0^1 \sum_{k=0}^{\infty} \frac{[x(yz(1-z)(1-y))^{1/2}]^{2k+1}}{(2k+1)^{n-2}} \frac{\mathrm{d}y\,\mathrm{d}z}{[yz(1-z)(1-y)]^{1/2}} \\
&= \int_0^1 \int_0^1 \frac{\mathrm{Ti}_{n-2}[x(yz(1-z)(1-y))^{1/2}]}{[yz(1-z)(1-y)]^{1/2}}\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}$$

We in the last step used the identity (1.4). Employing (1.4) again, the last equation can be written as

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{x^{2k+1}(k!)^4}{(2k+1)^n[(2k)!]^2} = \int_0^1 \int_0^1 \frac{\mathrm{Li}_{n-2}[x(yz(1-z)(1-y))^{1/2}]}{[yz(1-z)(1-y)]^{1/2}}\,\mathrm{d}y\,\mathrm{d}z - \frac{1}{2^{n-2}} \\
\times \int_0^1 \int_0^1 \frac{\mathrm{Li}_{n-2}[x^2 yz(1-z)(1-y)]}{[yz(1-z)(1-y)]^{1/2}}\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}$$

If we use the integral representation of Li_n given by (1.2), the last expression can also be written as

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{x^{2k+1}(k!)^4}{(2k+1)^n[(2k)!]^2} \\
&= \frac{(-1)^{n-3}}{(n-3)!} \int_0^1 \left[\int_0^1 \int_0^1 \frac{x \log^{n-2}\phi}{1 - x[yz(1-z)(1-y)]^{1/2}\phi}\,\mathrm{d}\phi\,\mathrm{d}y\right]\mathrm{d}z \\
&- \frac{(-1)^{n-3}}{(n-3)!} \frac{1}{2^{n-2}} \int_0^1 \left[\int_0^1 \int_0^1 \frac{x^2[yz(1-z)(1-y)]^{1/2} \log^{n-3}\phi}{1 - x^2[yz(1-z)(1-y)]\phi}\,\mathrm{d}\phi\,\mathrm{d}y\right]\mathrm{d}z.
\end{aligned}$$

One can easily justify in a similar fashion we did in the proof of Theorems 2.1 and 2.2 that both of the inner integrals at the right hand side can be invertable. Thus, we get

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\frac{x^{2k+1}(k!)^4}{(2k+1)^n[(2k)!]^2}\\
&=\frac{(-1)^{n-3}}{(n-3)!}\int_0^1\frac{1}{[z(1-z)]^{1/2}}\\
&\quad\times\left[\int_0^1\frac{\log^{n-3}\phi}{\phi}\left[\int_0^1\frac{\mathrm{d}y}{[\phi x[z(1-z)]^{1/2}]^{-1}-[y(1-y)]^{1/2}}\right]\mathrm{d}\phi\right]\mathrm{d}z \qquad (2.14)\\
&\quad-\frac{(-1)^{n-3}}{2^{n-2}(n-3)!}\int_0^1\frac{1}{[z(1-z)]^{1/2}}\\
&\quad\times\left[\int_0^1\frac{\log^{n-3}\phi}{\phi}\left[\int_0^1\frac{[y(1-y)]^{1/2}\,\mathrm{d}y}{[\phi x^2z(1-z)]^{-1}-y(1-y)}\right]\mathrm{d}\phi\right]\mathrm{d}z, \qquad (2.15)
\end{aligned}
$$

where we have inverted the order of the inner two integrals in both (2.14) and (2.15). The justification of this inversion may be done as the previous theorems. Now put, for convenience,

$$
I_1:=\int_0^1\frac{\mathrm{d}y}{[\phi x[z(1-z)]^{1/2}]^{-1}-[y(1-y)]^{1/2}} \quad \text{and}
$$

$$
I_2:=\int_0^1\frac{[y(1-y)]^{1/2}\,\mathrm{d}y}{[\phi x^2z(1-z)]^{-1}-y(1-y)}.
$$

It is not difficult to evaluate these two integrals and to show that, by a proper chain of change of variables,

$$
I_1=-\pi+\frac{8}{[4-x^2\phi^2z(1-z)]^{1/2}}\arctan\left[\frac{2+x\phi[z(1-z)]^{1/2}}{2-x\phi[z(1-z)]^{1/2}}\right]^{1/2} \quad \text{and}
$$

$$
I_2=-\pi+\frac{2\pi}{[4-\phi x^2z(1-z)]^{1/2}}.
$$

If we put these values of I_1 and I_2 into (2.14) and (2.15), we obtain

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{x^{2k+1}(k!)^4}{(2k+1)^n[(2k)!]^2}&=\frac{(-1)^{n-3}}{(n-3)!}\int_0^1\frac{1}{[z(1-z)]^{1/2}}\\
&\quad\times\left[\int_0^1\frac{\log^{n-3}\phi}{\phi}\left[-\pi+\frac{8}{[4-x^2\phi^2z(1-z)]^{1/2}}\right.\right.
\end{aligned}
$$

