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Some general theorems on the explicit evaluations of Ramanujan's cubic continued fraction

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Abstract

In 2001, Jinhee Yi found many explicit values of the famous Rogers–Ramanujan continued fraction by using modular equations and transformation formulas for theta-functions. In this paper, we use her method to find some general theorems for the explicit evaluations of Ramanujan's cubic continued fraction.

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1. Introduction

Let,

$$G(q) := \frac{q^{1/3}}{1} \, \frac{}{+} \, \frac{q+q^2}{1} \, \frac{}{+} \, \frac{q^2+q^4}{1} \, \frac{}{+} \, \frac{q^3+q^6}{1} \, \frac{}{+\cdots}, \tag{1.1}$$

denote the Ramanujan's cubic continued fraction for $|q| < 1$. This continued fraction was recorded by Ramanujan in his second letter to Hardy [9] and on p. 366 of his lost notebook [15], and claimed that there are many results of $G(q)$ which are analogous to the famous Rogers–Ramanujan continued fraction $R(q)$, defined by

$$R(q) := \frac{q^{1/5}}{1} \, \frac{q}{+1} \, \frac{q^2}{+1} \, \frac{q^3}{+1} \, \frac{}{+\cdots}, \quad |q| < 1.$$

Motivated by Ramanuajan's claims, Chan [10] proved three identities giving relations between $G(q)$ and the three continued fractions $G(-q)$, $G(q^2)$, and $G(q^3)$. Baruah [3] found two new identities

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giving relations between $G(q)$ and the two continued fractions $G(q^5)$ and $G(q^7)$. Chan [10] also found three reciprocity theorems for $G(q)$. He also evaluated $G(-e^{-\pi\sqrt{n}})$ for $n=1$ and 5 and $G(e^{-\pi\sqrt{n}})$ for n=1, 2, 4, and 2/9. Berndt et al. [8] have found general formulas for $G(-e^{-\pi\sqrt{n}})$ and $G(e^{-\pi\sqrt{n}})$ in terms of Weber–Ramanujan class invariants G_n and g_n, defined by

$$G_n := 2^{-1/4}q^{-1/24}(-q;q^2)_\infty$$

and

$$g_n := 2^{-1/4}q^{-1/24}(q;q^2)_\infty, \quad q = e^{-\pi\sqrt{n}},$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1.$$

They evaluated $G(-e^{-\pi\sqrt{n}})$ for n=1, 5, 13, and 37 and $G(e^{-\pi\sqrt{n}})$ for n=2, 10, 22, and 58. Ramanathan [13] has also found $G(e^{-\pi\sqrt{10}})$ by using Kronecker's limit formula. This value was recorded by Ramanujan on p. 366 of his lost notebook [15]. By evaluating some new Weber–Ramanujan class invariants Yi [17, pp. 120–124] has also evaluated some new values for $G(-e^{-\pi\sqrt{n}})$ and $G(e^{-\pi\sqrt{n}})$. By using modular equations and transformation formulas for theta-functions Adiga et al. [2,1], Vasuki and Shivashankara [16] have recently found $G(-e^{-\pi\sqrt{n}})$ for $n=1/147$, 1/75, 1/27, 1/13, 1/9, 1/7, 1/5, 1/3, 1, 3, 5, and 25/3, and $G(e^{-\pi\sqrt{n}})$ for $n=1/3$, 1, 4/3, 4, 16/3, and 16. Other values of $G(q)$ can be found by using the reciprocity theorems given by Chan [10] and Adiga et al. [1].

In this paper, we present some general theorems for evaluating $G(-e^{-\pi\sqrt{n}})$ and $G(e^{-\pi\sqrt{n}})$ by using modular equations and transformation formulas for theta-functions. Our theorems are motivated by Jinhee Yi's paper [18], in which she evaluates many explicit values $R(q)$.

We end this introduction by defining Ramanujan's special theta-functions. After Ramanujan, for $|q|<1$, we define the theta-functions,

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.2}$$

$$\psi(q) := \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

and

$$f(-q) := (q;q)_\infty =: q^{-1/24}\eta(z), \quad q = e^{2\pi i z}, \quad \operatorname{Im} z > 0, \tag{1.4}$$

where $\eta(z)$ denotes the Dedekind eta-function.

2. Modular equations

In this section, we state and prove some modular equations which will be used in finding theorems for the explicit evaluations of $G(q)$.

