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Robust inflation-forecast-based rules to shield against indeterminacy ☆

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Abstract

This paper provides a first attempt to quantify and at the same time utilize estimated measures of uncertainty for the design of robust interest rate rules. We estimate several variants of a linearized form of a New Keynesian model using quarterly US data. Both our theoretical and numerical results indicate that inflation-forecast-based (IFB) rules are increasingly prone to the problem of indeterminacy as the forward horizon increases. As a consequence the stabilization performance of optimized rules of this type worsens too. Robust IFB rules can be designed to avoid indeterminacy in an uncertain environment, but at an increasing utility loss as rules become more forward-looking.

JEL classification: E52; E37; E58

Keywords: Robustness; Taylor rules; Inflation-forecast-based rules; Indeterminacy

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1. Introduction

'Uncertainty is not just an important feature of the monetary policy landscape; it is the defining characteristic of that landscape' (Alan Greenspan[1]).

This paper adopts a consistently Bayesian approach to the measurement of uncertainty and the design of robust rules for the conduct of monetary policy. Employing a closed-economy dynamic stochastic general equilibrium (DSGE) model, the sources of uncertainty are the structural parameters and the volatility of the white noise disturbances. We estimate several variants of a linearized form of the model using quarterly US data. From these competing specifications we obtain estimates for posterior model probabilities and, for each model variant, estimates of the posterior densities of the parameters.

Throughout we focus on Taylor-type rules, and in particular on inflation-forecast-based (IFB) rules. These are 'simple' rules as in Taylor (1993), but where the policy instrument responds to deviations of expected, rather than current inflation from target. In most applications, the inflation forecasts underlying IFB rules are taken to be the endogenous rational-expectation forecasts conditional on an intertemporal equilibrium of the model. These rules are of interest because, as shown in Clarida et al. (2000) and Castelnuovo (2003), estimates of IFB-type rules appear to be a good fit to the actual monetary policy in the US and Europe of recent years. They are also of specific interest because similar reaction functions are used in the forecasting models of the Bank of Canada and the Reserve Bank of New Zealand, two prominent inflation-targeting central banks. In these countries and elsewhere, central bankers extol the virtues of IFB rules on the grounds that they 'pre-empt inflation' and 'enhance low-inflation credibility'.

The literature on IFB interest-rate rules highlights two forms of possible indeterminacy: if the response of interest rates to a rise in expected inflation is insufficient, then real interest rates fall, thus raising demand and confirming any exogenous expected inflation. But indeterminacy is also possible if the rule is overly aggressive or overly forward-looking or both (Bernanke and Woodford, 1997; Batini and Pearlman, 2002; Giannoni and Woodford, 2002; Carlstrom and Fuerst, 1999; Benhabib et al., 2001; Woodford, 2003; Batini et al., 2004a, BLP hereafter). We provide a theoretical analysis of both these forms of indeterminacy.

Using our estimated rival models, the estimated model probabilities and posterior densities of parameters, we then proceed to design IFB rules that are robust in two senses with respect to our *estimated* measures of uncertainty[2]: '*weakly robust*' rules are guaranteed to be stable and determinate in all the possible central variants of the model whereas '*strongly robust*' rules, also guarantee stable and unique equilibria and, in addition, use the probabilities to minimize an expected loss function of the

[1]Federal Reserve Bank of Kansas (2003), Opening Remarks.

[2]The literature gives multiple interpretations of the concept of 'robustness'. The way we intend it here is akin to that used by the literature on robust control, which defines robust monetary policy rules as rules designed to work well in worst-case scenarios thanks to their reduced sensitivity to parameter variations and modelling errors.

Table 1
Four robustness criteria

	Weak robustness	Strong robustness
M (Model)-robustness	(M,W)	(M,S)
P (Parameter)-robustness	(P,W)	(P,S)

central bank subject to this model uncertainty. Both these forms of robustness across models with estimates at their median values we refer to as 'M-robustness', weak or strong. A more demanding robustness requirement is minimize the expected loss across *all* possible parameter values drawn from a large sample constructed using the estimated posterior parameter distributions as well as the model probabilities. This we refer to as 'P-robustness', weak or strong. Table 1 summarizes this taxonomy.

The monetary rules studied in the paper are defined in terms of feedback parameters. Weakly robust rules then define a space of these parameters for which stability and determinacy is guaranteed across models with model parameters at median values, or across all possible parameter values. In each case, a strongly robust rule chooses from the set of weakly robust rules the rule that maximizes the policymaker's expected utility.

Our approach thus differs from existing work on the design of robust policy rules in a number of important respects. First, existing work that assumes structured model uncertainty typically posits the latter by arbitrarily calibrating the relative probability of alternative models being true representations of the economy (see for example Angeloni et al., 2003; Coenen, 2003; Levin et al., 2003).[3] This paper provides a first attempt to quantify and at the same time utilize *estimated* measures of uncertainty for the design of robust rules. Second, the literature taking the rival model approach typically confines itself to what we call strong M-robustness. Third, we examine robust policy in a unified framework that compares different simple rules with each other, and with their optimal counterparts.[4]

We find four main results. First, in each of our three model variants with the highest posterior model probabilities chosen for the policy exercise, there are significant gains from stabilization using an optimized inflation targeting rule with the interest rate feeding back on current inflation. Second, a strongly M-robust and P-robust current inflation rule can be designed that achieves almost all of the stabilization gain that would be achieved if there was no model uncertainty. Third, integral interest rate rules where the change in interest rates feeds back on current or

[3]This literature contrasts with the minmax framework of Hansen and Sargent (2002) that assumes unstructured model uncertainty. Walsh (2003) provides a useful overview of this approach and Tetlow and von zur Muehlen (2002) provides a comparison.

[4]In common with all DSGE models and despite a rival model approach, our work still suffers to from *misspecification* in that these models imposes invalid cross-coefficient restrictions on the moving average representation of the variables. As a consequence the forecasting performance of DGGE models is worse than VARs, though the latter are vulnerable to the Lucas critique. In recent work Del Negro and Schorfheide (2005) and address this issue by combining the two approaches in a framework that enables robust rules to be formulated to take into account misspecification.

expected future inflation perform better than non-integral rules.[5] Fourth, the optimized IFB rules perform increasingly less well as the forward horizon increases from $j = 0$ (the current inflation rule) to $j = 4$ quarters. We find a qualitative difference between optimized rules that respond to current inflation and one-quarter ahead expected inflation on the one hand, and rules that respond to expected inflation two or more quarters ahead ($j \geqslant 2$). For the former ($j = 0, 1$), little by way of utility outcome is lost by insisting on M-robustness or P-robustness. For the latter ($j > 2$), robustness in both senses is only achieved by sacrificing the utility outcome when each of the models in turn describes the true economic environment. This deterioration is especially marked if we insist on P-robustness as our design criterion.

The rest of the paper is organized as follows. Section 2 sets out our model. Section 3 provides a theoretical examination of the indeterminacy problem of IFB rules using the root locus method[6] employed by Batini and Pearlman (2002) and BLP. This analysis indicates which features of the model and the rule make them indeterminacy-prone. Section 4 first focuses on optimized IFB rules and optimal rules without uncertainty before we turn to the robust policy problem in Section 5. Section 6 concludes the paper.

2. The model

Our model is the closed economy version of BLP. There is one traded risk-free nominal bond. A final homogeneous good is produced competitively using a CES technology consisting of a continuum of differentiated non-traded goods. Intermediate goods producers and household suppliers of labor have monopolistic power. Nominal prices of intermediate goods are sticky. We incorporate habit formation in consumption, and Calvo price setting with indexing of prices for those firms who, in a particular period, do not re-optimize their prices. The latter two aspects of the model follow Christiano et al. (2001) and, as with these authors, our motivation is an empirical one: to generate sufficient inertia in the model so as to enable it, in calibrated form, to reproduce commonly-observed output, inflation and nominal interest rate responses to exogenous shocks. Our model is stochastic with two exogenous AR(1) stochastic processes for total factor productivity (TFP) in the intermediate goods sector and government spending.

2.1. Households

A representative household r maximizes

$$\mathscr{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{(C_t(r) - H_t)^{1-\sigma}}{1-\sigma} + \chi \frac{\left(M_t(r)/P_t\right)^{1-\varphi}}{1-\varphi} - \kappa \frac{N_t(r)^{1+\phi}}{1+\phi} + u(G_t) \right], \tag{1}$$

where $\mathscr{E}_t$ is the expectations operator indicating expectations formed at time t, $C_t(r)$ is an index of consumption, $N_t(r)$ are hours worked, H_t represents the habit, or

[5] This accords the results of Levin et al. (2003).

[6] This is a standard method for analyzing the stability of dynamic linear systems found in the engineering literature (see Evans, 1954 and Aoki, 1981 for an application to economics).

desire not to differ too much from other consumers, and we choose it as $H_t = hC_{t-1}$, where C_t is the average consumption index and $h \in [0,1)$ and $\sigma > 1$ is a risk-aversion parameter. $M_t(r)$ are end-of-period nominal money balances, $1+\phi$ is the elasticity of disutility with respect to labor supply and $u(G_t)$ is the utility from exogenous real government spending G_t.

The representative household r must obey a budget constraint:

$$P_tC_t(r) + D_t(r) + M_t(r) = W_t(r)N_t(r) + (1+i_{t-1})D_{t-1}(r) + M_{t-1}(r) + \Gamma_t(r) - P_t\tau_t, \tag{2}$$

where P_t is a price index, $D_t(r)$ are end-of-period holdings of riskless nominal bonds with nominal interest rate i_t over the interval $[t, t+1]$. $W_t(r)$ is the wage, $\Gamma_t(r)$ are dividends from ownership of firms and τ_t are lump-sum real taxes. In addition, if we assume that households' labor supply is differentiated with elasticity of supply η, then (as we shall see below) the demand for each consumer's labor is given by

$$N_t(r) = \left(\frac{W_t(r)}{W_t}\right)^{-\eta} N_t, \tag{3}$$

where $W_t = [\int_0^1 W_t(r)^{1-\eta}\,\mathrm{d}r]^{1/(1-\eta)}$ is an average wage index and N_t is average employment.

Maximizing (1) subject to (2) and (3) and imposing symmetry on households (so that $C_t(r) = C_t$, etc.) yields standard results:

$$1 = \beta(1+i_t)\mathscr{E}_t\left[\left(\frac{C_{t+1}-H_{t+1}}{C_t-H_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right], \tag{4}$$

$$\left(\frac{M_t}{P_t}\right)^{-\varphi} = \frac{(C_t-H_t)^{-\sigma}}{\chi P_t}\left[\frac{i_t}{1+i_t}\right], \tag{5}$$

$$\frac{W_t}{P_t} = \frac{\kappa}{(1-1/\eta)}N_t^{\phi}(C_t-H_t)^{\sigma}. \tag{6}$$

(4) is the familiar Keynes–Ramsey rule adapted to take into account the consumption habit. In (5), the demand for money balances depends positively on consumption relative to habit and negatively on the nominal interest rate. Given the central bank's setting of the latter, (5) is completely recursive to the rest of the system describing our macro-model and will be ignored in the rest of the paper. (6) reflects the market power of households arising from their monopolistic supply of a differentiated factor input with elasticity η.

2.2. Firms

Competitive final goods firms use a continuum of non-traded intermediate goods according to a constant returns CES technology to produce aggregate output

$$Y_t = \left(\int_0^1 Y_t(m)^{(\zeta-1)/\zeta}\,\mathrm{d}m\right)^{\zeta/(\zeta-1)}, \tag{7}$$

where ζ is the elasticity of substitution. This implies a set of demand equations for each intermediate good m with price $P_t(m)$ of the form

$$Y_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} Y_t, \tag{8}$$

where $P_t = [\int_0^1 P_t(m)^{1-\zeta}\,\mathrm{d}m]^{1/(1-\zeta)}$. P_t is an aggregate intermediate price index, but since final goods firms are competitive and the only inputs are intermediate goods, it is also the domestic price level.