$$\times \arctan\left[\frac{2+x\phi[z(1-z)]^{1/2}}{2-x\phi[z(1-z)]^{1/2}}\right]^{1/2} \mathrm{d}\phi\Bigg]\Bigg] \mathrm{d}z - \frac{(-1)^{n-3}}{2^{n-2}(n-3)!}\int_0^1 \frac{1}{[z(1-z)]^{1/2}}$$

$$\times\left[\int_0^1 \frac{\log^{n-3}\phi}{\phi}\left[-\pi+\frac{2\pi}{[4-\phi x^2 z(1-z)]}\right]\mathrm{d}\phi\right]\mathrm{d}z. \tag{2.16}$$

Making the change of variable $\phi = t^2$ in the inner integral of the second term on the right-hand side and then taking ϕ as the dummy variable again, we find that

$$\sum_{k=0}^{\infty}\frac{x^{2k+1}(k!)^4}{(2k+1)^n[(2k)!]^2} = \frac{(-1)^{n-3}}{(n-3)!}\int_0^1 \frac{1}{[z(1-z)]^{1/2}}$$

$$\times\left[\int_0^1 \frac{\log^{n-3}\phi}{\phi}\left[\frac{8}{[4-x^2\phi^2 z(1-z)]^{1/2}}\arctan\left[\frac{2+x\phi[z(1-z)]^{1/2}}{2-x\phi[z(1-z)]^{1/2}}\right]^{1/2}\right.\right.$$

$$\left.\left.-\frac{2\pi}{[4-x^2\phi^2 z(1-z)]^{1/2}}\right]\mathrm{d}\phi\right]\mathrm{d}z \tag{2.17}$$

after combining both integrals in (2.16) into a single integral. Now make the change of variable

$$u = \arctan\left[\frac{2+x\phi[z(1-z)]^{1/2}}{2-x\phi[z(1-z)]^{1/2}}\right]^{1/2}$$

in the inner integral in (2.17) to yield

$$\sum_{k=0}^{\infty}\frac{x^{2k+1}(k!)^4}{(2k+1)^n[(2k)!]^2} = -2\frac{(-1)^{n-3}}{(n-3)!}\int_0^1 \frac{1}{[z(1-z)]^{1/2}}$$

$$\times\left[\int_{\pi/4}^{f(x,z)} \log^{n-3}[-2\cos 2u/x(z-z^2)]\,\frac{4u-\pi}{\cos 2u}\,\mathrm{d}u\right]\mathrm{d}z,$$

where

$$f(x,z) = \arctan\left[\frac{2+x[z(1-z)]^{1/2}}{2-x[z(1-z)]^{1/2}}\right]^{1/2}.$$

Let $\phi = (4u - \pi)/2$ in the inner integral to find

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}(k!)^4}{(2k+1)^n[(2k)!]^2} = \frac{2(-1)^{n-3}}{(n-3)!} \int_0^1 \frac{1}{[z(1-z)]^{1/2}} \times \left[\int_0^{h(x,z)} \frac{\phi}{\sin\phi} \log^{n-3}[2\sin\phi/x(z-z^2)^{1/2}]\,\mathrm{d}\phi \right] \mathrm{d}z, \tag{2.18}$$

where $h(x,z) = \arcsin[x(z-z^2)^{1/2}/2]$, since

$$f(x,z) = \arctan\left[\frac{2 + x[z(1-z)]^{1/2}}{2 - x[z(1-z)]^{1/2}}\right]^{1/2} = \frac{1}{2}\arcsin\frac{[z-z^2]^{1/2}x}{2} + \frac{\pi}{4}.$$

We need to make two more change of variables to bring (2.18) in a simpler shape. If we make the successive change of variables $z = u + 1/2$ and $u = \sin y/2$ in (2.18), we get

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}(k!)^4}{(2k+1)^n[(2k)!]^2} = \frac{4(-1)^{n-3}}{(n-3)!} \int_0^{\pi/2} \int_0^{\arcsin(x\cos y/4)} \frac{\phi}{\sin\phi} \times \log^{n-3}[4\sin\phi/x\cos y]\,\mathrm{d}\phi\,\mathrm{d}y.$$

This finishes the proof of Theorem 2.3. □

Theorem 2.4. *For* $|x| \leqslant 2$ *and* $n = 1, 2, 3, \ldots$

$$\sum_{k=1}^{\infty} \frac{x^k \Gamma(k/2)^2}{k^n k!} = 4\frac{(-1)^n}{n!} \int_0^{\arcsin(x/2)} (\phi + \pi/2)\log^n(2\sin\phi/x)\,\mathrm{d}\phi.$$

Proof. We start by employing (1.1).