Theorem 2.1 (Berndt [5, p. 204, Entry 51]). *If*

$$P=\frac{f^2(-q)}{q^{1/6}f^2(-q^3)}\quad \textit{and}\quad Q=\frac{f^2(-q^2)}{q^{1/3}f^2(-q^6)},$$

then

$$PQ+\frac{9}{PQ}=\left(\frac{Q}{P}\right)^3+\left(\frac{P}{Q}\right)^3. \tag{2.1}$$

Theorem 2.2 (Berndt [4, p. 346, Entry 1(iv)]). *If*

$$P=\frac{f^3(-q)}{q^{1/4}f^3(-q^3)}\quad \textit{and}\quad Q=\frac{f^3(-q^3)}{q^{3/4}f^3(-q^9)},$$

then

$$\left(1+\frac{9}{PQ}\right)^3=1+\frac{27}{P^4}. \tag{2.2}$$

Theorem 2.3 (Berndt [5, p. 221, Entry 62]). *If*

$$P=\frac{f(-q)}{q^{1/12}f(-q^3)}\quad \textit{and}\quad Q=\frac{f(-q^5)}{q^{5/12}f(-q^{15})},$$

then

$$(PQ)^2+5+\frac{9}{(PQ)^2}=\left(\frac{Q}{P}\right)^3-\left(\frac{P}{Q}\right)^3. \tag{2.3}$$

Theorem 2.4 (Berndt [5, p. 236, Entry 69]). *If*

$$P=\frac{f(-q)}{q^{1/12}f(-q^3)}\quad \textit{and}\quad Q=\frac{f(-q^7)}{q^{7/12}f(-q^{21})},$$

then

$$(PQ)^3+\frac{27}{(PQ)^3}=\left(\frac{Q}{P}\right)^4-7\left(\frac{Q}{P}\right)^2+7\left(\frac{P}{Q}\right)^2-\left(\frac{P}{Q}\right)^4. \tag{2.4}$$

Theorem 2.5 (Berndt [6, p. 127]). *If*

$$P=\frac{f(-q)}{q^{1/12}f(-q^3)}\quad \textit{and}\quad Q=\frac{f(-q^{11})}{q^{11/12}f(-q^{33})},$$

then

$$(PQ)^5+\left(\frac{3}{PQ}\right)^5+11\left\{(PQ)^4+\left(\frac{3}{PQ}\right)^4\right\}+66\left\{(PQ)^3+\left(\frac{3}{PQ}\right)^3\right\}$$
$$+253\left\{(PQ)^2+\left(\frac{3}{PQ}\right)^2\right\}+693\left\{PQ+\frac{3}{PQ}\right\}+1386=\left(\frac{Q}{P}\right)^6+\left(\frac{P}{Q}\right)^6.$$

Theorem 2.6 (Berndt [5, p. 210, Entry 56]). *If*

$$P=\frac{f(-q^{1/3})}{q^{1/9}f(-q^3)}\quad and \quad Q=\frac{f(-q^{2/3})}{q^{2/9}f(-q^6)},$$

then

$$P^3+Q^3=P^2Q^2+3PQ. \tag{2.5}$$

Theorem 2.7. *If*

$$P=\frac{f(-q^{1/3})}{q^{1/9}f(-q^3)}\quad and \quad Q=\frac{f(-q)}{q^{1/3}f(-q^9)},$$

then

$$(PQ)^3+\left(\frac{9}{PQ}\right)^3+27\left(\left(\frac{P}{Q}\right)^3+\left(\frac{Q}{P}\right)^3\right)+9(P^3+Q^3)+243\left(\frac{1}{P^3}+\frac{1}{Q^3}\right)$$
$$+81=\left(\frac{Q}{P}\right)^6. \tag{2.6}$$

Proof. This easily follows from Theorem 2.2.

Theorem 2.8. *If*

$$P=\frac{\phi(q^{1/3})}{\phi(q^3)}\quad and \quad Q=\frac{\phi(q)}{\phi(q^9)},$$

then

$$PQ+\frac{9}{PQ}+3\left(\frac{P}{Q}+\frac{Q}{P}\right)-9\left(\frac{1}{P}+\frac{1}{Q}\right)-3(P+Q)+9=\left(\frac{Q}{P}\right)^2. \tag{2.7}$$

Proof. We use the first two identities of Entry 1 (iii) [4, p. 345].

Theorem 2.9. *If*

$$P=\frac{\phi(q^{1/3})}{\phi(q^3)}\quad and \quad Q=\frac{\phi(q^{5/3})}{\phi(q^{15})},$$

then

$$\left(\frac{P}{Q}\right)^3+\left(\frac{Q}{P}\right)^3+15\left(\frac{P^2}{Q}+\frac{Q^2}{P}+3\frac{P}{Q^2}+3\frac{Q}{P^2}\right)+5(P+Q)(6+PQ)$$
$$+45\left(\frac{1}{P}+\frac{1}{Q}\right)\left(2+\frac{3}{PQ}\right)=(PQ)^2+\frac{81}{(PQ)^2}+10(P+Q)^2+90\left(\frac{1}{P}+\frac{1}{Q}\right)^2$$
$$+15\left(\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right)+45\left(\frac{P}{Q}+\frac{Q}{P}\right)+40. \tag{2.8}$$

Proof. We use the first two identities of Entry 1 (iii) [4, p. 345] and Remark 1 of Theorem 2.1 in [3, p. 245, 247].