In the intermediate goods sector each good m is produced by a single firm m using only differentiated labor with another constant returns CES technology:

$$Y_t(m) = A_t\left(\int_0^1 N_t(r,m)^{(\eta-1)/\eta}\,\mathrm{d}r\right)^{\eta/(\eta-1)}, \tag{9}$$

where $N_t(r,m)$ is the labor input of type r by firm m and A_t is an exogenous shock capturing shifts to trend TFP in this sector. Minimizing costs $\int_0^1 W_t(r)N_t(r,m)\,\mathrm{d}r$ and aggregating over firms and denoting $\int_0^1 N_t(r,m)\,\mathrm{d}m = N_t(r)$ leads to the demand for labor as shown in (3). In an equilibrium of equal households and firms, all wages adjust to the same level W_t and it follows that $Y_t = A_tN_t$.

For later analysis it is useful to define the real marginal cost as the wage costs as a proportion of output. Using (6) and $Y_t = A_tN_t$ this can be written as

$$MC_t \equiv \frac{W_tN_t}{P_tY_t} = \frac{W_t}{A_tP_t} = \frac{\kappa}{(1-1/\eta)A_t}\left(\frac{Y_t}{A_t}\right)^{\phi}(C_t - H_t)^{\sigma}. \tag{10}$$

Now we assume that there is a probability of $1-\xi$ at each period that the price of each intermediate good m is set optimally to $P_t^0(m)$. If the price is not re-optimized, then it is indexed to last period's aggregate producer price inflation.[7] With indexation parameter $\gamma \geqslant 0$, this implies that successive prices with no re-optimization are given by $P_t^0(m)$, $P_t^0(m)(P_t/P_{t-1})^{\gamma}$, $P_t^0(m)(P_{t+1}/P_{t-1})^{\gamma}, \ldots$. Let $Q_{t,t+k} = 1/(1+i_t)1/(1+i_{t+1}))\cdots 1/(1+i_{t+k-1})$ be the discount factor over the interval $[t, t+k]$. Then each intermediate producer m at time t chooses $\{P_t(m)\}$ to maximize discounted profits

$$\mathscr{E}_t \sum_{k=0}^{\infty} \xi^k Q_{t,t+k} Y_{t+k}(m)\left[P_t^0(m)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma} - \frac{W_{t+k}}{A_t}\right] \tag{11}$$

given i_t (since firms are atomistic), subject to (8). The solution to this is

$$\mathscr{E}_t \sum_{k=0}^{\infty} \xi^k Q_{t,t+k} Y_{t+k}(m)\left[P_t^0(m)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma} - \frac{1}{(1-1/\zeta)}\frac{W_{t+k}}{A_t}\right] = 0 \tag{12}$$

[7] Thus we can interpret $1/(1-\xi)$ as the average duration for which prices are left unchanged.

and by the law of large numbers the evolution of the price index is given by

$$P_{t+1}^{1-\zeta} = \xi\left(P_t\left(\frac{P_t}{P_{t-1}}\right)^{\gamma}\right)^{1-\zeta} + (1-\xi)(P_{t+1}^0)^{1-\zeta}. \tag{13}$$

2.3. Equilibrium

In equilibrium, goods markets, money markets and the bond market all clear. Equating the supply and demand of the consumer good we obtain

$$Y_t = A_t N_t = C_t + G_t. \tag{14}$$

A balanced budget government budget constraint

$$G_t = \tau_t + \frac{M_t - M_{t-1}}{P_t} \tag{15}$$

completes the model. Given interest rates i_t (expressed later in terms of an optimal or IFB rule) the money supply is fixed by the central banks to accommodate money demand. By Walras' Law we can dispense with the bond market equilibrium condition and therefore the government budget constraint that determines taxes τ_t. Then the equilibrium is defined at $t = 0$ by stochastic processes C_t, D_t, P_t, M_t, W_t, Y_t, N_t, given past price indices and exogenous TFP and government spending processes.

2.4. Linearization and state-space representation

We now linearize about the deterministic zero-inflation steady state. Output is then at its sticky-price, imperfectly competitive natural rate and from the Keynes–Ramsey condition (4) the nominal rate of interest is given by $\bar{\imath} = 1/\beta - 1$. Define all lower case variables as proportional deviations from this baseline steady state.[8] Then the linearization takes the form:

$$\pi_t = \frac{\beta}{1+\beta\gamma}\,\mathscr{E}_t\pi_{t+1} + \frac{\gamma}{1+\beta\gamma}\,\pi_{t-1} + \frac{(1-\beta\xi)(1-\xi)}{(1+\beta\gamma)\xi}\,mc_t, \tag{16}$$

$$mc_t = -(1+\phi)a_t + \frac{\sigma}{1-h}(c_t - hc_{t-1}) + \phi y_t, \tag{17}$$

$$c_t = \frac{h}{1+h}\,c_{t-1} + \frac{1}{1+h}\,\mathscr{E}_t c_{t+1} - \frac{1-h}{(1+h)\sigma}(i_t - \mathscr{E}_t\pi_{t+1}), \tag{18}$$

$$y_t = \frac{\bar{C}}{\bar{Y}}\,c_t + \frac{\bar{G}}{\bar{Y}}\,g_t, \tag{19}$$

[8] That is, for a typical variable X_t, $x_t = (X_t - \bar{X})/\bar{X} \simeq \log(X_t/\bar{X})$ where $\bar{X}$ is the baseline steady state. The interest rate however is now expressed as an absolute deviation about $\bar{\imath}$.

$$g_{t+1} = \rho_g g_t + \varepsilon_{g,t+1}, \tag{20}$$

$$a_{t+1} = \rho_a a_t + \varepsilon_{a,t+1}. \tag{21}$$

Variables y_t, c_t, mc_t, a_t, g_t are proportional deviations about the steady state. $[\varepsilon_{g,t}, \varepsilon_{a,t}]$ are i.i.d. zero-mean disturbances. π_t and i_t are absolute deviations about the steady state. For later use we require the *output gap* (the difference between output for the sticky-price model obtained above and output when prices are flexible), $y_{n,t}$ say. The latter, obtained by setting $\xi = 0$ in (16) to (19), is in deviation form given by[9]

$$\frac{\sigma}{1-h}(c_{n,t} - hc_{n,t-1}) + \phi y_{n,t} = (1+\phi)a_t, \tag{22}$$

$$y_{n,t} = \frac{\bar{C}}{\bar{Y}} c_{n,t} + \frac{\bar{G}}{\bar{Y}} g_t. \tag{23}$$

We can write this system in state-space form as

$$\begin{bmatrix} \mathbf{z}_{t+1} \\ \mathscr{E}_t \mathbf{x}_{t+1} \end{bmatrix} = A \begin{bmatrix} \mathbf{z}_t \\ \mathbf{x}_t \end{bmatrix} + Bi_t + C \begin{bmatrix} \varepsilon_{g,t+1} \\ \varepsilon_{a,t+1} \end{bmatrix}, \tag{24}$$

$$\begin{bmatrix} y_t \\ y_{n,t} \end{bmatrix} = E \begin{bmatrix} \mathbf{z}_t \\ \mathbf{x}_t \end{bmatrix}, \tag{25}$$

where $\mathbf{z}_t = [g_t, a_t, c_{t-1}, c_{n,t-1}, \pi_{t-1}]$ is a vector of predetermined variables at time t and $\mathbf{x}_t = [c_t, \pi_t]$ are non-predetermined variables. Rational expectations are formed assuming an information set $\{z_s, x_s, \varepsilon_{g,t}, \varepsilon_{a,t}\}$, $s \leqslant t$, the model and the monetary rule. To help the reader through the rest of the paper, Table 2 provides a summary of our notation.

2.5. Estimation

2.5.1. Overview

In this section we estimate four main variants of model (16)–(21) using Bayesian methods. In particular, we estimate: the most general specification of the model with both inflation and habit persistence (we label this variant '*Z*'); a version of the model without inflation persistence but with persistence in habits ($\gamma = 0$, variant '*G*'); a version without habit persistence but with persistence in inflation ($h = 0$, variant '*H*'); and finally a version with neither inflation nor habit persistence ($\gamma = h = 0$, variant '*GH*'). We close the model with a one-quarter ahead IFB rule of form (27) that is the subject of the next section.

Bayesian estimation of the model has the specific advantage that it provides a posterior distribution of the parameter values that allows us to make probabilistic statements about the functionals of the model(s)' parameters. Furthermore, it

[9]Note that the zero-inflation steady states of the sticky and flexi-price steady states are the same.

Table 2
Summary of notation (variables in deviation form)

π_t	Producer price inflation over interval $[t-1, t]$
i_t, r_t	Nominal and real interest rates over interval $[t, t+1]$
mc_t	Marginal cost
y_t, $y_{n,t}$	Output with sticky prices and flexi-prices
$o_t = y_t - y_{n,t}$	Output gap
$g_{t+1} = \rho_g g_t + \varepsilon_{g,t+1}$	AR(1) process government spending shock, g_t
$a_{t+1} = \rho_a a_t + \varepsilon_{a,t+1}$	AR(1) process for factor productivity shock, a_t
β	Discount parameter
γ	Indexation parameter
h	Habit parameter
$1-\xi$	Probability of price re-optimization
σ	Risk-aversion parameter
ϕ	Disutility of labor supply parameter
$\chi = \dfrac{(1-\beta\xi)(1-\xi)}{(1+\beta\xi)\xi}$	Slope of Phillip's curve
ρ, Θ_y, Θ	Feedback in IFB rules on lagged interest rate, output and future expected inflation, respectively $\Theta = \rho(1-\theta)$ for non-integral rules ($\rho < 1$)

provides us with the odds on models that allow us to quantify how likely it is that the data would have come from a model with both habit and inflation persistence as opposed to a framework with just one of these mechanisms or neither. In this sense the estimation method per se supplies us with a consistent measure of both parameter (posterior distribution of the parameters) and model (posterior odds) uncertainty.[10]

The subsections below offer: a discussion of the specification of the prior distributions that we need to carry out (Section 2.5.2); the results from the estimation of our four model specifications (Section 2.5.3); and a formal comparison of models (Section 2.5.4). This subsection shows how we obtain the posterior model probabilities that we use as weights for the competing model specifications in the analysis of robust IFB rules under uncertainty. Section 2.5.5 addresses issues associated with identification in DSGE models, the convergence of our Markov chains and the fact that in our results we have constrained the estimation to the region of determinacy $\theta > 1$. Appendix A provides further details of the methods used in estimation.

2.5.2. Data and priors

We estimate the model(s) using quarterly US data on real GDP (detrended using a Hodrick–Prescott filter, see Lubik and Schorfheide, 2003, Juillard et al., 2004), the Federal Funds rate (annualized, in percentage points), and the annualized log

[10] Justiniano and Preston (2004) discuss the many additional advantages of using Bayesian methods to estimate dynamic stochastic general equilibrium models. These include overcoming convergence problems with numerical routines to maximize the likelihood as well as providing measures of uncertainty that need not assume a symmetric distribution.

difference of the consumer price index (CPI) for the sample 1984:I–2003:IV.[11] All series were obtained from DataStream International.

Following Lubik and Schorfheide (2004) rather than de-meaning the series, we estimate the mean of inflation and the (unobservable) real interest rate, π^* and r^*, respectively, together with the model(s) parameters. In turn, this gives the following mapping between observables (superscript obs) and the variables following the solution of the model:

$$\begin{pmatrix} \pi_t^{\text{obs}} \\ y_t^{\text{obs}} \\ i_t^{\text{obs}} \end{pmatrix} = \begin{pmatrix} \pi^* \\ 0 \\ \pi^* + r^* \end{pmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{pmatrix} \pi_t \\ y_t \\ i_t \end{pmatrix}.$$

In addition, the mean of the real rate gives us an estimate of the discount factor $\beta = 1/\sqrt[4]{1 + r^*/100}$.