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{x^k \Gamma(1+k/2)^2}{k^n k!} &= \frac{1}{2}\sum_{k=1}^{\infty} \frac{x^k}{k^{n-1}} \frac{\Gamma(k/2)\Gamma(1+k/2)}{\Gamma(k+1)} \\
&= \frac{1}{2}\sum_{k=1}^{\infty} \frac{x^k}{k^{n-1}} \beta(k/2, 1+k/2) \\
&= \frac{1}{2}\sum_{k=1}^{\infty} \frac{x^k}{k^{n-1}} \int_0^1 u^{k/2-1}(1-u)^{k/2}\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 \sum_{k=1}^{\infty} \frac{[x(u-u^2)^{1/2}]^k}{k^{n-1}} \frac{\mathrm{d}u}{u},
\end{aligned}$$

where we have interchanged the order of the integration and summation. On using (1.2) we have

$$\sum_{k=1}^{\infty} \frac{x^k \Gamma(1+k/2)^2}{k^n k!} = \frac{1}{2} \int_0^1 \mathrm{Li}_{n-1}[x(u-u^2)^{1/2}] \frac{\mathrm{d}u}{u}.$$

Using (1.2) once more we get

$$\sum_{k=1}^{\infty} \frac{x^k \Gamma(1+k/2)^2}{k^n k!} = \frac{1}{2} \frac{(-1)^n}{(n-2)!} \int_0^1 \left[\int_0^1 \frac{x(u-u^2)^{1/2} \log^{n-2}\phi}{1 - x(u-u^2)^{1/2}\phi} \mathrm{d}\phi \right] \frac{\mathrm{d}u}{u}.$$

If we can invert the order of integrations, this is

$$\sum_{k=1}^{\infty} \frac{x^k \Gamma(1+k/2)^2}{k^n k!} = \frac{1}{2} \frac{(-1)^n}{(n-2)!} \int_0^1 \frac{\log^{n-2}\phi}{\phi} \left[\int_0^1 \frac{(u-u^2)^{1/2}}{[x\phi]^{-1} - (u-u^2)^{1/2}} \frac{\mathrm{d}u}{u} \right] \mathrm{d}\phi.$$

The inversion of the order of integrations may be justified as in Theorems 2.1 and 2.2. It is not difficult to evaluate and to show that, by a proper sequence of change of variables

$$\int_0^1 \frac{(u-u^2)^{1/2}}{[x\phi]^{-1} - (u-u^2)^{1/2}} \frac{\mathrm{d}u}{u} = \frac{4x\phi}{[4-(x\phi)^2]^{1/2}} + \arctan\left[\frac{2+x\phi}{2-x\phi}\right]^{1/2}.$$

Hence we have

$$\sum_{k=1}^{\infty} \frac{x^k \Gamma(1+k/2)^2}{k^n k!} = \frac{2(-1)^n}{(n-2)!} \times \int_0^1 \frac{\log^{n-2}\phi}{\phi} \left[\frac{x\phi}{[4-(x\phi)^2]^{1/2}} + \arctan\left[\frac{2+x\phi}{2-x\phi}\right]^{1/2} \right] \mathrm{d}\phi.$$

If we let

$$u = \arctan\left[\frac{2+x\phi}{2-x\phi}\right]^{1/2},$$

we obtain

$$\sum_{k=1}^{\infty} \frac{x^k \Gamma(1+k/2)^2}{k^n k!} = \frac{(-1)^{n-2}}{(n-2)!} \int_{\pi/4}^{\arcsin(x/2)/2+\pi/4} u \log^{n-2}(-2\cos u/x)\, \mathrm{d}u,$$

since

$$\arctan\left[\frac{2+x\phi}{2-x\phi}\right]^{1/2} = \frac{1}{2}\arcsin(x/2) + \frac{\pi}{4}.$$

Make the change of variable $u = \frac{\phi}{2} + \frac{\pi}{4}$ to find

$$\sum_{k=1}^{\infty} \frac{x^k \Gamma(1+k/2)^2}{k^n k!} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{x^k \Gamma(k/2)^2}{k^{n-2} k!}$$
$$= \frac{(-1)^{n-2}}{(n-2)!} \int_0^{\arcsin(x/2)} (\phi + \pi/2) \log^{n-2}(2\sin\phi/x)\,\mathrm{d}\phi.$$