Theorem 2.10. *If*

$$P=\frac{\psi(-q^{1/3})}{q^{1/3}\psi(-q^3)} \quad \textit{and} \quad Q=\frac{\psi(-q)}{q\psi(-q^9)},$$

then

$$PQ+\frac{9}{PQ}+3\left(\frac{P}{Q}+\frac{Q}{P}\right)+9\left(\frac{1}{P}+\frac{1}{Q}\right)+3(P+Q)+9=\left(\frac{Q}{P}\right)^2. \tag{2.9}$$

Proof. We use the first and last identities of Entry 1 (ii) [4, p. 345].

Theorem 2.11 (Baruah [3, p. 253]). *If*

$$P=\frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \textit{and} \quad Q=\frac{\psi(q^5)}{q^{5/4}\psi(q^{15})},$$

then

$$(PQ)^4+\frac{81}{(PQ)^4}+15\left(\left(\frac{Q}{P}\right)^4+\left(\frac{P}{Q}\right)^4\right)+120-10(P^4+Q^4)$$
$$-90\left(\frac{1}{P^4}+\frac{1}{Q^4}\right)=\left(\frac{P}{Q}\right)^2\left(\left(\frac{Q}{P}\right)^8+\left(\frac{P}{Q}\right)^4+15\left(\frac{Q}{P}\right)^4+15\right). \tag{2.10}$$

Theorem 2.12 (Baruah [3, p. 250]). *If*

$$P=\frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \textit{and} \quad Q=\frac{\psi(q^7)}{q^{7/4}\psi(q^{21})},$$

then

$$k_1(PQ)^3+k_2(PQ)=k_3(PQ)^2+k_4\left(\frac{P}{Q}\right)^2-k_5, \tag{2.11}$$

where

$$k_1=\left(\frac{P}{Q}\right)^8-1,\quad k_2=14P^4\left(\left(\frac{P}{Q}\right)^4-1\right),\quad k_3=P^4(7-P^4),$$

$$k_4=7P^4(P^4-3),\quad \textit{and}\quad k_5=27\left(\frac{P}{Q}\right)^4-7P^4\left(3+3\left(\frac{P}{Q}\right)^4-P^4\right). \tag{2.12}$$

3. Transformation formulas for theta-functions

In this section, we state some transformation formulas for theta-functions which will be used in the last section. In these formulas it is assumed that α and β are such that the modulus of each exponential argument is less than 1.

Theorem 3.1 (Berndt [4, p. 43, Entry 27 (i)]). *If* $\alpha\beta=\pi$ *then*

$$\sqrt{\alpha}\phi(e^{-\alpha^2})=\sqrt{\beta}\phi(e^{-\beta^2}).$$

Theorem 3.2 (Berndt [4, p. 43, Entry 27 (iii)]). *If* $\alpha\beta=\pi^2$ *then*

$$e^{-\alpha/12}\sqrt[4]{\alpha}f(-e^{-2\alpha})=e^{-\beta/12}\sqrt[4]{\beta}f(-e^{-2\beta}).$$

Theorem 3.3 (Berndt [4, p. 43, Entry 27 (iv)]). *If* $\alpha\beta=\pi^2$ *then*

$$e^{-\alpha/24}\sqrt[4]{\alpha}f(e^{-\alpha})=e^{-\beta/24}\sqrt[4]{\beta}f(e^{-\beta}).$$

Theorem 3.4 (Adiga et al. [1]). *If* $\alpha\beta=\pi^2$ *then*

$$e^{-\alpha/8}\sqrt[4]{\alpha}\psi(-e^{-\alpha})=e^{-\beta/8}\sqrt[4]{\beta}\psi(-e^{-\beta}).$$

4. Theorems for explicit evaluations of $G(q)$

Theorem 4.1. (i) *For* $q=e^{-\pi\sqrt{n/3}}$, *let*

$$\lambda_n=\frac{1}{3\sqrt{3}}\frac{f^6(q)}{\sqrt{q}f^6(q^3)}.$$

Then

$$3\left(1-\lambda_n^2\right)^{1/3}=4w^2+\frac{1}{w},$$

where $w=G(-q)$.