To proceed with the Bayesian estimation we need a prior distribution for the parameters. Details on our priors are presented in Table B1 in Appendix B reporting the type of density, mean and standard deviation for each coefficient.[12] The last two columns also provide the 1% and 99% percentiles of the prior ordinates. In choosing these densities we considered the entire spectrum of prior existing empirical estimates or calibrations. As a result, some of our priors are more widely dispersed, and therefore less tight than those chosen by other authors.[13]

The degree of habit formation (h), price indexation (γ) and interest smoothing in the IFB-type rule (ρ), as well as the autoregressive coefficients of the shocks (ρ_g and ρ_a) are all constrained to the unit interval, motivating our choice of Beta densities for these priors. The priors for h and γ are centered at 0.7, on the assumption that output and inflation are considerably inertial, in line with findings by Fuhrer and Moore (1995), Fuhrer (2000), Banerjee and Batini (2003, 2004) and Smets and Wouters (2004) (SW, 2004), among others. Likewise, our prior for the mean of ρ is rather high and close to the estimates from Clarida et al., 2000 (CGG, 2000).

Priors for the risk-aversion and labor supply parameters, σ and ϕ, respectively, are shaped in the form of a Gamma density and are chosen to be fairly flat, reflecting the wide dispersion of existing empirical estimates and calibrations of these parameters in the literature (see Nelson and Nikolov, 2002).

The slope of the Phillips' curve, $\chi = (1 - \beta\xi)(1 - \xi)/(1 + \beta\gamma)\xi$ is a function of the degree of price stickiness in the economy, ξ, and the discount factor. So we selected the prior for χ in line with the assumption that the quarterly discount factor is equal

[11]Eight observations, corresponding to the period 1982:I–1983:IV are used to initialize the Kalman filter.

[12]In principle, is would be possible to specify flat or non-informative priors. However, in addition to being able to choose priors based on coefficients values available in the literature, flat priors are not well suited for model comparisons.

[13]Throughout the estimation of different models, the share of government expenditures in output is calibrated at 0.22, which represents the sample average of this coefficient for our sample.

to 0.99 and prices are sticky for three quarters, as suggested by survey evidence on the average duration of US price contracts (see, for example, Blinder et al., 1998).[14]

Finally, the prior for the feedback parameter, θ, accounts for the breadth of the spectrum of estimated responses to expected inflation by the US Federal Reserve. More specifically, our specification contains the 90% posterior intervals of Lubik and Schorfheide (2004)[15] and is looser than the prior specified by SW for the same parameter.[16]

2.5.3. Estimation results

Table B2 in Appendix B, summarizes the results off estimating the four model variants (G, H, GH and Z). The three columns for each specification report the median, first and ninth decile of the 100,000 draws generated using the Random Walk-Metropolis algorithm used to approximate the posterior densities.

A few important things emerge from the table. First, estimates of the policy coefficients are fairly robust across specifications. Posterior estimates of the interest rate smoothing parameter ρ are tightly concentrated on values that suggest a substantial degree of interest smoothing, in accordance with results reported by CGG amongst other authors. Meanwhile, the posterior density for the feedback coefficient θ is remarkably similar (that is both in medians and percentiles) across the first three specifications, implying a very aggressive response by the US Federal Reserve to expected inflation, in line with findings by CGG for a similar rule and sample.

The median estimates for the real interest rate r^* translate into a median value of 0.995 for the stochastic discount factor which, in turn, implies plausible estimates for the degree of price stickiness based on the inferred values for the Phillips curve slope χ. The implied point estimates of the price stickiness parameter ξ range from 0.36 up to 0.67, increasing, as expected, depending on whether or not price indexation is allowed for.[17] These higher values are in accordance with Blinder et al. (1998) and Rotemberg and Woodford (1998), but contrast with the high degree of price rigidity estimated by SW (2004).

Our estimates of the risk-aversion parameter σ are rather large. With no habits, these estimates map directly with the intertemporal elasticity of substitution and suggest that this may be quite small.[18] A common theme in papers estimating DSGE

[14] It is worth noting that the results of the estimation from assuming a prior directly on the Calvo coefficient ξ are somewhat different. This may be because with a prior on lambda, as we have used now, the link between ξ and the discount factor in determining the slope of the PC is not imposed. We plan to re-run the estimation with this alternative prior as a robustness check.

[15] Note that in contrast to these authors however we constrain the estimation to the region of determinacy and therefore truncate the prior for θ. The results of their paper suggest, however, that at least for a Taylor rule on current inflation, indeterminacy has not been an issue for our sample. We discuss this issue further in Section 2.5.5.

[16] In their paper, however, SW include the output gap in the Taylor rule.

[17] Using $\chi \equiv (1-\beta\xi)(1-\xi)/(1+\beta\gamma)\xi$ we obtain $\xi = 0.67, 0.36, 0.60, 0.53$ corresponding to contract lengths, $1/(1-\xi)$, of 3.06, 1.57, 2.50 and 2.13 quarters for models G, H, GH and Z, respectively.

[18] This result is attributable to a prior density centered on high values for σ. Redoing the estimation using the SW priors leads to point estimates far closer to one, clearly revealing that inference on this parameters is sensitive to the choice of priors.

models is the difficulty in pinning down the labor supply parameter ϕ. Therefore, it is not surprising that, inference on the inverse Frisch elasticity of labor supply is susceptible to the specification of the model, and exhibits wide posterior probability intervals.

Turning to the coefficient governing habit formation, h is tightly estimated and suggests rather inertial consumption and output processes. Reported posterior intervals for h are almost identical to the ones obtained by Juillard et al. (2004) and higher than the estimates by SW. By contrast, the posterior density of the indexation parameter γ lies to the left of our chosen prior, suggesting, in contrast to studies mentioned earlier, that inflation is intrinsically not very persistent – a result that accords with findings in Erceg and Levin (2001), Taylor (2000) and Cogley and Sargent (2001).

Estimates of the shock processes reveal that both the technology and the government expenditure shock are highly persistent, and this holds true regardless of the exact model specification. Posterior estimates clearly attribute greater volatility of shocks to the government expenditure component rather than to disturbances in technology.[19] As usual, exogenous disturbances to the monetary policy equation appear much less important than technology and government expenditure shocks in driving inflation, consumption and output processes.[20]

2.5.4. Model comparison

Since the goal of this paper is to characterize the design of robust rules under uncertainty, it is important to investigate which specification seems to be best supported by the data. In doing so we do not intend to select any particular model as being the 'true' one, but rather wish to compute posterior probabilities to place odds on the different models. Bayesian methods for model comparisons allows us to obtain these posterior model probabilities in order to discriminate or aggregate across competing specifications, thereafter providing coefficient estimates that explicitly account for model uncertainty. Let us define m_k to be one possible element from the (discrete) set of competing models $\mu = \{G, H, GH, Z\}$. The posterior model probability for $p(m_k|Y^T)$ summarizes the evidence provided by the data in favor of m_k and is then given by

$$p(m_k|Y^T) = f(Y^T|m_k)p(m_k)/f(Y^T), \tag{26}$$

where $p(m_k)$ stands for the prior probability assigned to model k, that in our case equals $\frac{1}{4}$ since we treat each model as equiprobable a priori. The first expression in the numerator is the marginal likelihood (or marginal data density) and equals the denominator in Eq. (A.1).

We compute the posterior model probabilities using the Reversible Jump MCMC algorithm (RJMCMC) of Dellaportas et al. (2002) which belongs to the

[19] Indeed, the first posterior decile of the former exceeds the nineth decile of the latter, for all models, despite similar prior densities for the innovation standard deviations.

[20] Note that the correlation of shocks is important as well. So far, as it is standard in most models, we have constrained the disturbances to be i.i.d.

class of product space search methods that have become very popular for model comparisons in the statistics literature. This class of trans-dimensional MCMC algorithms adds a model indicator variable to be estimated jointly with the parameters allowing the sampler consequently to move both within and across models and to compute posterior model probabilities directly without evaluating the marginal likelihood.[21]

Estimates of $p(m_k|Y^T)$ obtained with the RJMCMC for our four model variants are presented in Table B3. In line with results discussed above, the specification with habit persistence and no price indexation (G) attains highest posterior probability. Model Z, which allows for both of these intrinsic mechanisms, follows in probability ranking. In contrast, model H with no habit persistence is 9 times less likely than those specifications (Z and G) with endogenous persistence in consumption. Finally, the most restrictive model, GH attains the lowest posterior model probability further providing evidence of the need to incorporate at least one of the two intrinsic mechanisms imparting greater inertia to the model. Therefore, these results can be interpreted as suggesting that the addition of endogenous mechanisms of persistence, particularly habit in consumption, improve the fit of the model. These posterior odds will be used to weight the models for our analysis of uncertainty on the robustness of policy rules.

2.5.5. *Identification, convergence and indeterminacy issues*

In this section we discuss three important issues related to our estimation of the model(s), and more generally to the estimation of DSGE models of the kind used here. First, we examine the issue of parameter identification; second, we review some key diagnostic issues; and finally we look at the implications for our results of choosing priors in estimation to lay outside the region of indeterminacy.

On the question of *parameter identification*, Beyer and Farmer (2003, 2004) have drawn attention to the problem of identifying specific parameters in DSGE models and, more specifically, ask the question whether existing estimation methods can correctly distinguish between determinate and indeterminate solutions of such models. How big of a problem is this for our estimates? In our case the marginal likelihood is based on obtaining the saddle-path solution to (16)–(23). In principle, we too thus face the risk of observational equivalence in that two sets of parameters could yield exactly the same path for the variables, and therefore the same likelihoods. In practice, however, identification does not seem to be an issue here. Indeed had there been a problem of identification this would naturally have manifested itself at an early stage in the form of non-invertibility of the Hessian from the posterior density obtained from our initial maximization algorithm. The fact that this did not occur for any of our four models does not, of course, preclude the possibility that the priors may be masking some directions of the parameter space in

[21] An alternative method is the Modified Harmonic Mean estimator proposed by Geweke (1999). Lopes and West (2004) assess different methods for model comparison via simulation which reveal that Reversible-jump algorithms can be very accurate in the computation of posterior model odds and therefore advocate their use. These results hold true even in very high dimensional spaces as in Justiniano (2004). These papers compare the performance of various methods in the context of static and dynamic factor models.

which the likelihood is flat. Yet, if this was the case, this would have implied that our priors were not updated – something that is not true in our case, given results from our comparisons of prior and posterior probability bands in Tables B1 and B2.[22]

Turning to *diagnostic issues*, convergence diagnostics are important in their own right as they serve to gauge whether the generated draws provide an accurate characterization of the posterior distributions. Following the recommendations in several reviews of MCMC diagnostics in the statistics literature, we monitor convergence through a battery of diagnostics tools (e.g. kernel and trace plots) and more prominently the evolution of the variance potential scale reduction factors (PSRFs) proposed originally by Gelman and Rubin (1992) and confidence interval variants refined in Brooks and Gelman (1998).[23]

Finally, what are the implications of *truncating priors out of the region of indeterminacy*? One last important estimation issue relates to the fact that in our results we have constrained the estimation to the region of determinacy $\theta > 1$ for our one-quarter ahead IFB rule, IFB1 and therefore truncated the prior accordingly. At first pass, one way of gauging whether the truncation is binding is to re-estimate the model with a prior that assigns a non-trivial mass to values of $\theta < 1$ and counting the number of draws the fall in the indeterminacy region.[24] To check this we re-estimate all model variants with a normal prior on θ with mean 1.5 and standard deviation 0.5. This alternative prior density has a cdf of 0.16 for values of theta below 1. Should the data prefer values of theta closer to 1 we would thus expect a substantial share of draws to generate indeterminate solutions and our constraint of limiting the estimation to the region of indeterminacy to be binding.[25] For each model, chains of 60,000 draws were generated and failed to produce a single draw close to $\theta = 1$, suggesting that the truncation in our prior for our baseline results is not binding.