Replacing $n-2$ by n here finishes the proof. □

Theorem 2.5. *For* $|x| \leqslant 2$ *and* $n = 3, 4, \ldots$ *we have*

$$\sum_{k=1}^{\infty} \frac{H_k (k!)^2}{(2k)^n (2k)!} x^{2k} = \frac{(-1)^n}{(n-3)!} \int_0^{\arcsin(|x|/2)} \psi_n(y) \cot y \log[1 - 4\sin^2 y/x^2]\,\mathrm{d}y,$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$, *the nth term of harmonic sum, and*

$$\psi_n(y) = \int_0^y \phi \log^{n-3}[\sin\phi/\sin y]\,\mathrm{d}\phi.$$

Proof. It is well known that

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \int_0^1 \frac{1-u^n}{1-u}\,\mathrm{d}u = -k \int_0^1 y^{k-1} \log(1-y)\,\mathrm{d}y.$$

Hence

$$\sum_{k=1}^{\infty} \frac{H_k (k!)^2}{k^n (2k)!} x^{2k} = \sum_{k=1}^{\infty} \frac{H_k \Gamma(k)\Gamma(k+1)}{k^{n-1}\Gamma(2k+1)} x^{2k} = \sum_{k=1}^{\infty} \frac{x^{2k}}{k^{n-1}} H_k \beta(k, k+1)$$
$$- \sum_{k=1}^{\infty} \frac{x^{2k}}{k^{n-2}} \int_0^1 y^{k-1} \log(1-y)\,\mathrm{d}y \int_0^1 z^{k-1}(1-z)^k\,\mathrm{d}z$$
$$= -\sum_{k=1}^{\infty} \frac{x^{2k}}{k^{n-2}} \int_0^1 \int_0^1 (yz)^{k-1}(1-z)^k \log(1-y)\,\mathrm{d}z\,\mathrm{d}y$$
$$= -\int_0^1 \int_0^1 \frac{\log(1-y)}{yz} \sum_{k=1}^{\infty} \frac{[x^2 yz(1-z)]^k}{k^{n-2}}\,\mathrm{d}z\,\mathrm{d}y$$
$$= -\int_0^1 \int_0^1 \frac{\log(1-y)}{yz} \mathrm{Li}_{n-2}[x^2 yz(1-z)]\,\mathrm{d}z\,\mathrm{d}y.$$

Here we used (1.2). On using (1.2) once more, after some manipulations, we write

$$\sum_{k=1}^{\infty}\frac{H_k(k!)^2}{k^n(2k)!}x^{2k}=\frac{(-1)^n}{(n-3)!}\times\int_0^1\frac{\log(1-y)}{y}\left[\int_0^1\int_0^1\frac{\log^{n-3}\phi}{\phi}\,\frac{1-z}{[x^2\phi y]^{-1}-z+z^2}\mathrm{d}\phi\,\mathrm{d}z\right]\mathrm{d}y$$
$$=\frac{(-1)^n}{(n-3)!}\int_0^1\frac{\log(1-y)}{y}\left[\int_0^1\frac{\log^{n-3}\phi}{\phi}\int_0^1\frac{1-z}{[x^2\phi y]^{-1}-z+z^2}\mathrm{d}z\,\mathrm{d}\phi\right]\mathrm{d}y. \tag{2.19}$$

One may easily justify the inversion of the order of the inner two integrals here as in the proof of Theorems 2.1 and 2.2. Observe that

$$\int_0^1\frac{1-z}{[x^2\phi y]^{-1}-z+z^2}\mathrm{d}z=\left[\frac{4x^2y\phi}{4-x^2y\phi}\right]^{1/2}\arctan\left[\frac{x^2y\phi}{4-x^2y\phi}\right]^{1/2}.$$

Replacing this into (2.19) we get

$$\sum_{k=1}^{\infty}\frac{H_k(k!)^2}{k^n(2k)!}x^{2k}=\frac{(-1)^n}{(n-3)!}\times\int_0^1\frac{\log(1-y)}{y}\left[\int_0^1\frac{\log^{n-3}\phi}{\phi}\left[\frac{4x^2y\phi}{4-x^2y\phi}\right]^{1/2}\arctan\left[\frac{x^2y\phi}{4-x^2y\phi}\right]^{1/2}\mathrm{d}\phi\right]\mathrm{d}y. \tag{2.20}$$