(ii) *For* $q = e^{-2\pi\sqrt{n/3}}$, *let*

$$\mu_n = \frac{1}{3\sqrt{3}} \frac{f^6(-q)}{\sqrt{q} f^6(-q^3)}.$$

Then

$$3\left(1+\mu_n^2\right)^{1/3} = 4v^2 + \frac{1}{v},$$

where $v = G(q)$.

Proof. We use the first identity of Entry 1(iv) [4, p. 345].

Several values of λ_n were recorded by Ramanujan on p. 212 of his lost notebook [15]. All of those values were proved by Berndt et al. [7]. They also evaluated many new values by using modular j-invariants, Weber–Ramanujan class invariants, modular equations, Kronecker's limit formula, and an empirical process. Thus, one can use Theorem 4.1 to find the values of $G(-e^{-\pi\sqrt{n/3}})$ if the corresponding values of λ_n are known. For example, by noting that [7, p. 281]

$$\lambda_{1/n} = \frac{1}{\lambda_n},$$

we have, $\lambda_1 = 1$. Hence, Theorem 4.1 implies that

$$G(-e^{-\pi/\sqrt{3}}) = -\frac{1}{\sqrt[3]{4}}.$$

This was also evaluated by Adiga et al. [2,1]. Many other values of $G(-e^{-\pi\sqrt{n/3}})$ can be computed by using the known values of λ_n. Clearly, we can also compute $G(-e^{-\pi/\sqrt{3n}})$ if we know $G(-e^{-\pi\sqrt{n/3}})$.

The function μ_n was first defined by Ramanathan [14].

Theorem 4.2. *If* μ_n *is as defined in Theorem* 4.1, *then*

$$\mu_{1/n} = \frac{1}{\mu_n}.$$

Proof. We use the definition of μ_n and Theorem 3.2.

Corollary 4.3. $\mu_1 = 1$.

Theorem 4.4. *If* μ_n *is as defined in Theorem* 4.1, *then*

$$\text{(i)}\quad 3\left((\mu_n\mu_{4n})^{1/3} + \frac{1}{(\mu_n\mu_{4n})^{1/3}}\right) = \frac{\mu_n}{\mu_{4n}} + \frac{\mu_{4n}}{\mu_n},$$

$$\text{(ii)}\quad \left(1 + \left(\frac{3}{\mu_n\mu_{9n}}\right)^{1/2}\right)^3 = 1 + \frac{1}{\mu_n^2},$$

$$\text{(iii)}\quad 3\left((\mu_n\mu_{25n})^{1/3}+\frac{1}{(\mu_n\mu_{25n})^{1/3}}\right)+5=\left(\frac{\mu_{25n}}{\mu_n}\right)^{1/2}-\left(\frac{\mu_n}{\mu_{25n}}\right)^{1/2},$$

$$\text{(iv)}\quad 3\sqrt{3}\left((\mu_n\mu_{49n})^{1/2}+\frac{1}{(\mu_n\mu_{49n})^{1/2}}\right)$$
$$=\left(\frac{\mu_{49n}}{\mu_n}\right)^{2/3}-\left(\frac{\mu_n}{\mu_{49n}}\right)^{2/3}-7\left(\frac{\mu_{49n}}{\mu_n}\right)^{1/3}+7\left(\frac{\mu_n}{\mu_{49n}}\right)^{1/3},$$

$$\text{(v)}\quad 9\sqrt{3}\left((\mu_n\mu_{121n})^{5/6}+\frac{1}{(\mu_n\mu_{121n})^{5/6}}\right)+99\left((\mu_n\mu_{121n})^{2/3}+\frac{1}{(\mu_n\mu_{121n})^{2/3}}\right)$$
$$+198\sqrt{3}\left((\mu_n\mu_{121n})^{1/2}+\frac{1}{(\mu_n\mu_{121n})^{1/2}}\right)+759\left((\mu_n\mu_{121n})^{1/3}+\frac{1}{(\mu_n\mu_{121n})^{1/3}}\right)$$
$$+693\sqrt{3}\left((\mu_n\mu_{121n})^{1/6}+\frac{1}{(\mu_n\mu_{121n})^{1/6}}\right)+1386=\left(\frac{\mu_{121n}}{\mu_n}\right)+\left(\frac{\mu_n}{\mu_{121n}}\right).$$

Proof. The theorem follows from the definition of μ_n and Theorems 2.1–2.5
Theorems 4.4 (i)–(iv) were also found by Yi [17].