Yet this finding does not rule out the possibility that richer dynamics in the region of indeterminacy – brought about either by changes in the transmission of fundamental shocks or the presence of sunspots – could result in other modes that our methodology has not explored and which might still affect our posterior model odds (see Lubik and Schorfheide, 2004, henceforth LS). So how likely is this in our case? Over the sample that we consider here, LS find no tangible signs of indeterminacy in the United States when

[22]This is further confirmed by looking at kernel estimates of the prior and posterior draws. And even if a mere comparison of prior and posterior probability regions for individual parameters still cannot rule out identification issues in (possibly non-linear) combinations of parameters, our estimation of multiple (as opposed to single) chains initialized from dispersed starting values and the application of convergence diagnostics can help us further rule out general identification issues. For example, identification problems would manifest themselves also through ridges in the likelihood that would hinder the convergence of chains generated with the Random Walk algorithm. We find none of this in our analysis.

[23]For a review of convergence diagnostics for MCMC see Brooks and Roberts (1998). These PRSFs lie below 1.2 for our estimates, indicative of convergence of the multiple chains and providing further evidence against problems with identifications in the parameters.

[24]The proposed densities in our estimation procedure are not truncated and can therefore generate draws in the indeterminate region if this is not penalized by the prior. In fact, any draw that yields an indeterminate solution is not discarded and an indicator variable tracks the number of these draws for all the chains.

[25]As emphasized later, we would not expect this to be the case as the forecast horizon become larger.

they estimate their model closing it with a current-looking Taylor type rule. Clearly, their results on IFB0 are not directly applicable to our IFB1 setting, unless it is possible to show that the dynamic properties of our model are robust with respect to whether an IFB0 or an IFB1 rule is specified. To check for this as well, and also assess the empirical validity of and IFB1 rule, we hence re-estimate our models on the alternative assumption that the IFB rule responds to current rather than to one-period ahead inflation using our benchmark priors. Table B4 shows that 90% posterior probability bands from the IFB0 model encompass the point estimates of our IFB1 baseline for our preferred models G and Z, indicating that all parameters are robust across specifications. These additional findings further support the view that, like in LS, indeterminacy is not an issue in our sample, leading us to believe that imposing this constraint on the priors for θ should not have dramatic implications neither for our empirical results nor, as a consequence, for our design of robust rules based on those estimates.

Certainly, indeterminacy can arise through a number of mechanisms in an economy (see for example Carlstrom and Fuerst, 2000 and Benhabib et al., 2001), and not simply through an ill-designed monetary policy rule. The idea here, however, is to focus on indeterminacy brought about by ill-designed rules and to, thus, develop a method that can help central banks design rules in such a way that determinacy is preserved within a model economy that is otherwise determinate.[26]

We summarize this discussion by observing that although we strongly believe that indeterminacy is not an issue for our purposes, there are important estimation issues concerning the determinacy and indeterminacy of DSGE models that deserve further research.

3. The stability and determinacy of IFB rules

3.1. Theory

This section studies an IFB rule of the form

$$\begin{aligned} i_t &= \rho i_{t-1} + \theta(1-\rho)\mathscr{E}_t \pi_{t+j}, \quad \rho \in [0,1), \ \ \theta > 0 \\ &= i_{t-1} + \Theta \mathscr{E}_t \pi_{t+j}, \quad \rho = 1, \ \ \Theta > 0, \end{aligned} \tag{27}$$

[26]Moreover, even if we had reason to believe that the estimation encompassing the region would be a fruitful exercise, using the LS approach raises a number of problems. First, from the Beyer–Farmer papers, there is an observational equivalence between indeterminate solutions of a given forward-looking model and determinate solutions of a related model with more complex dynamics. This equivalence only holds if the non-fundamental shocks that drive the indeterminate solutions have the same distribution (typically normal) as the fundamental shocks. LS make this strong assumption and then proceed to estimate a model in which the likelihood function is dependent on the possibility of non-fundamental shocks occurring when the model parameters lie within a region of indeterminacy. Second, as LS point out, the observational equivalence issue coupled with the normal distribution assumption for non-fundamental shocks implies that one should estimate a particular model only if one is wedded to it on theoretical grounds. But this is precisely what we are trying to avoid in this paper when we estimate competing models; otherwise our less complex model GH could end up ruling out the more complex model Z on the spurious grounds that the non-fundamental shocks are normally distributed!

where $j \geqslant 0$ is the forecast horizon, which is a feedback on single-period inflation over the period $[t+j-1, t+j]$.[27] With rule (27), policymakers set the nominal interest rate so as to respond to deviations of the inflation term from target. In addition, policymakers smooth rates, in line with the idea that central banks adjust the short-term nominal interest rate only partially towards the long-run inflation target, which is set to zero for simplicity. The parameter $\rho \in [0,1]$ measures the degree of interest rate smoothing. If $\rho = 1$ we have an *integral rule* that guarantees that the long-run inflation target (zero in our set-up) is met, provided the rule stabilizes the economy. For $\rho < 1$, (27) can be written as $\Delta i_t = (1-\rho)/\rho[\theta \mathscr{E}_t \pi_{t+j} - i_t]$ which is a partial adjustment to a static IFB rule $i_t = \theta \mathscr{E}_t \pi_{t+j}$. j is the feedback horizon of the central bank. When $j = 0$, the central bank feeds back from current dated variables only. When $j > 0$, the central bank feeds back instead from deviations of forecasts of variables from target. Finally, $\theta, \Theta > 0$ are the feedback parameters for the non-integral and integral rules, respectively: the larger is θ or Θ, the faster is the pace at which the central bank acts to eliminate the gap between expected inflation and its target value.

Stability and indeterminacy of a dynamic system are associated with the roots of the system's characteristic equation, or equivalently, the eigenvalues of its state space setup. If the number of unstable roots (outside the unit circle) exactly matches the number of non-predetermined variables, there is a unique solution path. Fewer unstable roots than these lead to indeterminacy, while more roots lead to instability.

The mechanism through which indeterminacy arises can be illustrated in the context of the following simplified version of our model. Assume there is no government spending, habit persistence and indexing of prices. Then $g_t = h = \gamma = 0$. Also put $a_t = 0$. Then the supply side of our model, (16) and (17) with $c_t = y_t$ takes the form of a familiar Phillips curve (i) $\pi_t = \mathscr{E}_t \pi_{t+1} + \chi y_t$. On the demand side we replace the Keynes–Ramsey condition (18) with an ad hoc IS curve (ii) $y_t = -\alpha(i_t - \mathscr{E}_t \pi_{t+1})$.

Suppose that the central bank employs a non-integral rule without interest rate smoothing ($\rho = 0$), so that (27) becomes $i_t = \theta \mathrm{E}_t \pi_{t+1}$. Substituting out for y_t and i_t we arrive at the following process for inflation:

$$\mathscr{E}_t(\pi_{t+1}) = \frac{1}{1 - \chi\alpha(\theta - 1)} \pi_t. \tag{28}$$

Consider the case in which private sector expectations are driven by a non-fundamental shock process and anticipate that inflation next period will be equal to 1. This will lead to an increase in real interest rates, with a consequent reduction in demand of $\alpha(\theta - 1)$. Given (28), price-setting behavior will thus imply a current inflation rate of $1 - \chi\alpha(\theta - 1)$, which we define as π_0.

[27] To set the model up with this rule in state-space form for $j \leqslant 1$ we need to augment the state vector with a lagged term i_{t-1}. For $j = 2$ replace t with $t+1$ in (16), (17) and take expectations at time t. Then the state-space presentations remains of the same dimension. For $j > 2$ replace t with $t+j-1$ in (16), (17) and take expectations at time t. The state vector must then be augmented with $\mathscr{E}_t \pi_{t+1} \cdots \mathscr{E}_t \pi_{t+j-2}$.

Now assume that θ is chosen so that $0<\pi_0<1$, which is the case if $1+1/(\chi\alpha)>\theta>1$. If we then lead Eq. (28) forward in time and take expectations, consistency requires that the sequence of successive inflationary expectations is given by $1, 1/\pi_0, 1/\pi_0^2, 1/\pi_0^3, \ldots$. However, these inflation expectations tend to infinity – a solution that clashes with private sector expectations. Thus the unique possible solution is $\pi_t = y_t = i_t = 0$ for all $t>0$. On the other hand, suppose that the central bank is *not aggressive*, and $\theta<1$. The interest rate then does not even react one-for-one to inflation. In this case $\pi_0>1$, and hence the sequence of inflationary expectations tends to zero – a solution that fulfils private sector expectations making these 'self-fulfilling'. Now suppose the central bank is *over-aggressive* such that $\pi_0<-1$. This happens when $\theta>1+2/\chi\alpha$. In this case the economy experiences cycles of positive and negative inflation but again the sequence of inflationary expectations tends to zero and fulfils private sector expectations. Self-fulfilling expectations imply that any initial private sector expectation leads to an acceptable path for inflation – hence indeterminacy. Furthermore, if these (non-fundamental) shocks to private sector expectations follow a stochastic process, then 'sunspot equilibria' are generated. These are typically welfare-reducing because they induce increased volatility in the system.

In Appendix C we derive conditions for θ (for non-integral rules or Θ (for integral rules) that ensure determinacy of the equilibrium in our model. The main results are summarized as[28]:

Result 1. For an *integral rule* feeding back on current inflation ($j=0$), $\Theta>0$ is a necessary and sufficient condition for stability and determinacy. For higher feedback horizons ($j\geqslant 1$), $\Theta>0$ is a necessary but not sufficient condition for stability and determinacy.

Result 2. For *j*-period ahead *integral IFB rules*, $j\geqslant 1$, there exists a range $\Theta \in [0, \bar{\Theta}(j)]$ with $\bar{\Theta}(j)>0$ such that the model is stable and determinate.

Result 3. For a *non-integral rule* feeding back on current inflation ($j=0$), $\theta>1$ is a necessary and sufficient condition for stability and determinacy. For higher feedback horizons ($j\geqslant 1$), $\theta>1$ is a necessary but not sufficient condition for stability and determinacy.

Result 4. For *j*-period ahead *non-integral IFB rules*, $j\geqslant 1$, there exists a some lead J such that for $j>J$ there is indeterminacy *for all values of* θ.

This comparison between integral and non-integral rules shows the benefit of the latter: integral rules can always ensure determinacy as long as the feedback parameter on expected inflation deviations from target is chosen within a specific range; while non-integral rules cannot ensure determinacy if the horizon is too forward, no matter what feedback the central bank chooses on inflation. To get a feel for these results we now provide numerical results for threshold values $\bar{\theta}$ for

[28]Appendix C of a working paper version, Batini et al. (2004b) derives conditions for θ (for non-integral rules) or Θ (for integral rules) that ensure determinacy of the equilibrium in our model.

Table 3
Critical upper bounds for $\bar{\theta}(j)$ and $\bar{\Theta}(j)$ for models (a) G, (b) GH, (c) Z, and (d) H

Threshold	ρ	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
(a)							
$\bar{\theta}(j)$	0.8	222	24	5.5	1.8	1.3	Indeterminacy
$\bar{\Theta}(j)$	1	51	7.4	2.2	1.0	0.62	0.43
(b)							
$\bar{\theta}(j)$	0.8	102	12	3.4	1.7	1.0	Indeterminacy
$\bar{\Theta}(j)$	1	23	3.6	1.2	0.68	0.46	0.34
(c)							
$\bar{\theta}(j)$	0.8	119	19	3.2	1.6	1.0	Indeterminacy
$\bar{\Theta}(j)$	1	23	5.1	1.2	0.57	0.40	0.31
(d)							
$\bar{\theta}(j)$	0.8	55	8.4	2.5	1.5	1.0	Indeterminacy
$\bar{\Theta}(j)$	1	12	2.4	0.8	0.49	0.37	0.29

non-integral rules and $\bar{\Theta}$ for integral rules. Tables 3a–d set parameter values at their median values for models G, GH, Z and H, respectively. For non-integral rules we set $\rho = 0.8$.

These numerical results confirm the predictions of our theory. For each model the indeterminacy problem becomes more acute as the horizon j increases imposing a tighter constraint on the range of IFB rules available. For non-integral rules with $\rho = 0.8$, the maximum horizon J is just over five quarters. In accordance with result 2, for integral rules as j increases there is always some feedback coefficient on expected inflation $0<\Theta<\bar{\Theta}$ such that the IFB rule yields stability and determinacy. For $j \geqslant 1$, model H with inflation but no output persistence is most prone to indeterminacy.[29]

3.2. Weakly robust IFB rules

We are now in a position to identify weakly robust rules; i.e., those IFB rules that guarantee stability and determinacy. Weakly M-robust rules give stability and determinacy across parameter specifications corresponding to median values in models G, GH and Z. Weakly P-robust rules give stability and determinacy for all possible parameter specifications across a large number of draws.