Make the change of variable

$$u=\arctan\left[\frac{x^2y\phi}{4-x^2y\phi}\right]^{1/2}$$

to find that

$$\int_0^1\frac{\log^{n-3}\phi}{\phi}\left[\frac{4x^2y\phi}{4-x^2y\phi}\right]^{1/2}\arctan\left[\frac{x^2y\phi}{4-4x^2y\phi}\right]^{1/2}\mathrm{d}\phi$$
$$=4\int_0^{\eta(x,y)}u\log^{n-3}[4\sin^2u/x^2y]\,\mathrm{d}u,$$

where

$$\eta(x,y) = \arctan\left[\frac{x^2y}{4-x^2y}\right]^{1/2}.$$

Substituting this into (2.20), we write

$$\sum_{k=1}^{\infty}\frac{H_k(k!)^2}{k^n(2k)!}x^{2k} = \frac{4(-1)^n}{(n-3)!} \times \int_0^1 \frac{\log(1-y)}{y}\left[\int_0^{\eta(x,y)} u\log^{n-3}[4\sin^2 u/x^2y]\,\mathrm{d}u\right]\mathrm{d}y.$$

If we substitute

$$z = \arctan\left[\frac{x^2y}{4-x^2y}\right]^{1/2},$$

we get

$$\sum_{k=1}^{\infty}\frac{H_k(k!)^2}{(2k)^n(2k)!}x^{2k} = \frac{(-1)^n}{(n-3)!}\int_0^{\arcsin(|x|/2)}\psi_n(z)\cot z\log[1-4\sin^2 z/x^2]\,\mathrm{d}z,$$

where $\psi_n(z) = \int_0^z \phi\log^{n-3}[\sin\phi/\sin z]\,\mathrm{d}\phi$, since $\arctan[\frac{x^2}{4-x^2}]^{1/2} = \arcsin(|x|/2)$.
This completes the proof of Theorem 2.5.

3. Some generalizations of the theorems

Now making use of the factorization formula (1.3) for polylogarithm series, we can make a number of generalizations to the previous theorems. Our first theorem in this section generalizes Theorem 2.4.

Theorem 3.1. *For* $|x| \leqslant 4$ $m = 1,2,\ldots$ *and* $n = 3,4,5,\ldots$ *we have*

$$\sum_{k=1}^{\infty}\frac{x^{mk}[(mk)!]^4}{k^n[(2mk)!]^2} = \frac{m^{n-1}(-1)^{n-3}}{(n-3)!}\sum_{j=1}^{m}\int_0^{\pi/2}\frac{1}{\sin y} \times \int_0^{\arcsin(\omega^{j/2}x^{m/2}\sin y/4)}\phi\log^{n-3}[4\sin\phi/\omega^{j/2}x^{m/2}\sin y]\,\mathrm{d}\phi\,\mathrm{d}y,$$

where $\omega = \mathrm{e}^{2\pi i/m}$.

Proof. We start by employing (1.1) and (1.2).

$$\sum_{k=1}^{\infty}\frac{x^{mk}[(mk)!]^4}{k^n[(2mk)!]^2}=\sum_{k=1}^{\infty}\frac{x^{mk}}{k^n}\frac{\Gamma(mk+1)^4}{\Gamma(2mk+1)^2}=m^2\sum_{k=1}^{\infty}\frac{x^{mk}}{k^{n-2}}\frac{\Gamma(mk)^2\Gamma(mk+1)^2}{\Gamma(2mk+1)^2}$$
$$=m^2\sum_{k=1}^{\infty}\frac{x^{mk}}{k^{n-2}}\beta(mk,mk+1)^2$$
$$=m^2\sum_{k=1}^{\infty}\frac{x^{mk}}{k^{n-2}}\left[\int_0^1 u^{mk-1}(1-u)^{mk}\,\mathrm{d}u\right]\left[\int_0^1 z^{mk-1}(1-z)^{mk}\,\mathrm{d}z\right]$$
$$=m^2\sum_{k=1}^{\infty}\frac{x^{mk}}{k^{n-2}}\int_0^1\int_0^1(uz)^{mk-1}[(1-z)(1-u)]^{mk}\,\mathrm{d}u\,\mathrm{d}z.$$

Interchanging the order of summation and integration, we find that

$$\sum_{k=1}^{\infty}\frac{x^{mk}[(mk)!]^4}{k^n[(2mk)!]^2}=m^2\int_0^1\int_0^1\sum_{k=1}^{\infty}\frac{[[xuz(1-u)(1-z)]^m]^k}{k^{n-2}}\frac{\mathrm{d}u\,\mathrm{d}z}{uz}$$
$$=m^2\int_0^1\int_0^1\mathrm{Li}_{n-2}[(xuz(1-u)(1-z))^m]\frac{\mathrm{d}u\,\mathrm{d}z}{uz}.$$

Now use (1.5) to get

$$\sum_{k=1}^{\infty}\frac{x^{mk}[(mk)!]^4}{k^n[(2mk)!]^2}=m^{n-1}\sum_{j=1}^{m}\int_0^1\int_0^1\mathrm{Li}_{n-2}[\omega^j xuz(1-u)(1-z)]\frac{\mathrm{d}u\,\mathrm{d}z}{uz}$$
$$=m^{n-1}\sum_{j=1}^{m}\int_0^1\int_0^1\sum_{k=1}^{\infty}\frac{[\omega^j xuz(1-u)(1-z)]^k}{k^{n-2}}\frac{\mathrm{d}u\,\mathrm{d}z}{uz}.$$