Theorem 4.5. We have

(i) $\mu_2=\sqrt{2}+1$

(ii) $\mu_4=\dfrac{3\sqrt{3}+5}{\sqrt{2}},$

(iii) $\mu_3=\sqrt{6\sqrt{3}+9},$

(iv) $\mu_9=\dfrac{3}{(\sqrt[3]{2}-1)^2},$

(v) $\mu_5=\dfrac{11+5\sqrt{5}}{2},$

(vi) $\mu_{25}=\left(\dfrac{a+\sqrt{a^2+4}}{2}\right)^6, \quad \textit{where} \quad a=1+4\sqrt[3]{10},$

(vii) $\mu_7=\left(\dfrac{3+2\sqrt{3}+2\sqrt{7}+\sqrt{21}}{2}\right)^{3/2},$

(viii) $\mu_{49}=\left(\dfrac{a+\sqrt{a^2-4}}{2}\right)^3, \quad \textit{where} \quad a=\dfrac{43}{3}+\dfrac{13}{3}\sqrt[3]{4}\sqrt[3]{7}+\dfrac{8}{3}\sqrt[3]{2}\sqrt[3]{49},$

(ix) $\mu_{11}=\left(1551+900\sqrt{3}+470\sqrt{11}+270\sqrt{33}\right)^{1/2}.$

Proof. We put $n=1/2$ and 1, $n=1/3$ and 1, $n=1/5$ and 1, $n=1/7$ and 1, and $n=1/11$, in Theorems 4.4 (i)–(v), respectively. We obtain the results by appealing to Theorem 4.2 and Corollary 4.3, and then solving the resulting polynomial equations.

The values of $\mu_{1/n}$ for $n = 2$, 3, 4, 5, 7, 9, 11, 25, and 49 can easily be found by applying Theorems 4.2 and 4.5

Theorems 4.5 (i)–(viii) can also be found in [17]

Theorem 4.6. (i) *For* $q = \mathrm{e}^{-2\pi\sqrt{n}}$, *let*

$$J_n = \frac{f(-q^{1/3})}{\sqrt{3}q^{1/9} f(-q^3)}.$$

Then

$$3 + 3\sqrt{3}J_n^3 = 4v^2 + \frac{1}{v},$$

where $v = G(q)$.

(ii) *For* $q = \mathrm{e}^{-\pi\sqrt{n}}$, *let*

$$D_n = \frac{f(q^{1/3})}{\sqrt{3}q^{1/9} f(q^3)}.$$

Then

$$3 - 3\sqrt{3}D_n^3 = 4w^2 + \frac{1}{w},$$

where $v = G(-q)$.

Proof. We use Entry 1 (iv) [4, p. 345].

Theorem 4.7. *If* J_n *and* D_n *are as defined in Theorem* 4.6, *then*

$$J_{1/n} = \frac{1}{J_n} \text{ and } D_{1/n} = \frac{1}{D_n}.$$

Proof. We use the definitions of J_n and D_n and Theorems 3.2 and 3.3.

Corollary 4.8. $J_1 = 1$ *and* $D_1 = 1$.

Theorem 4.9. *If* J_n *and* D_n *are as defined in Theorem* 4.6, *then*

$$\text{(i)} \quad J_n^3 + J_{4n}^3 = \sqrt{3}J_n J_{4n}(J_n J_{4n} + 1),$$

$$\text{(ii)} \quad J_n^3 + D_n^3 = \sqrt{3}J_n D_n(J_n D_n - 1),$$

$$\text{(iii)}\quad (J_nJ_{9n})^3+\frac{1}{(J_nJ_{9n})^3}+\left(\frac{J_n}{J_{9n}}\right)^3+\left(\frac{J_{9n}}{J_n}\right)^3$$
$$+\sqrt{3}\left(J_n^3+J_{9n}^3+\frac{1}{J_n^3}+\frac{1}{J_{9n}^3}\right)+3=\left(\frac{J_{9n}}{\sqrt{3}J_n}\right)^6,$$

$$\text{(iv)}\quad (D_nD_{9n})^3+\frac{1}{(D_nD_{9n})^3}+\left(\frac{D_n}{D_{9n}}\right)^3+\left(\frac{D_{9n}}{D_n}\right)^3$$
$$-\sqrt{3}\left(D_n^3+D_{9n}^3+\frac{1}{D_n^3}+\frac{1}{D_{9n}^3}\right)+3=\left(\frac{D_{9n}}{\sqrt{3}D_n}\right)^6.$$

Proof. We use the definitions of J_n and D_n and Theorems 2.6 and 2.7.