Consider first non-integral rules. Regions to the south-east of each contour corresponding to a choice of IFB horizon j in Fig. 1 show the regions for parameters

[29] Further insight into these results can be provided by writing the expected value of future inflation approximately as $E_t\pi_{t+j} = (\lambda^{\max})^j \pi_t$ where $\lambda^{\max}$ is the largest stable eigenvalue of the system under control. Then as j increases, $(\lambda^{\max})^j$ decreases, so that the feedback effect becomes negligible, and the system exhibits indeterminate similar to the $\theta<1$ type.

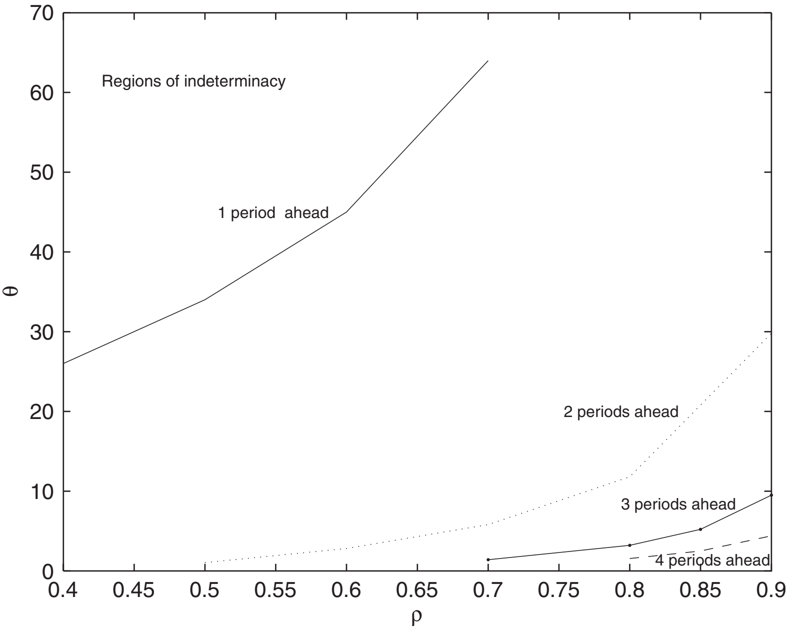

Fig. 1. Non-integral rules: regions of weak M-robustness.

ρ and θ that yield weakly M-robust rules. Fig. 2 is based on 10,000 draws of parameter combinations across all possible models using the estimated posterior parameter distributions of Section 2.5. Regions to the south-west of each contour then represent 100% confidence regions of determinacy for this sample and give rules that are weakly P-robust. For both M-robustness and P-robustness, the declining size of this region as the forward horizon j increases confirms the earlier theoretical results that show that IFB rules with unique and stable equilibria are increasingly constrained in the choice of (ρ, θ) with a qualitative change taking place between $j = 1$ and $j = 2$.

Consider next integral rules. Weakly M-robust rules can be identified from Tables 3a–d by picking out the minimal thresholds $\bar{\Theta}(j)$ across the three models. These are dictated by the most indeterminacy-prone model H. For weakly P-robust we pick out the minimal thresholds based on 10,000 draws of parameter combinations as before. The results in Table 4 confirm that the requirement of M-robustness, and especially P-robustness, increasingly constrain the IFB rule as j increases.

4. Optimal policy and optimized IFB rules without model uncertainty

Without model uncertainty, the policy problem of the central bank at time $t = 1$ is to choose in each period $t = 1, 2, 3, \ldots$ an interest rate i_t so as to minimize a standard expected loss function that depends on the variation of the output gap $o_t = y_t - y_{n,t}$

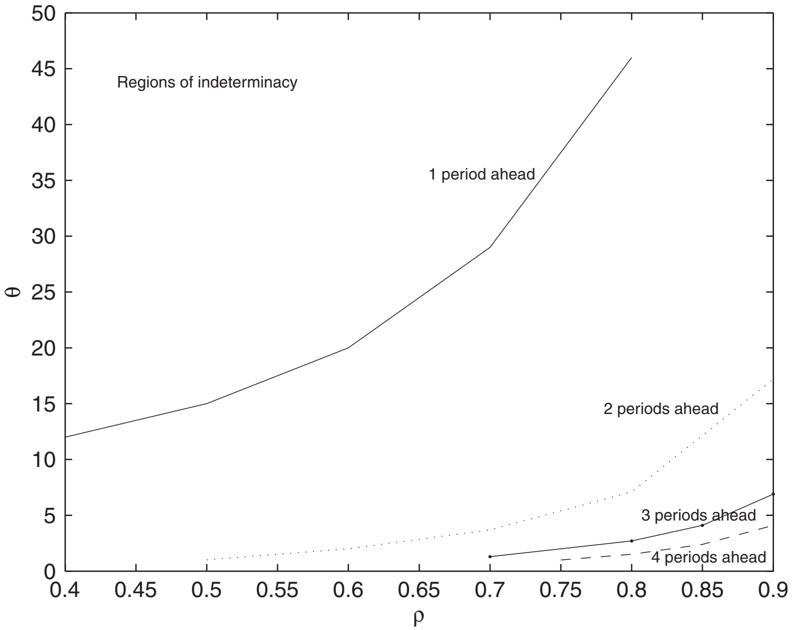

Fig. 2. Non-integral rules: regions of weak P-robustness.

Table 4
Integral rules: critical upper bounds for $\bar{\Theta}(j)$

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
M-robustness: $\bar{\Theta}(j)$	12	2.4	0.8	0.49	0.37	0.29
P-robustness: $\bar{\Theta}(j)$	5.3	2.0	0.66	0.46	0.35	0.27

relative to an flexi-price output target k, inflation and the change in the nominal interest rate[30]:

$$\Omega_0 = \mathscr{E}_0\left[\frac{1}{2}\sum_{t=0}^{\infty}\beta_c^t[(o_t-k)^2+b\pi_t^2+c(i_t-i_{t-1})^2]\right], \tag{29}$$

where β_c is the discount factor of the central bank. The term k is an ambitious target for output that exceeds the natural level of output. It arises because the natural level of output is not efficient (owing to mark-up pricing in a monopolistically competitive intermediate goods sector, market power in the labor market and habit persistence).

[30]Notice this is a central bankers' loss function, not a welfare function. It describes the actual policy objectives banks have (or are instructed to have) rather than what they should have.

4.1. Optimal policy with and without commitment

We first compute the optimal policies where the policymaker can commit, and the optimal discretionary policy where no commitment mechanism is in place.[31] In our linear-quadratic framework optimal policies (including those for optimal IFB rules) conveniently decompose into deterministic and stochastic components. Let target variables in (29) be written as sums of a deterministic stochastic components such as $y_t = \bar{y}_t + \tilde{y}_t$ where all variables are expressed in deviation form about the baseline zero-inflation deterministic steady state of the known model. Then the expected loss function decomposes as

$$\Omega_0 = \frac{1}{2}\sum_{t=0}^{\infty} \beta_c^t[(\bar{o}_t - k)^2 + b\bar{\pi}_t^2 + c(\bar{i}_t - \bar{i}_{t-1})^2 + \mathscr{E}_0[\tilde{o}_t^2 + b\tilde{\pi}_t^2 + c(\tilde{i}_t - \tilde{i}_{t-1})^2]]$$
$$= \bar{\Omega}_0 + \tilde{\Omega}_0 \tag{30}$$

say. The policymaker can then design an optimal policy consisting of an open-loop trajectory that minimizes $\bar{\Omega}_0$ subject to the deterministic model plus a feedback rule that minimizes $\tilde{\Omega}_0$ subject to a stochastic model expressing stochastic deviations about the open-loop trajectory. By the property of *certainty equivalence* for full optimal policies, but *not* optimized simple rules, the feedback rule is independent of both the initial values of the predetermined variables and the variance–covariance matrix of the disturbances.

The optimal policy under commitment provides a benchmark with which to compare the loss in other policy rules. We use the optimal discretionary policy with an ambitious output target $k = 5\%$ to calibrate b to result in an annual inflationary bias (the long-run inflation rate) of 5%. This gave $b = 2.5, 1.5, 0.85, 0.7$ for models G, GH, Z and H, respectively. The discount factor of the central bank is set at $\beta_c = 0.988$ which corresponds to an annual discount rate of 5%. We then set the weight c just sufficiently high to avoid a negative interest rate anywhere along the trajectory of the optimal policies. This requires $c_G = c_{GH} = 3$, $c_Z = 2$ and $c_H = 1$.[32]

4.2. Optimized IFB rules

We now turn to optimized IFB rules and optimal Taylor-type rules feeding back on either current inflation alone or on inflation and the output gap. The general form of the rule that covers integral and non-integral IFB as well as the Taylor-type rules is given by

$$i_t = \rho i_{t-1} + \Theta\mathscr{E}_t\pi_{t+j} + \Theta_y(y_t - y_{n,t}), \quad \rho \in [0, 1], \;\; \Theta, \Theta_y > 0, \;\; j \geqslant 0. \tag{31}$$

In all the results from this point onwards we focus exclusively on stabilization policy by putting $k = 0$ so there is no deterministic component of policy in response

[31] Full details of the procedures used to compute optimal policies and optimized IFB rules are available from the authors and included in the Conference version of Batini et al. (2004b).

[32] A working paper version of the paper Batini et al. (2004b) provides simulations to show that these calibration requirements are met for b_i and c_i; $i = G, GH, Z, H$.

Table 5
Model *G*: optimal rules and optimized simple rules compared[a]

Rule	ρ	Θ	Θ_y	Loss function	y_e	π_e
Minimal feedback on π_t	1	0.001	0	39.1	0.62	0.39
IFB0	1	1.43	0	2.27	0.11	0.07
Taylor rule	1	0.81	1.00	1.86	0.09	0.06
IFB1	1	3.99	0	2.63	0.12	0.08
IFB2	1	5.00	0	3.19	0.15	0.09
IFB3	1	2.17	0	9.78	0.30	0.19
IFB4	1	1.02	0	44.0	0.66	0.41
Optimal commitment	n.a.	n.a.	n.a	1.07	0	0
Optimal discretion	n.a.	n.a.	n.a	2.98	0.14	0.08

[a]IFBj denotes a *j*-period ahead IFB rule. y_e and π_e (given by (32) and (33)) are the output and per period (quarterly) inflation equivalent losses, respectively, relative to optimal commitment. Optimized simple rules were restricted to the ranges $\rho \in [0, 1]$ and $\Theta \in [1, 5]$.

to an ambitious output target.[33] Given the estimated variance–covariance matrix of the white noise disturbances, an optimal combination (Θ, ρ) can be found for each rule defined by the time horizon $j \geqslant 0$, and for the Taylor rule, and optimal combination (Θ, Θ_y, ρ). The results are shown in Tables 5–8 for the estimated models *G*, *GH* and *Z* of Section 2.5. The Taylor rule is for $j = 0$ only. In these tables the output and inflation equivalent welfare differences, y_e and π_e, respectively, compared with the optimal commitment policy are computed as follows. Suppose the welfare loss difference is X. This is equivalent to a permanent output gap of y_e if $(1/2(1-\beta))y_e^2 = X$, and to a permanent inflationary bias of π_e if $(1/2(1-\beta))b\pi_e^2 = X$; i.e.,

$$y_e = \sqrt{2(1-\beta)X}, \tag{32}$$

$$\pi_e = \sqrt{\frac{2(1-\beta)X}{b}}. \tag{33}$$

A number of interesting observations emerge from these tables. First, from the output equivalent loss (relative to the optimal commitment outcome) of 'minimal control', the closest saddle-path stable rule using current inflation to no feedback rule at all, we see that there are significant though not dramatic gains from stabilization of between 0.45–0.62% output equivalent and 0.37–0.53% quarterly inflation equivalent across the three models. Second, the output equivalent loss from optimal discretion indicate only small stabilization gains from commitment if the latter policy rule is implementable. By far the main gain from commitment is the elimination of the inflationary bias which has been ruled out in these results by putting $k = 0$. Third, if the policymaker can commit using a simple rule, the best one in this respect is a Taylor integral rule, and this realizes

[33]Since the IFB rule assumes a commitment mechanism, the policymaker in principle should be able to implement a policy $i_t = \bar{i}_t$ plus a feedback component such as (27) or (31) relative to $\bar{i}_t$, where the latter is the optimal deterministic trajectory found in the previous section.