Interchanging the order of summation and integration once more, this is

$$\sum_{k=1}^{\infty}\frac{x^{mk}[(mk)!]^4}{k^n[(2mk)!]^2}=m^{n-1}\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{\omega^{jk}x^k}{k^{n-2}}\int_0^1\int_0^1[u^{k-1}z^{k-1}(1-u)^k(1-z)^k\,\mathrm{d}u\,\mathrm{d}z$$
$$=m^{n-1}\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{\omega^{jk}x^k}{k^{n-2}}\left[\int_0^1 u^{k-1}(1-u)^k\,\mathrm{d}u\right]$$
$$\times\left[\int_0^1 z^{k-1}(1-z)^k\,\mathrm{d}z\right]=m^{n-1}\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{\omega^{jk}x^k}{k^{n-2}}\beta(k,k+1)^2$$
$$=m^{n-1}\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{\omega^{jk}x^k}{k^{n-2}}\frac{\Gamma(k)^2\Gamma(k+1)^2}{\Gamma(2k+1)^2}$$
$$=m^{n-1}\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{\omega^{jk}x^{mk}[(k)!]^4}{k^n[(2k)!]^2},$$

where $\omega = e^{2\pi i/m}$. Replacing $\omega^{j/2}x^{m/2}$ by x in Theorem 2.1 finishes the proof. □

Theorem 3.2. *For* $|x| \leqslant 2$ $m = 1, 2, \ldots$ *and* $n = 2, 3, 4, \ldots$ *we have*

$$\sum_{k=1}^{\infty} \frac{x^{mk}[(mk)!]^2}{k^n (2mk)!} = \frac{m^{n-1}(-1)^{n-2}}{(n-2)!} \times \sum_{j=1}^{m} \int_0^{\arcsin(\omega^{j/2}x^{m/2}/2)} \phi \log^{n-2}[2\sin\phi/\omega^{j/2}x^{m/2}]\, d\phi$$

with $\omega = e^{2\pi i/m}$.

Proof. We start with (1.1) and (1.2)

$$\sum_{k=1}^{\infty} \frac{x^{mk}[(mk)!]^2}{k^n[(2mk)!]} = \sum_{k=1}^{\infty} \frac{x^{mk}}{k^n} \frac{\Gamma(mk+1)^2}{\Gamma(2mk+1)} = m\sum_{k=1}^{\infty} \frac{x^{mk}}{k^{n-1}} \frac{\Gamma(mk)\Gamma(mk+1)}{\Gamma(2mk+1)}$$
$$= m\sum_{k=1}^{\infty} \frac{x^{mk}}{k^{n-1}} \beta(mk, mk+1)$$
$$= m\sum_{k=1}^{\infty} \frac{x^{mk}}{k^{n-1}} \int_0^1 u^{mk-1}(1-u)^{mk}\, du.$$

Interchanging the order of summation and integration, we find

$$\sum_{k=1}^{\infty} \frac{x^{mk}[(mk)!]^2}{k^n (2mk)!} = m\int_0^1 \sum_{k=1}^{\infty} \frac{[(xu(1-u))^m]^k}{k^{n-1}} \frac{du}{u}$$
$$= m\int_0^1 Li_{n-1}[(xu(1-u))^m] \frac{du}{u}.$$

Now use (1.5) to get

$$\sum_{k=1}^{\infty} \frac{x^{mk}[(mk)!]^2}{k^n[(2mk)!]} = m^{n-1}\sum_{j=1}^{m} \int_0^1 \mathrm{Li}_{n-1}[\omega^j xu(1-u)]\, du$$
$$= m^{n-1}\sum_{j=1}^{m} \int_0^1 \sum_{k=1}^{\infty} \frac{[\omega^j xu(1-u)]^k}{k^{n-1}} \frac{du}{u}.$$

Interchanging the order of summation and integration once more, this is

$$\sum_{k=1}^{\infty}\frac{x^{mk}[(mk)!]^2}{k^n[(2mk)!]} = m^{n-1}\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{\omega^{jk}x^k}{k^{n-1}}\int_0^1 u^{k-1}(1-u)^k\,\mathrm{d}u$$

$$= m^{n-1}\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{\omega^{jk}x^k}{k^{n-1}}\beta(k,k+1)$$

$$= m^{n-1}\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{\omega^{jk}x^k}{k^{n-1}}\frac{\Gamma(k)\Gamma(k+1)}{\Gamma(2k+1)}$$

$$= m^{n-1}\sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{\omega^{jk}x^k(k!)^2}{k^n(2k)!}. \tag{3.1}$$