Theorem 4.10. *We have*

(i) $J_2=(\sqrt{3}+\sqrt{2})^{1/3}$,

(ii) $D_2=\dfrac{\sqrt{3}a^3-\sqrt{2}-\sqrt{3}}{\sqrt{2}a}$, *where* $a=(\sqrt{2}+\sqrt{3})^{1/3}$,

(iii) $J_4=\dfrac{\sqrt{3}+1+\sqrt{2\sqrt{3}}}{2}$,

(iv) $D_4=\dfrac{2\sqrt{30+15\sqrt{2\sqrt{3}}+16\sqrt{3}+9\sqrt{2\sqrt{27}}}-4-3\sqrt{2\sqrt{3}}-2\sqrt{3}-\sqrt{2\sqrt{27}}}{4}$,

(v) $J_3=\left(\sqrt{3}\left(1+2^{1/3}+2^{2/3}\right)\right)^{1/3}$,

(viii) $D_3=3^{1/6}$,

(vi) $J_9=\left(3\left(6+3\sqrt{3}+(738+426\sqrt{3})^{1/3}+(776+448\sqrt{3})^{1/3}\right)\right)^{1/3}$,

(vii) $D_9=\left(3\left(6-3\sqrt{3}+(738-426\sqrt{3})^{1/3}+(776-448\sqrt{3})^{1/3}\right)\right)^{1/3}$.

Proof. Putting $n=1/2$ and 1 in Theorem 4.9 (i), and then solving the polynomial equations we obtain J_2 and J_4. Putting $n=2$ and 4 in Theorem 4.9 (ii) and then solving the polynomial equations we find D_2 and D_4. Putting $n=1/3$ and 1 in Theorems 4.9 (iii) and (iv) and then again solving the resulting polynomial equations we obtain J_3, D_3, J_9, and D_9.

The values of $J_{1/n}$ and $D_{1/n}$ for $n=2,3,4$, and 9 can easily be calculated by applying Theorems 4.7 and 4.10.

Theorem 4.11. *For* $q=\mathrm{e}^{-\pi n}$, *let*

$$S_n=\frac{\phi(q^{1/3})}{\sqrt{3}\phi(q^3)}.$$

Then

$$2G(-q) = 1 - \sqrt{3}S_n.$$

Proof. We use Entry 1 (ii) [4, p. 345].

Theorem 4.12. If S_n is as defined in Theorem 4.11, then

$$S_{1/n} = \frac{1}{S_n}.$$

Proof. We use Theorem 3.1 and the definition of S_n.

Corollary 4.13. $S_1 = 1$.

Theorem 4.14. *If S_n is as defined in Theorem* 4.11, *then*

$$\text{(i)} \quad 3\left(S_nS_{3n} + \frac{1}{S_nS_{3n}}\right) + 3\left(\frac{S_n}{S_{3n}} + \frac{S_{3n}}{S_n}\right) - 3\sqrt{3}\left(\frac{1}{S_n} + \frac{1}{S_{3n}} + S_n + S_{3n}\right) + 9 = \left(\frac{S_{3n}}{S_n}\right)^2,$$

$$\begin{aligned} \text{(ii)} \quad & \left(\frac{S_n}{S_{5n}}\right)^3 + \left(\frac{S_{5n}}{S_n}\right)^3 + 15\sqrt{3}\left(S_n^2 + \frac{1}{S_n^2}\right)\left(S_{5n} + \frac{1}{S_{5n}} - 10\right) + 15\sqrt{3}\left(S_n + \frac{1}{S_n}\right) \\ & \times \left(S_{5n}^2 + \frac{1}{S_{5n}^2} + 2\right) + 60\left(S_nS_{5n} + \frac{1}{S_nS_{5n}}\right) = 15\left(\left(\frac{S_n}{S_{5n}}\right)^2 + \left(\frac{S_{5n}}{S_n}\right)^2\right) \\ & +9\left(S_n^2S_{5n}^2 + \frac{1}{S_n^2S_{5n}^2}\right) + 45\left(\frac{S_n}{S_{5n}} + \frac{S_{5n}}{S_n}\right) + 30\left(S_{5n}^2 + \frac{1}{S_{5n}^2}\right) + 30\sqrt{3}\left(S_{5n} + \frac{1}{S_{5n}}\right) + 40. \end{aligned}$$

Proof. We use the definition of S_n and Theorems 2.8 and 2.9.

Theorem 4.15. *We have*

(i) $S_3 = 2 - \sqrt{3} - \dfrac{2(-5 + 3\sqrt{3})}{a^{1/3}} + a^{1/3}$, *where* $a = 56 - 32\sqrt{3}$,

(ii) $S_5 = \dfrac{1}{2(2 - \sqrt{3})}(28 - 15\sqrt{3} + 7\sqrt{15} - 12\sqrt{5} + \sqrt{40530 - 23400\sqrt{3} - 18138\sqrt{5} + 10472\sqrt{15}})$.