Table 6
Model *GH*: optimal rules and optimized simple rules compared

Rule	ρ	Θ	Θ_y	Loss function	y_e	π_e
Minimal control π_t	1	0.001	0	30.3	0.54	0.44
IFB0	1	1.40	0	1.44	0.09	0.07
Taylor rule	1	0.77	1.00	1.37	0.09	0.07
IFB1	1	5.00	0	1.61	0.10	0.08
IFB2	1	3.59	0	2.70	0.14	0.12
IFB3	1	1.23	0	21.3	0.45	0.37
IFB4	1	0.66	0	147	1.21	0.99
Optimal commitment	n.a.	n.a.	n.a	0.64	0	0
Optimal discretion	n.a.	n.a.	n.a	1.78	0.20	0.09

Table 7
Model *Z*: optimal rules and optimized simple rules compared

Rule	ρ	Θ	Θ_y	Loss function	y_e	π_e
Minimal feedback on π_t	1	0.001	0	22.45	0.47	0.51
IFB0	1	1.25	0	0.88	0.07	0.07
Taylor rule	1	1.25	0.11	0.88	0.07	0.07
IFB1	1	2.96	0	1.16	0.08	0.09
IFB2	1	5.0	0	1.45	0.10	0.11
IFB3	1	1.17	0	28.03	0.53	0.57
IFB4	0.87	0.40	0	2314	4.81	5.22
Optimal commitment	n.a.	n.a.	n.a	0.45	0	0
Optimal discretion	n.a.	n.a.	n.a	0.93	0.07	0.08

a large part of the potential stabilization gain. Third, for each model we search for optimized rules within those that satisfy the determinacy conditions on ρ and θ for non-integral rules and on Θ for integral rules. Our theory has shown that this requirement severely constrains the range of possible stabilizing rules as the horizon j increases and as a result compared with the Taylor rule, IFBj rules *perform increasingly less well*. In our results the transition from IFB3 to IFB4 is particularly dramatic involving an output equivalent loss of between 0.66% (for model *G*) and 4.85% (model *Z*). This is what our theory leads us to expect from Table 3 since the determinacy requirement imposes the tightest constraint on model *Z*.

5. Robust IFB rules with model uncertainty

5.1. Theory

In this section we consider model uncertainty in the form of uncertain estimates of the non-policy parameters of the model, $\Theta = (\beta, \gamma, \xi, \phi, \sigma, h, \rho_a, \rho_b, \zeta, \eta, \kappa, \sigma^2_{at}, \sigma^2_{gt})$.

Table 8
Model H: optimal rules and optimized simple rules compared

Rule	ρ	Θ	Θ_y	Loss function	y_e	π_e
Minimal feedback on π_t	1	0.001	0	20.16	0.45	0.53
IFB0	1	2.16	0	0.41	0.04	0.05
Taylor rule	1	2.18	0.02	0.40	0.04	0.05
IFB1	1	5.0	0	0.50	0.05	0.06
IFB2	1	2.64	0	2.67	0.16	0.19
IFB3	1	0.78	0	1003	3.17	3.78
IFB4	1	0.47	0	633	2.52	3.01
Optimal commitment	n.a.	n.a.	n.a	0.23	0	0
Optimal discretion	n.a.	n.a.	n.a	0.49	0.05	0.06

Suppose the state of the world s is described by a model with $\Theta = \Theta^s$ expressed in state-space form as

$$\begin{bmatrix} \mathsf{z}^s_{t+1} \\ \mathscr{E}_t \mathsf{x}^s_{t+1} \end{bmatrix} = A^s \begin{bmatrix} \mathsf{z}^s_t \\ \mathsf{x}^s_t \end{bmatrix} + B^s i^s_t + C^s \begin{bmatrix} \varepsilon_{gt+1} \\ \varepsilon_{at+1} \end{bmatrix}, \tag{34}$$

$$o^s_i = E^s \begin{bmatrix} \mathsf{z}^s_t \\ \mathsf{x}^s_t \end{bmatrix}, \tag{35}$$

where $\mathsf{z}^s_t = [a^s_t, g^s_t, c^s_{t-1}, c^s_{n,t-1}, \pi^s_{t-1}]$ is a vector of predetermined variables at time t and $\mathsf{x}_t = [c^s_t, \pi^s_t]$ are non-predetermined variables in state s of the world. In (34) and (35) it is important to stress that variables are in deviation form about a zero-inflation steady state of the model in state s. For example output in deviation form is given by $y^s_t = (Y^s_t - \bar{Y}^s)/\bar{Y}_s$ where $\bar{Y}^s$ is the steady state of the model in state s defined by parameters Θ^s and $i^s_t = i_t - \bar{i}^s$ where the natural rate of interest in model s, $\bar{i}_s = 1/\beta^s - 1$.

For M-robustness, in general one sets up a *composite model* of outputs from each of the states $s = 1, 2, \ldots, n$ corresponding to the rival models and minimizes the expected loss across these states using estimated posterior probabilities. Because each model is linearized about a different steady state, we must now set up the model in state s in terms of the *actual* interest rate, not the deviation about the steady state. Then augmenting the state vector to become $\mathsf{z}^s_t = [1, a^s_t, g^s_t, c^s_{t-1}, c^s_{n,t-1}, \pi^s_{t-1}]$ we still have a state-space form (34) and (35) and we minimize

$$\begin{aligned} \Omega_0 = \frac{1}{2} \sum_{t=0}^{\infty} \beta^t_c \sum_{s=1}^{n} p_s[(\bar{o}^s_t - k)^2 + b_s(\bar{\pi}^s_t)^2 + c_s(\bar{i}_t - \bar{i}_{t-1})^2 \\ + \mathscr{E}_0[(\tilde{o}^s_t)^2 + b_s(\tilde{\pi}^s_t)^2 + c_s(\tilde{i}_t - \tilde{i}_{t-1})^2]]. \end{aligned} \tag{36}$$

For P-robustness (36) is replaced with the average expected utility loss across a large number of draws from all models constructed using both the posterior model probabilities and the posterior parameter distributions for each model.

In (36) the output target in state s of the world is given by $o_t^s = y_{nt}^s + k^s$ where the ambitious output target k^s depends on s. In fact we will continue to assume that the central bank has no ambitious output targets and set $k^s = 0$ in its loss function. However, with model uncertainty there is still a deterministic component of policy arising from differences in the natural rate of interest compatible with zero inflation in the steady state, $\bar{i}^s = 1/\beta^s - 1$.[34] A non-integral rule specifying $i_t = \bar{i}^s$ in the long-run will only result in zero inflation in model s. From the consumers' Euler equation (4) in model r with $\beta^r > \beta^s$, implementing the rule designed for model s with $\bar{i} = \bar{i}^s = 1/\beta^s - 1$ gives a steady state inflation rate $\bar{\pi}^r$ that is no longer zero but given by

$$\frac{\beta^r(1+\bar{i}^s)}{(1+\bar{\pi}^r)} = \frac{\beta^r}{\beta^s(1+\bar{\pi}^r)} = 1 \quad \text{i.e., } \bar{\pi}^r = \frac{\beta^r}{\beta^s} - 1 > 0. \tag{37}$$

Our robust non-integral rule designed for any model specifies a natural zero-inflation rate of interest $\bar{i}_R$, corresponding to a discount factor $\beta_R = 1/(1+\bar{i}_R)$ to result in an expected long-run inflation rate across models of zero. This implies β_R is determined by

$$\sum_{s=1}^{n} p_s \left[\frac{\beta_s}{\beta_R} - 1\right] \;\Rightarrow\; \beta_R = \sum_{s=1}^{n} p_s \beta_s. \tag{38}$$

That is, β_R is the expected value of β_s across the model variants. The need to specify a natural rate of interest, $\bar{i}_R$, only applies to non-integral rules. By contrast, a further benefit of integral rules is that the economy is automatically driven to a zero-inflation steady state whatever the state of the world without having to specify $\bar{i}_R$.

There is one final consideration first raised by Levine (1986) that is usually ignored in the literature. Up to now we have assumed that private sector expectations $\mathscr{E}_t \mathbf{x}_{t+1}^s$ are state s model-consistent expectations. In other words in each state of the world the private sector knows the state and faces no model uncertainty. In a more general formulation of the problem we can relax this assumption and assume that both the policymaker and the private sector faces model uncertainty. Suppose that in state s of the world the latter believes model s' with probability $q_{ss'}$. Then $\mathscr{E}_t \mathbf{x}_{t+1}^s$ must be replaced by the composite expectation $\sum_{s'=1}^{n} q_{ss'} \mathscr{E}_t \mathbf{x}_{t+1}^{s'}$ and the composite model no longer decomposes into independent systems. In the results that follow we bypass this complication and confine ourselves to model-consistent expectations in each state of the world.

5.2. Strongly robust IFB rules

We now present the strongly M-robust and P-robust IFB rules with horizon $j = 0, \ldots, 4$. Table 9 reports these rules, which turn out to be of the integral type in almost all cases, as our theory leads us to expect. An interesting feature of these results is that our more stringent criterion of P- rather than M-robustness has the consequence that the optimized rule responds *less* aggressively to expected inflation, and increasingly so as the horizon j increases, a result for monetary policy in an uncertain environment that goes back to Brainard (1967).

[34] In fact estimated differences in β^s between models $s = G, GH, Z$ are not great, so the point we make here is only potentially important.

Table 9
Strongly robust IFB rules

Rule	ρ	Θ
M-robust IFB0	1	1.42
P-robust IFB0	1	1.37
M-robust IFB1	1	3.953
P-robust IFB1	1	3.53
M-robust IFB2	1	2.37
P-robust IFB2	1	1.85
M-robust IFB3	1	0.79
P-robust IFB3	1	0.20
M-robust IFB4	0.92	0.49
P-robust IFB4	1	0.26

The diagonal elements of Table 10 gives the policymaker's losses obtained previously in Tables 5–8 when the optimized rule designed for model $s = G, GH, Z, H$ is implemented in that model. Figures in brackets refer to output equivalent % losses relative to the optimal commitment policy. We refer to these rules as IFBj(s) for horizon j. The off-diagonal entries show the loss outcome when the rule designed for model s is implemented on model $r \neq s$. Two striking results emerge from this table: first, in contrast with much previous literature (e.g. Levin et al., 2001) current-looking policy rules are always superior to rules that feed off future variables. Second, whereas the current inflation rule IFB0 and the IFB1 rule are remarkably robust across models, this is no longer true for IFBj for $j \geqslant 2$. IFB0 and IFB1 rules designed for the wrong model perform well in terms of their stabilization properties and the requirements of M-robustness or even P-robustness have little impact on policy design.

For IFBj rules with $j \geqslant 2$, optimized rules designed for the wrong model can lead to indeterminacy. M-robust and P-robust rules avoid this indeterminacy by design.[35] However, robustness comes at a cost especially as the horizon j goes beyond $j = 2$. For $j = 3$, M-robustness imposes output equivalent costs of as much as 3.2% which happens if the world turns out to be correctly represented by model H. For $j = 4$ this robustness cost rises to 6.2% if the true state of the world is model Z. The corresponding worst-case scenarios for the most stringent robustness criterion, P-robustness are output costs of 6.7% and 6.6% for $j = 3, 4$, respectively.