We can easily show by the same procedure given in the proof of the theorems in the second section that

$$\sum_{k=1}^{\infty}\frac{x^{2k}[(k!)]^2}{k^n[(2k)!]} = \frac{(-1)^{n-2}}{(n-2)!}\int_0^{\arcsin(x/2)}\phi\log^{n-2}[2\sin\phi/x]\,\mathrm{d}\phi.$$

Thus, replacing x by $\omega^{j/2}x^{1/2}$ here and then substituting it into (3.1) we get

$$\sum_{k=1}^{\infty}\frac{x^{mk}[(mk)!]^2}{k^n(2mk)!} = \frac{m^{n-1}(-1)^{n-2}}{(n-2)!}\times\sum_{j=1}^{m}\int_0^{\arcsin(\omega^{j/2}x^{1/2}/2)}\phi\log^{n-2}[2\sin\phi/\omega^{j/2}x^{1/2}]\,\mathrm{d}\phi.$$

By the same method we can make similar generalizations for Theorems 2.1, 2.3–2.5. □

4. Some examples to the previous theorems

By giving particular values for x and n in Theorems 2.1–2.5, we may obtain many interesting formulas some of which contain Catalan constant G, $\zeta(2)$ and $\zeta(3)$.

1. For $|x|\leqslant\pi/4$,

$$\sum_{k=0}^{\infty}\frac{(2\sin 2x)^{2k+1}(k!)^2}{(2k+1)^2(2k)!} = 2\sum_{k=0}^{\infty}\frac{\tan^{2k+1}x}{(2k+1)^2}.$$

2. For $|x|\leqslant\pi/4$,

$$\sum_{k=0}^{\infty}\frac{(2\sin 2x)^{2k+1}(k!)^2}{(2k+1)^3(2k)!} = 2\sum_{k=0}^{\infty}\frac{(-1)^{k-1}\tan^{2k+1}x}{(2k+1)^3} + 4\sum_{k=0}^{\infty}\frac{a_k(x)}{(2k+1)^2} - 3\zeta(2)x$$

with

$$a_k(x) = \sum_{j=0}^{k} \frac{(-1)^j \tan^{2j+1} x}{(2j+1)}.$$

3.

$$\sum_{k=1}^{\infty} \frac{4^{2k}(k!)^4}{k^3[(2k)!]^2} = 8\pi G - 14\zeta(3).$$

4.

$$\sum_{k=0}^{\infty} \frac{4^{2k+1}(k!)^4}{(2k+1)^3[(2k)!]^2} = 14\zeta(3) - 4\pi G.$$

5.

$$\zeta(3) = \frac{4}{7} + \frac{1}{14} \sum_{k=1}^{\infty} \frac{4^{2k}(1+6k+12k^2+16k^3)(k!)^4}{[k(2k+1)]^3[(2k)!]^2}.$$

6.

$$G = \frac{1}{\pi} + \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{4^{2k}(1+6k+12k^2+12k^3)(k!)^4}{[k(2k+1)]^3[(2k)!]^2}.$$

7.

$$\sum_{k=1}^{\infty} \frac{x^k \Gamma(k/2)^2}{k!} = 2[\arcsin(x/2)]^2 + 2\pi \arcsin(x/2), \quad \text{for } |x| \leqslant 2.$$

Proof of (1). Replacing x by $2\sin 2x$ in Theorem 2.1 with $n = 2$, we find that

$$\sum_{k=0}^{\infty} \frac{(2\sin 2x)^{2k+1}(k!)^2}{(2k+1)^2(2k)!} = \int_0^{2x} \frac{\phi}{\sin\phi}\, \mathrm{d}\phi.$$

If we make the change of variable $\phi = 2\arctan y$, this becomes

$$\sum_{k=0}^{\infty} \frac{(2\sin 2x)^{2k+1}(k!)^2}{(2k+1)^2(2k)!} = 2\int_0^{\tan x} \frac{\arctan y}{y}\, \mathrm{d}y.$$

Expanding $\arctan y$ into its power series at $y=0$, the result follows from integrating term by term. This result is well known and recorded by Ramanujan (see [2, vol. 1 p. 288, entry: 32]) and Bradley (see [4, Theorem 2]). □

Proof of (2). Due to Theorem 2.2, we have, for $|x|\leqslant 2$.