Proof. We put $n = 1$ in the above theorem and then solve the resulting polynomial equations to obtain the results.

The values of $S_{1/3}$ and $S_{1/5}$ follow from Theorems 4.12 and 4.15.

Theorem 4.16. *For* $q = e^{-\pi\sqrt{n}}$, *let*

$$L_n = \frac{\psi(-q^{1/3})}{\sqrt{3}q^{1/3}\psi(-q^3)}.$$

Then

$$-G(-q)=\frac{1}{1+\sqrt{3}L_n}.$$

Proof. We use Entry 1 (i) [4, p. 345].

Theorem 4.17. *If* L_n *is as defined in Theorem* 4.16, *then*

$$L_{1/n}=\frac{1}{L_n}.$$

Proof. We use Theorem 3.4 and the definition of L_n.

Corollary 4.18. $L_n=1$.

Theorem 4.19. *If* L_n *is as defined in Theorem* 4.16, *then*

$$3\left(L_nL_{9n}+\frac{1}{L_nL_{9n}}\right)+3\left(\frac{L_n}{L_{9n}}+\frac{L_{9n}}{L_n}\right)+3\sqrt{3}\left(\frac{1}{L_n}+\frac{1}{L_{9n}}+L_n+L_{9n}\right)+9=\left(\frac{L_{9n}}{L_n}\right)^2.$$

Proof. We use the definition of L_n and Theorem 2.10.

Theorem 4.20. *We have*

(i) $L_3=\frac{1}{\sqrt{3}}+\frac{2.2^{1/3}}{\sqrt{3}}+\frac{2^{2/3}}{\sqrt{3}}$

(ii) $L_9=2+\sqrt{3}+(38+22\sqrt{3})^{1/3}+2(2+\sqrt{3})^{2/3}$.

Proof. Putting $n=1/3$ and $n=1$ in the above theorem and then solving the resulting polynomial equations we obtain the results.

The values of $L_{1/3}$ and $L_{1/9}$ follow from Theorems 4.17 and 4.20.

Theorem 4.21. *For* $q=e^{-\pi\sqrt{n/3}}$, *let*

$$B_n=\frac{\psi^4(-q)}{3q\psi^4(-q^3)}.$$

Then

$$-G^3(-q)=\frac{1}{1+3B_n}.$$

Proof. We use Entry 1 (i) [4, p. 345].

Theorem 4.22. *If B_n is as defined in Theorem* 4.21, *then*

$$B_{1/n} = \frac{1}{B_n}.$$

Proof. We use Theorem 3.4 and the definition of B_n.

Corollary 4.23. $B_1 = 1$.

Theorem 4.24. *If B_n is as defined in Theorem* 4.21, *then*

$$\text{(i)}\quad \sqrt{3}\left((B_nB_{9n})^{1/4} + \frac{1}{(B_nB_{9n})^{1/4}}\right) + 3 = \left(\frac{B_{9n}}{B_n}\right)^{1/2},$$

$$\text{(ii)}\quad 9\left(B_nB_{25n} + \frac{1}{B_nB_{25n}}\right) + 15\left(\frac{B_n}{B_{25n}} + \frac{B_{25n}}{B_n}\right) + 30\left(\frac{1}{B_n} + \frac{1}{B_{25n}} + B_n + B_{25n}\right) + 120$$
$$= \left(\frac{B_{25n}}{B_n}\right)^{3/2} + \left(\frac{B_n}{B_{25n}}\right)^{3/2} + 15\left(\left(\frac{B_{25n}}{B_n}\right)^{1/2} + \left(\frac{B_n}{B_{25n}}\right)^{1/2}\right),$$

$$\text{(iii)}\quad a_1(B_nB_{49n})^{3/4} - a_2(B_nB_{49n})^{1/4} + a_3(B_nB_{49n})^{1/2} + a_4\left(\frac{B_n}{B_{49n}}\right)^{1/2} + a_5 = 0,$$

where

$$a_1 = \left(\frac{B_n}{B_{49n}}\right)^2 - 1, \quad a_2 = 14B_n\left(\frac{B_n}{B_{49n}} - 1\right), \quad a_3 = \sqrt{3}B_n(7 + 3B_n),$$

$$a_4 = 7\sqrt{3}B_n(B_n + 1), \quad \textit{and} \quad a_5 = 3\sqrt{3}\,\frac{B_n}{B_{49n}} + 7\sqrt{3}B_n\left(\frac{B_n}{B_{49n}} + B_n + 1\right).$$

Proof. We replace q by $-q$ in Theorems 2.11 and 2.12, and then use the definition of B_n.

