THE HARDY CLASS OF A SPIRAL-LIKE FUNCTION AND ITS DERIVATIVE

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ABSTRACT. A determination is made of the Hardy classes to which a spiral-like univalent function and its derivative belong. An estimate for the size of the Taylor coefficients is deduced.

Let $f(z)$ be analytic in D, the unit disc $|z| < 1$, and let α be a real number such that $|\alpha| < \pi/2$. If $f(0) = 0$, $f'(0) \neq 0$, and if

$$\operatorname{Re}[e^{i\alpha}zf'(z)/f(z)] > 0, \qquad z \in D,$$

then $f(z)$ is univalent [5] and is said to be spiral-like. Under these conditions we have

$$e^{i\alpha}zf'(z)/f(z) = Q(z),$$

where $\operatorname{Re} Q(z) > 0$ and $Q(0) = e^{i\alpha}$. Defining $P(z) = Q(z) \sec \alpha - i \tan \alpha$, we may write

$$(1) \qquad zf'(z)/f(z) = e^{-i\alpha}[P(z) \cos \alpha + i \sin \alpha],$$

where $\operatorname{Re} P(z) > 0$, $P(0) = 1$.

If $f'(0) = 1$,[2] if $f(z)$ satisfies (1), and if $\operatorname{Re} P(z) \geq \rho \geq 0$, $z \in D$, we shall say that $f(z)$ belongs to the class $S_{\alpha,\rho}$ [3]. In particular, with $\alpha = 0$, $S_{0,\rho}$ coincides with the class of normalized starlike functions of order ρ. The relationship between $S_{\alpha,\rho}$ and $S_{0,\rho}$ is indicated in the following lemma.

LEMMA 1. $f(z) \in S_{\alpha,\rho}$ *if and only if there is a* $g(z) \in S_{0,\rho}$ *such that*

$$(2) \qquad [f(z)/z]^{\exp(i\alpha)} = [g(z)/z]^{\cos \alpha},$$

where the branches are chosen so that each side of the equation has the value 1 when $z = 0$.

PROOF. If $g(z) \in S_{0,\rho}$ then $zg'(z)/g(z) = P(z)$, where $\operatorname{Re} P(z) \geq \rho$, $P(0) = 1$. If $f(z)$ is defined by (2) then, differentiating logarithmically and multiplying by $ze^{-i\alpha}$, we obtain (1), which shows that $f(z) \in S_{\alpha,\rho}$.

[2] This normalization is made only for convenience. The conclusion of Theorem 2 and its corollary remain valid for the classes $S_{\alpha,\rho}$, when defined without normalization.

Conversely, if $f(z) \in S_{\alpha,\rho}$ and if we define $g(z)$ by

$$g(z)/z = [f(z)/z]^{1+i \tan \alpha},$$

then a similar calculation shows that $g(z) \in S_{0,\rho}$.

For a real $\lambda > 0$, we say that a function $h(z)$, analytic in D, belongs to the class H^λ if

$$\int_{-\pi}^{\pi} | h(re^{i\theta}) |^\lambda d\theta < K$$

for $0 \leq r < 1$, where K is a constant depending on $h(z)$ and λ.

The following theorem is equivalent to Theorem 6 in [1].

THEOREM 1. *If $g(z) \in S_{0,\rho}$ is not of the form*

$$g(z) = z(1 - ze^{i\tau})^{2\rho-2}$$

for some real τ, then
  (i) *there exists $\delta = \delta(g) > 0$ such that $g(z)/z \in H^{(1+\delta)/2(1-\rho)}$; and*
  (ii) *there exists $\epsilon = \epsilon(g) > 0$ such that $g'(z) \in H^{(1+\epsilon)/(3-2\rho)}$.*

The object of this note is to extend this theorem to the classes $S_{\alpha,\rho}$. To do this we require some further lemmas.

LEMMA 2. *If $g(z) \in S_{0,0}$, then*

$$| \arg(g(z)/z) | < \pi, \qquad z \in D,$$

where the principal value of the argument is taken.

LEMMA 3. *If $Q(z)$ is analytic and $\operatorname{Re} Q(z) > 0$ in D, then $Q(z) \in H^\lambda$ for all $\lambda < 1$.*

LEMMA 4. *If $h(z) \in H^\lambda$, $0 < \lambda < 1$, and $h(z) = \sum_0^\infty a_n z^n$, then*

$$a_n = o(n^{(1/\lambda)-1}).$$

Lemma 2 is in [4], Lemma 4 in [2]. Lemma 3 is well known.

The following theorem contains Theorem 1.