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(2\sin 2x)^{2k+1}(k!)^2}{(2k+1)^3(2k)!} &= \int_0^{2x}\frac{\phi}{\sin\phi}\log(\sin\phi/\sin 2x)\,\mathrm{d}\phi\\ &= \int_0^{2x}\frac{\phi\log(\sin\phi)}{\sin\phi}\,\mathrm{d}\phi - \log(\sin 2x)\int_0^{2x}\frac{\phi\,\mathrm{d}\phi}{\sin\phi}.\end{aligned}$$

Make the substitution $\phi = 2\arctan y$ to get, after some simplification,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(2\sin 2x)^{2k+1}(k!)^2}{(2k+1)^3(2k)!} &= 2\log(2/\sin 2x)\int_0^{\tan x}\frac{\arctan y}{y}\,\mathrm{d}y\\ &\quad + 2\int_0^{\tan x}\frac{\arctan y}{y}\log y\,\mathrm{d}y\\ &\quad - 2\int_0^{\tan x}\frac{\arctan y}{y}\log(1+y^2)\,\mathrm{d}y.\end{aligned}$$

If we expand $\arctan y$ into its power series at $y=0$, we get

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(2\sin 2x)^{2k+1}(k!)^2}{(2k+1)^3(2k)!} &= 2\log(2/\sin 2x)\sum_{k=0}^{\infty}\frac{(-1)^k\tan^{2k+1}x}{(2k+1)^2}\\ &\quad + 2\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\int_0^{\tan x}y^{2k}\log y\,\mathrm{d}y\\ &\quad - 2\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\int_0^{\tan x}y^{2k}\log(1+y^2)\,\mathrm{d}y.\end{aligned}$$

Now integration by parts in both integrals yields, after some simplification

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(2\sin 2x)^{2k+1}(k!)^2}{(2k+1)^3(2k)!} &= -2\sum_{k=0}^{\infty}\frac{(-1)^k\tan^{2k+1}x}{(2k+1)^3}\\ &\quad + 4\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\int_0^{\tan x}\frac{y^{2k+2}}{1+y^2}\,\mathrm{d}y. \qquad (4.1)\end{aligned}$$

Now

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\int_0^{\tan x}\frac{y^{2k+2}}{1+y^2}\,\mathrm{d}y = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\sum_{j=0}^{\infty}\frac{(-1)^{k+j}(\tan x)^{2k+2j+3}}{2k+2j+3}$$
$$= \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\sum_{n=k+1}^{\infty}\frac{(-1)^{n-1}\tan^{2n+1}x}{2n+1}$$
$$= \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\left[\sum_{n=0}^{\infty}\frac{(-1)^{n-1}\tan^{2n+1}x}{2n+1} + \sum_{n=0}^{k}\frac{(-1)^n\tan^{2n+1}x}{2n+1}\right]$$
$$= -\frac{3x}{4}\zeta(2) + \sum_{k=0}^{\infty}\frac{a_k(x)}{(2k+1)^2}$$

with

$$a_k(x) = \sum_{n=0}^{k}\frac{(-1)^n\tan^{2n+1}x}{2n+1}.$$

Substitution this into (4.1) completes the proof of (2). □

Proof of (3). Employing Theorem 2.2 with $n=3$ and $x=4$, we find that

$$\sum_{k=1}^{\infty}\frac{4^{2k}(k!)^4}{k^3[(2k)!]^2} = 4\int_0^{\pi/2}\frac{y^2}{\sin y}\,\mathrm{d}y.$$

These kinds of integrals considered by Ramanujan (see [2, vol. 1 p. 261, entry: 14]) and by his formula given there we find that

$$\int_0^{\pi/2}\frac{y^2}{\sin y}\,\mathrm{d}y = 2\pi G - \frac{7}{2}\zeta(3).$$

Replacing this value of the integral above we get the desired result. □

Proof of (4). Employing Theorem 2.3 with $n=3$ and $x=4$ again, we find that

$$\sum_{k=0}^{\infty}\frac{4^{2k+1}(k!)^4}{(2k+1)^3[(2k)!]^2} = 4\int_0^{\pi/2}\int_0^{y}\frac{\phi\,\mathrm{d}\phi}{\sin\phi}\,\mathrm{d}y.$$

Integration by parts yields

$$\sum_{k=0}^{\infty}\frac{4^{2k+1}(k!)^4}{(2k+1)^3[(2k)!]^2} = 2\pi\int_0^{\pi/2}\frac{y\,\mathrm{d}y}{\sin y} - 4\int_0^{\pi/2}\frac{y^2}{\sin y}\,\mathrm{d}y.$$

The result follows from [2, vol. 1, p. 261, entry: 14] again since

$$\int_0^{\pi/2} \frac{y\,\mathrm{d}y}{\sin y} = 2G. \qquad \square$$

Proof of (5) and (6). They follow immediately from (3) and (4). $\square$

Proof of (7). It follows immediately from Theorem 2.4 with $n = 0$. $\square$

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