6. Conclusions

Both our theoretical results on IFB rules in Section 3 and our numerical results of that and later sections indicate that they become increasingly prone to the problem

[35] In our computations a very large loss utility loss is assigned to rules that lead to instability or indeterminacy.

Table 10
Value of loss function for different rules with model uncertainty[a]

Rule	Model G	Model GH	Model Z	Model H
IFB0 (G)	2.27 (0.11)	1.44 (0.09)	0.88 (0.07)	0.43 (0.04)
IFB0 (GH)	2.27 (0.11)	1.44 (0.09)	0.88 (0.07)	0.43 (0.04)
IFB0 (Z)	2.29 (0.11)	1.45 (0.09)	0.88 (0.07)	0.44 (0.05)
IFB0 (H)	2.39 (0.11)	1.50 (0.09)	0.94 (0.07)	0.41 (0.04)
IFB0 (M-robust)	2.27 (0.11)	1.44 (0.09)	0.88 (0.07)	0.43 (0.04)
IFB1 (P-robust)	2.24 (0.11)	1.44 (0.09)	0.88 (0.07)	0.43 (0.04)
IFB1 (G)	2.63 (0.12)	1.63 (0.10)	1.18 (0.08)	0.53 (0.05)
IFB1 (GH)	2.66 (0.13)	1.61 (0.10)	1.18 (0.08)	0.50 (0.05)
IFB1 (Z)	2.69 (0.13)	1.72 (0.10)	1.16 (0.08)	0.58 (0.06)
IFB1 (H)	2.66 (0.13)	1.61 (0.10)	1.21 (0.09)	0.50 (0.05)
IFB1 (M-robust)	2.63 (0.12)	1.63 (0.10)	1.18 (0.08)	0.53 (0.05)
IFB1 (P-robust)	2.64 (0.13)	1.66 (0.10)	1.17 (0.08)	0.54 (0.06)
IFB2 (G)	3.19 (0.15)	Indeterminacy	1.45 (0.10)	Indeterminacy
IFB2 (GH)	3.66 (0.16)	2.70 (0.14)	1.61 (0.11)	Indeterminacy
IFB2 (Z)	3.19 (0.15)	Indeterminacy	1.45 (0.10)	Indeterminacy
IFB2 (H)	4.38 (0.18)	3.41 (0.17)	1.88 (0.12)	2.67 (0.16)
IFB2 (M-robust)	4.17 (0.18)	3.75 (0.18)	2.02 (0.13)	2.74 (0.16)
IFB2 (P-robust)	5.70 (0.22)	4.76 (0.20)	2.48 (0.14)	4.00 (0.19)
IFB3 (G)	9.78 (0.30)	Indeterminacy	Indeterminacy	Indeterminacy
IFB3 (GH)	17.3 (0.40)	21.3 (0.45)	Indeterminacy	Indeterminacy
IFB3 (Z)	18.2 (0.41)	22.8 (0.47)	28.0 (0.52)	Indeterminacy
IFB3 (H)	28.1 (0.52)	38.7 (0.62)	66.7 (0.81)	1003 (3.17)
IFB3 (M-robust)	27.7 (0.52)	38.1 (0.61)	64.9 (0.80)	1005 (3.17)
IFB3 (P-robust)	114 (1.06)	207 (1.44)	912 (3.02)	4437 (6.66)
IFB4 (G)	44.0 (0.66)	Indeterminacy	Indeterminacy	Indeterminacy
IFB4 (GH)	72.3 (0.84)	147 (1.21)	Indeterminacy	Indeterminacy
IFB4 (Z)	190 (1.37)	424 (2.06)	2246 (4.74)	Indeterminacy
IFB4 (H)	106 (1.02)	231 (1.52)	3840 (6.20)	633 (2.52)
IFB4 (M-robust)	101 (1.00)	219 (1.48)	3858 (6.21)	635 (2.52)
IFB4 (P-robust)	199 (1.40)	482 (2.19)	4383 (6.62)	1583 (3.98)

[a]IFBj(s) denotes the outcome from the j-horizon IFB rule designed for model s. Each row then gives the value of the loss function for models $s = G, GH, Z, H$. M-robust rules use the posterior model probabilities $p_G = 0.56$, $p_{GH} = 0.03$, $p_Z = 0.32$ and $p_H = 0.09$. The % output equivalent losses are in brackets. Diagonal elements correspond to losses in Tables 5–8.

of indeterminacy as the forward horizon increases from $j = 2$ to $j = 4$. This seems to be in line with results found in the literature that monetary policy inertia is 'good' for ensuring determinacy both in a certain and an uncertain environment (e.g. Batini and Pearlman, 2002; Woodford, 1999). As a consequence the stabilization performance of optimized rules of this type worsens too. M-robust and P-robust rules avoid indeterminacy in an uncertain environment, but at an increasing utility loss as rules become more forward-looking. Another striking result in contrast with much previous literature is that, at least in our analytical set-up, current-looking policy rules are always superior to rules that feed off future variables.

In view of these results the question arises: why do central banks pursue forward-looking targeting rules in the first place? Two main reasons for doing so are commonly cited. First, the delayed response of inflation to interest rate changes obliges monetary authorities to react in a pre-emptive fashion to expected inflation in the future. Second, by targeting inflation in the future in a simple and accountable fashion, the central bank can respond to shocks whilst at the same time providing the private sector with assurances that inflation will eventually return to its long-run target of zero inflation.

Of these two reasons only the second makes any sense in terms of our analysis. Central banks can only target *forecasts* of future inflation and these can only be conditional on information available at the time the interest rate is set, i.e., the state vector at time t in (24). By committing to a rule that feeds back on inflation $j \geqslant 1$ periods ahead, since this forecast can be expressed as a linear combination of these state variables, the authority is severely constraining how the interest rate should in effect respond to this information, and it is this constraint that lies at the heart of the poor performance of these rules. It may well be the case that a forward-looking inflation-targeting rule with a long horizon *does* help to establish a reputation for low long-run inflation, but the formalization of such a commitment mechanism remains a challenge.

Appendix A. Details of Bayesian estimation methods

Each model indexed by k and denoted m_k, has an associated set of unknown parameters $\omega_k \in \Omega_k$. Following a Bayesian approach, our aim is to characterize the posterior distribution of the models' parameters, $p(\omega_k | Y^T, m_k)$, where Y^T stands for the full sample of observed data (T denotes the number of observations). Having specified a (perhaps model specific) prior density, $p(\omega_k|m_k)$, the posterior of the parameters is given by

$$p(\omega_k|m_k) = \frac{\mathscr{L}(Y^T|\omega_k, m_k)p(\omega_k|m_k)}{\int \mathscr{L}(Y^T|\omega_k, m_k)p(\omega_k|m_k)\,\mathrm{d}\omega_k}, \tag{A.1}$$

where $\mathscr{L}(Y^T|\omega_k, m_k)$ is the likelihood obtained under the assumption of normally distributed disturbances from the state-space representation implied by the solution of the linear rational-expectations model. The denominator in Eq. (A.1) corresponds to the marginal likelihood (also known as the 'marginal data density') and, as explained later, plays a key role in model comparisons.

The solution of the model is a non-linear function of the parameters which does not allow for any closed-form expression for the posterior density. Furthermore, the high dimension of the parameters space renders numerical integration inefficient. Markov Chain Monte Carlo (MCMC) methods, however, provide a feasible and accurate approximation to this density.

Following Schorfheide (2000) the estimation follows a two step approach. In the first step, a numerical algorithm is used to approximate the posterior mode by combining the likelihood $\mathscr{L}(Y^T|\omega_k, m_k)$ with the prior. In the second step, the obtained posterior mode is then used as starting value (ω_k^0) for a Random Walk Metropolis algorithm that generates draws from the posterior $p(\omega_k|Y^T, m_k)$. At each step i of the Markov Chain, the proposal density used to draw a new candidate parameter ω_k^* is a normal centered at the current state of the chain, $\mathrm{N}(\omega_k^i, c\Sigma_k)$. A new draw is then accepted with probability

$$\alpha = \min\left(1, \frac{\mathscr{L}(Y^T|\omega_k^*, m_k)p(\omega_k^*|m_k)}{\mathscr{L}(Y^T|\omega_k^i, m_k)p(\omega_k^i|m_k)}\right).$$

If accepted, $\omega_k^{i+1} = \omega_k^*$; otherwise, $\omega_k^{i+1} = \omega_k^i$. This may be viewed as a stochastic climbing algorithm. Whenever a new draw results in higher posterior probability than the current state of the chain, the draw is retained. Otherwise, there is probability (α) that one will jump to a point of lower posterior density. We generate chains of 130,000 draws in this manner discarding the first 30,000 iterations.[36]

Point estimates of the parameters ω_k can be obtained from the generated values by using various location measures, such as mean or, as in this paper, medians. Similarly, measures of uncertainty follow from computing the percentiles of the draws.

Appendix B. Estimation results

Estimation results are given in Tables B1–B4.

Appendix C. Stability analysis by the root locus method

To understand better how the precise combination of the pairs (j, θ) or (j, Θ) in IFB rules can lead the economy into instability or indeterminacy consider the deterministic model economy (24) and (25) with interest rate rules of form (27). g_t and a_t are exogenous stable processes and play no part in the stability analysis. For convenience, we therefore set them to zero. Let z be the forward operator. Taking z-transforms of (16), (17), (18) and (27), the characteristic equation for the system is given by

$$(z-\rho)\left[(z-1)(z-h)(\beta z-1)(z-\gamma) - \frac{\lambda}{\mu} z^2(\tilde{\phi} z + \mu(z-h))\right] + \frac{\lambda\theta}{\mu}(1-\rho)(\tilde{\phi} z + \mu(z-h))z^{j+2} = 0 \tag{C.1}$$

[36]This initial burn-in phase is intended to remove any dependence of the chain from its starting values.

Table B1
Priors for baseline model

	Distribution	Mean	Standard deviation	Percentiles	
				1%	99%
ρ_i	B	0.75	0.15	0.538	0.981
θ	G	1.7	0.5	1.099	3.074
X	G	0.15	0.1	0.044	0.473
φ	G	1.75	0.5	1.148	3.118
σ	G	1.5	0.8	0.609	3.952
γ	B	0.7	0.1	0.566	0.897
h	B	0.7	0.1	0.566	0.897
ρ_a	B	0.7	0.15	0.492	0.959
ρ_g	B	0.7	0.15	0.492	0.959
π^*	G	4	2	1.745	10.045
r^*	G	2	1	0.872	5.023
sd_g	IG1	1.7	inf	0.635	9.260
sd_e	IG1	1	inf	0.372	5.699
sd_a	IG1	1.7	inf	0.635	9.260

For all models, g is calibrated to 0.22. Distributions: G (Gamma), B (Beta), and), IG1 (Inverse Gamma-1). ρ corresponds to the autoregressive coefficient of an AR(1) process. *sd* denotes the standard deviation of the shocks. Last two columns, report the inverse cumulative distribution function of each prior ordinate for the 0.01 and 0.99 percentiles.