THEOREM 2. *If $f(z) \in S_{\alpha,\rho}$ is not of the form*

(3) $f(z) = z(1 - ze^{i\tau})^{-a}, \qquad a = 2(1 - \rho)(\cos \alpha - i \sin \alpha) \cos \alpha$

for some real τ, then
  (i) *there exists $\delta = \delta(f) > 0$ such that*

$$f(z)/z \in H^\mu, \qquad \mu = (1 + \delta) \sec^2 \alpha / 2(1 - \rho);$$

and

(ii) *there exists* $\epsilon = \epsilon(f) > 0$ *such that*

$$f'(z) \in H^\nu, \qquad \nu = (1 + \epsilon)/(1 + 2(1 - \rho) \cos^2 \alpha).$$

PROOF. (i) By Lemma 1, there is a function $g(z) \in S_{0,\rho}$ such that

$$f(z)/z = [g(z)/z]^{\cos^2 \alpha - i \sin \alpha \cos \alpha}.$$

Taking moduli we obtain

$$\left| f(z)/z \right| = \left| g(z)/z \right|^{\cos^2 \alpha} \exp[\sin \alpha \cos \alpha \, \arg(g(z)/z)],$$

so

$$\left| f(z)/z \right|^\mu = \left| g(z)/z \right|^{(1+\delta)/2(1-\rho)} \exp[(1 + \delta) \tan \alpha / 2(1 - \rho) \, \arg(g(z)/z)].$$

By Lemma 2, since $S_{0,\rho} \subset S_{0,0}$, the exponential factor is bounded, and the conclusion follows immediately from part (i) of Theorem 1.

(ii) Writing $e^{i\alpha} z f'(z)/f(z) = Q(z)$, and applying Lemma 3, we see that

$$(4) \qquad\qquad z f'(z)/f(z) \in H^\lambda, \qquad \text{all } \lambda < 1.$$

Next, writing

$$f'(z) = (f(z)/z)\,(z f'(z)/f(z)),$$

and apply Hölder's inequality with conjugate indices p, q to $\left| f'(z) \right|^\lambda$, with $z = re^{i\theta}$ we obtain

$$\int_{-\pi}^{\pi} \left| f'(z) \right|^\lambda d\theta \leqq \left(\int_{-\pi}^{\pi} \left| \frac{f(z)}{z} \right|^{\lambda p} d\theta \right)^{1/p} \left(\int_{-\pi}^{\pi} \left| \frac{z f'(z)}{f(z)} \right|^{\lambda q} d\theta \right)^{1/q} = I_1 \cdot I_2,$$

say. By part (i), I_1 is bounded if we choose λ, p so that $\lambda p = \mu$, and by (4), I_2 is bounded if we make the further restriction $\lambda q < 1$. This is achieved by taking $\lambda = \nu$ provided that

$$\nu = \frac{1 + \epsilon}{1 + 2(1 - \rho) \cos^2 \alpha} < \frac{1 + \delta}{1 + \delta + 2(1 - \rho) \cos^2 \alpha},$$

which holds if ϵ is sufficiently small.

From part (ii) of the theorem and Lemma 4 we deduce the following:

COROLLARY. *If* $f(z) \in S_{\alpha,\rho}$ *is not of the form* (3) *for some real* τ, *and if*

$$f(z) = \sum_{0}^{\infty} a_n z^n,$$

then there exists $\eta = \eta(f) > 0$ *such that*

$$a_n = o(n^{2(1-\rho) \cos^2 \alpha - 1 - \eta}).$$

In conclusion we remark that for the function (3) we have

$$a_n = \frac{\Gamma(a + n)}{\Gamma(n + 1)\Gamma(a)} e^{ni\tau},$$

$$\left| a_n \right| \sim \left| n^{a-1} \right| / \left| \Gamma(a) \right| = n^{2(1-\rho) \cos^2 \alpha - 1} / \left| \Gamma(a) \right|.$$

This shows that (3) is indeed exceptional in the corollary, and therefore also in part (ii) of Theorem 2; a computation shows also that for the function (3) we have $f(z)/z \in H^\lambda$ if and only if $\lambda < \frac{1}{2}\sec^2 \alpha/(1-\rho)$, so that this function is also exceptional in part (i) of Theorem 2.

A weaker consequence of the corollary, well known for starlike functions, is that *for all spiral-like functions $f(z)$ with the exception only of those of the form $z(1 - ze^{i\tau})^{-2}$, we have $a_n = o(n^{1-\eta})$ for some $\eta = \eta(f) > 0$.*

ADDED IN PROOF (June 16, 1970). Application of Theorem 2 and of Theorem E of [1] yields the following result for functions of the class C_α [6].

If $f(z) \in C_\alpha$ is not of the form

$$f(z) = e^{-i\tau}(1 - ze^{i\tau})^{-a+1}/(a-1) + c, \quad a = 2(\cos \alpha - i \sin \alpha) \cos \alpha$$

for some complex c and real τ, then there exists $\delta = \delta(f) > 0$ such that
  (i) $f'(z) \in H^\beta, \beta = \frac{1}{2}(1 + \delta) \sec^2\alpha$;
  (ii) *if* $|\alpha| < \pi/4$ *then* $f(z) \in H^\gamma, \gamma = (1 + \delta)\sec^2\alpha/[2 - (1 + \delta) \sec^2\alpha]$.

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