Table B2
Parameter estimates for $j = 1$

Model	Model G ($\gamma = 0$)		Model H ($h = 0$)		Model GH ($\gamma = h = 0$)		Model Z	
	Posterior distribution		Posterior distribution		Posterior distribution		Posterior distribution	
Coefficient	Median	10% 90%	Median	10% 90%	Median	10% 90%	Median	10% 90%
ρ_i	0.80	[0.73, 0.85]	0.67	[0.55, 0.75]	0.72	[0.64, 0.79]	0.77	[0.71, 0.83]
θ	2.55	[2.15, 2.99]	2.68	[2.18, 3.29]	2.64	[2.14, 3.20]	2.25	[1.86, 2.58]
X	0.16	[0.08, 0.30]	0.71	[0.49, 0.98]	0.27	[0.14, 0.44]	0.47	[0.25, 0.74]
φ	1.46	[1.06, 1.94]	2.40	[1.70, 3.15]	2.16	[1.47, 2.77]	1.32	[0.99, 1.81]
σ	3.29	[2.28, 4.98]	2.50	[1.47, 4.29]	3.91	[2.86, 5.76]	3.23	[2.55, 4.21]
γ			0.59	[0.45, 0.72]			0.54	[0.40, 0.68]
h	0.85	[0.74, 0.91]					0.85	[0.73, 0.92]
ρ_a	0.91	[0.85, 0.94]	0.91	[0.87, 0.94]	0.90	[0.85, 0.93]	0.94	[0.89, 0.96]
ρ_g	0.92	[0.88, 0.95]	0.91	[0.88, 0.93]	0.90	[0.87, 0.93]	0.93	[0.89, 0.96]
π^*	3.00	[2.43, 3.60]	2.72	[2.14, 3.32]	2.96	[2.40, 3.51]	2.88	[1.75, 3.55]
r^*	1.86	[1.38, 2.49]	1.82	[1.26, 2.48]	1.90	[1.33, 2.46]	1.80	[1.25, 2.38]
sd_g	2.19	[1.99, 2.39]	3.23	[2.53, 4.17]	2.75	[2.35, 3.11]	2.23	[2.05, 2.47]
sd_e	0.17	[0.15, 0.19]	0.20	[0.18, 0.23]	0.17	[0.16, 0.19]	0.19	[0.17, 0.21]
sd_a	0.78	[0.61, 1.01]	0.51	[0.42, 0.64]	0.59	[0.48, 0.74]	0.72	[0.57, 0.93]

Median and posterior deciles of the draws generated with a Random Walk Metropolis algorithm. We discarded the first 30,000 draws and retained the remaining 100,000 values.

Table B3
Model comparisons

Reversible Jump MCMC	
Model	Posterior odds, $P(m\|\text{data})$
G ($y = 0$)	0.56
H ($h = 0$)	0.09
GH ($y = h = 0$)	0.03
Z	0.32

Reversible MCMC of Dellaportas et al. (2002). 100,000 draws to obtain the proposal densities. For the Metropolis step, we discarded the first 20,000 values and retained the remaining 180,000 draws. Posterior odds $P(m|\text{data})$ based on assigning each model equal prior probability. Model proposal density assigns equal probability to the jump to any of four possible models, regardless of the current state of the chain.

Table B4
Parameter estimates for $j = 0$

Model	Model G ($y = 0$)		Model Z	
	Posterior distribution		Posterior distribution	
Coefficient	Median	10% 90%	Median	10% 90%
ρ_i	0.83	[0.79, 0.86]	0.84	[0.80, 0.87]
θ	2.21	[1.84, 2.66]	2.23	[1.85, 2.67]
X	0.28	[0.14, 0.51]	0.45	[0.25, 0.74]
φ	1.46	[1.08, 1.95]	1.54	[1.15, 2.06]
σ	3.25	[2.15, 4.71]	3.24	[2.13, 4.77]
γ			0.51	[0.37, 0.66]
h	0.83	[0.71, 0.91]	0.83	[0.69, 0.90]
ρ_a	0.91	[0.85, 0.95]	0.89	[0.82, 0.94]
ρ_g	0.92	[0.87, 0.95]	0.91	[0.86, 0.95]
π^*	2.95	[2.40, 3.42]	3.03	[2.51, 3.51]
r^*	1.83	[1.26, 2.40]	1.90	[1.30, 2.45]
sd_g	2.26	[2.03, 2.54]	2.26	[2.04, 2.53]
sd_e	0.18	[0.16, 0.21]	0.18	[0.16, 0.20]
sd_a	0.72	[0.58, 0.92]	0.72	[0.58, 0.94]

Median and posterior deciles of the draws generated with a Random Walk Metropolis algorithm, using the same priors as when $j = 1$. We discarded the first 30,000 draws and retained the remaining 100,000 values.

for non-integral rules and

$$(z-1)\left[(z-1)(z-h)(\beta z-1)(z-\gamma)-\frac{\lambda}{\mu}z^2(\tilde{\phi}z+\mu(z-h))\right]$$
$$+\frac{\lambda\Theta}{\mu}(\tilde{\phi}z+\mu(z-h))z^{j+2}=0 \tag{C.2}$$

for integral rules. In these characteristic equations we have defined $\lambda \equiv (1-\beta\xi)(1-\xi)/\xi$, $\tilde{\phi} \equiv (\bar{C}/\bar{Y})\phi$ and $\mu \equiv \sigma/(1-h)$. Eqs. (C.1) and (C.2) show that the minimal state-space form of the system has dimension $\max(5, j+3)$. Since there are three predetermined variables in the system, it follows that the saddle-path condition for a unique stable rational expectations solution is that the number of roots inside the unit circle of the complex plane is 3 and the number outside the unit circle is $\max(2, j)$.

In the analysis that follows we focus on integral rules with characteristic equation (C.2).[37] To identify values of (j, Θ) that involve exactly three roots of equation (C.2) we graph the root locus of (Θ, z) pairs that traces how the roots change as Θ varies between 0 and ∞.[38] Other parameters in the system, including the feedback horizon parameter j in the IFB rule, are kept constant. We generate separate charts, each conditioning on a different horizon assumption. Each chart shows the complex plane (indicated by the solid thin line),[39] the unit circle (indicated by the dashed line), and the root locus tracking zeroes of Eq. (C.2) as Θ varies between 0 and ∞ (indicated by the solid bold line). The arrows indicate the direction of the arms of the root locus as Θ increases. Throughout we experiment with both a 'high' and a 'low' λ/μ, as defined after (C.1). The economic interpretation of these cases is that the high λ/μ case corresponds to low ξ (i.e., more flexible prices) and low $\sigma/(1-h)$ (low risk aversion and habit formation).

If the nominal interest rate rule and feeds back on current rather than expected inflation, i.e. $j=0$, then the root locus technique yields a pattern of zeroes as depicted in Fig. 3. Integral control brings about a lag in the short-term nominal interest rate and the system is now stable if it has exactly *three* stable roots (as we now have three predetermined variables in the system). The figure demonstrates that if $\Theta > 0$ one arm of the root locus starting originally at $z=1$ exits the unit circle, turning one root from unity to unstable so that there are now three – as required – instead of four stable roots and the system has a determinate equilibrium. As $\Theta \to \infty$, there are roots at $\pm i\infty$, two roots at 0, and one at $\mu h/(\tilde{\phi}+\mu)$, the latter shown as a square.

Thus, we conclude that for a rule feeding back on current inflation, the system exhibits determinacy if and only if $\Theta > 0$. For higher values of $j \geqslant 1$ we can draw the sequence of root locus diagrams shown in Figs. 4 and 5. Our diagrams show that an arm of the root locus re-enters the unit circle for some high $\Theta > 1$ and indeterminacy re-emerges. Therefore, $\Theta > 0$ is necessary but not sufficient for stability and determinacy. Our results up to this point are summarized in Result 1 of the main text.

Now let $\bar{\Theta}(j)$ be the upper critical value of Θ for the system for a feedback horizon j. Fig. 4 shows that for the case $j=1$ indeterminacy occurs when this portion of the root locus enters the unit circle at $z=-1$. Proceeding on to j-period ahead IFB rules, for $j \geqslant 2$ the analysis is more difficult. For $j=2$, Fig. 5 shows that indeterminacy

[37] The corresponding analysis for non-integral rules is to be found in BLP.

[38] All the graphs can be drawn by following standard rules found in the control systems literature (see, for example, Evans, 1954 and a 'users guide' in Appendix A of BLP).

[39] In this plane, the horizontal axis depicts real numbers, and the vertical axis depicts imaginary numbers. If a root is complex, i.e. $z = x + iy$, then its complex conjugate $x - iy$ is also a root. Thus the root locus is symmetric about the real axis.

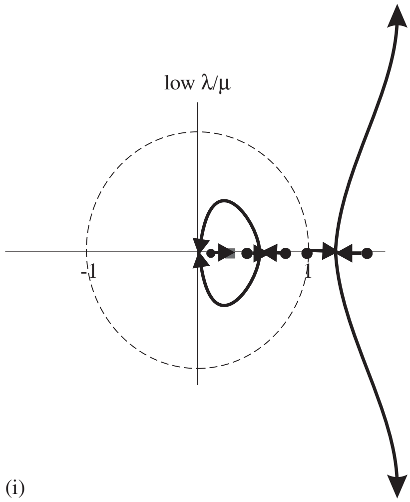

Fig. 3. Integral control IFB rule on current inflation: position of zeroes as Θ changes.

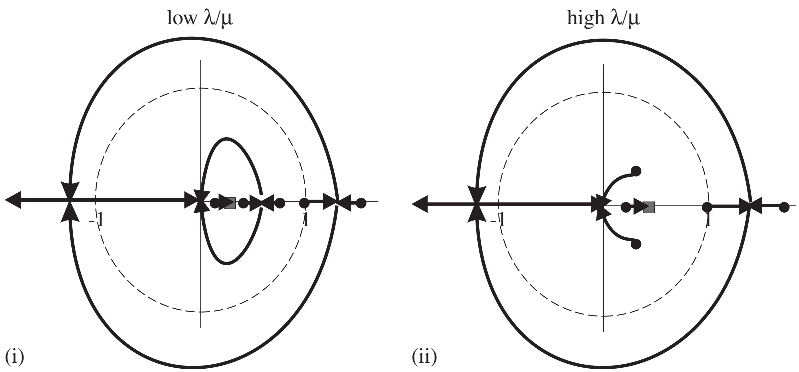

Fig. 4. Integral control IFB rule on one-period ahead expected inflation: position of zeroes as Θ changes.

occurs when the root locus enters the unit circle at $z = \cos(\psi) + \mathrm{i}\,\sin(\psi)$ for some $\psi \in (0, \pi/2)$. A similar reasoning applies to $j > 2$. These results are summarized in Result 2 of the main text.

These results for integral IFB rules contrast with those for non-integral rules (C.1) studied in BLP. There we found that Result 1 is modified to a generalized 'Taylor principle': $\theta > 1$ is necessary and sufficient for a $j = 0$ IFB non-integral rule, but only necessary for $j \geqslant 1$. For non-integral rules as j increases a more interesting result emerges, namely there is always some lead J such that for

$$j > J = \frac{1}{1-\rho} + \frac{(1-\beta)(1-\gamma)\sigma}{\lambda(\tilde{\phi}+\sigma)} \tag{C.3}$$

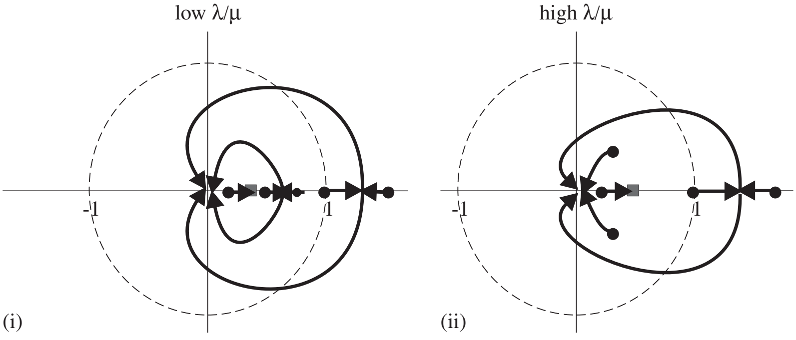

Fig. 5. Integral control IFB rule on two-period ahead expected inflation: position of zeroes as Θ changes.

there is indeterminacy *for all values of* θ.[40] We summarize these results for integral rules in Results 3 and 4.

To obtain numerical results in Tables 3a–d, write the characteristic equation as

$$\sum_{k=1}^{\max(5,j+3)} a_k(\Theta)z^k = 0 \tag{C.4}$$

noting that some of the a_k are dependent on Θ. The root locus meets the unit circle at $z = \cos(\psi) + \mathrm{i}\ \sin(\psi)$. Using De Moivre's theorem $z^k = \cos(k\psi) + \mathrm{i}\ \sin(k\psi)$ and equating real and imaginary parts we arrive at two equations which can be solved numerically for $\bar{\Theta}$ and ψ.

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[40] There are some conditions for this result to hold discussed in BLP. Numerical results indicate these conditions hold for all realistic values of the parameters, and certainly those estimated in Section 2.

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