Theorem 4.25. *We have*

$$\text{(i)}\quad B_3 = \sqrt{3}(2 + \sqrt{3}),$$

$$\text{(ii)}\quad B_9 = \frac{(1 + \sqrt[3]{2})^2}{\sqrt{3}},$$

$$\text{(iii)}\quad B_5 = 9 + 4\sqrt{5},$$

$$\text{(iv)}\quad B_{25} = \left(\frac{2(a - 17)}{a + b}\right)^2, \quad \textit{where } a = (5761 + \sqrt{421121})^{1/3} \textit{ and } b = \sqrt{68 - 4a + a^2},$$

$$\text{(v)}\quad B_7 = \frac{1}{9 - 6\sqrt{3} + 2\sqrt{49 - 28\sqrt{3}}}.$$

Proof. Putting $n=1/3$ and 1, $n=1/5$ and 1, and $n=1/7$, in Theorems 4.24 (i), (ii) and (iii), respectively, using Theorem 4.22 and Corollary 4.23, and then solving the resulting polynomial equations, we obtain the results.

The values of $S_{1/n}$ for n=3, 5, 7, 9, and 25 follow from Theorems 4.22 and 4.25.

Remark 4.26. (i) Theorem 4.4 implies that if we know μ_n, then we can evaluate μ_{4n}, $\mu_{n/4}$, μ_{9n}, $\mu_{n/9}$, μ_{25n}, $\mu_{n/25}$, μ_{49n}, $\mu_{n/49}$, μ_{121n}, or $\mu_{n/121}$. Thus, by Theorem 4.1 (ii), if we know $G(\mathrm{e}^{-2\pi\sqrt{n/3}})$, then we can also evaluate $G(\mathrm{e}^{-4\pi\sqrt{n/3}})$, $G(\mathrm{e}^{-\pi\sqrt{n/3}})$, $G(\mathrm{e}^{-2\pi\sqrt{3n}})$, $G(\mathrm{e}^{-2\pi\sqrt{n/27}})$, $G(\mathrm{e}^{-10\pi\sqrt{n/3}})$, $G(\mathrm{e}^{-2\pi\sqrt{n/75}})$, $G(\mathrm{e}^{-14\pi\sqrt{n/3}})$, $G(\mathrm{e}^{-2\pi\sqrt{n/147}})$, $G(\mathrm{e}^{-22\pi\sqrt{n/3}})$, or $G(\mathrm{e}^{-2\pi\sqrt{n/363}})$.

(ii) Using cubic Russell-type modular equations of degrees $p=13$, 17, 19, 23, 29, 41, 47, 53, and 59, derived by Chan and Liaw [11] and Liaw [12] (see also [9]), one can also find relations connecting μ_n and μ_{p^2n}.

(iii) Theorem 4.9 implies that if we know J_n, then we can evaluate J_{4n}, $J_{n/4}$, D_n, D_{4n}, $D_{n/4}$, J_{9n}, $J_{n/9}$, D_{9n}, or $D_{n/9}$. So using Theorem 4.6, if we know $G(\mathrm{e}^{-2\pi\sqrt{n}})$, then we can also evaluate $G(\mathrm{e}^{-4\pi\sqrt{n}})$, $G(\mathrm{e}^{-\pi\sqrt{n}})$, $G(-\mathrm{e}^{-\pi\sqrt{n}})$, $G(-\mathrm{e}^{-2\pi\sqrt{n}})$, $G(-\mathrm{e}^{-\pi\sqrt{n}/2})$, $G(\mathrm{e}^{-6\pi\sqrt{n}})$, $G(\mathrm{e}^{-2\pi\sqrt{n/9}})$, $G(-\mathrm{e}^{-3\pi\sqrt{n}})$, or $G(-\mathrm{e}^{-\pi\sqrt{n/9}})$.

(iv) Theorem 4.14 implies that if we know S_n, then we can evaluate S_{3n}, $S_{n/3}$, S_{5n}, or $S_{n/5}$. Thus, by Theorem 4.11, if we know $G(-\mathrm{e}^{-\pi n})$, then we can also evaluate $G(-\mathrm{e}^{-3\pi n})$, $G(-\mathrm{e}^{-\pi n/3})$, $G(-\mathrm{e}^{-5\pi n})$, or $G(-\mathrm{e}^{-\pi n/5})$.

(v) Theorem 4.19 implies that if we know L_n, then we can compute L_{9n} or $L_{n/9}$, that is, by Theorem 4.16, if we know $G(-\mathrm{e}^{-\pi\sqrt{n}})$, then we can also evaluate $G(-\mathrm{e}^{-3\pi\sqrt{n}})$ or $G(-\mathrm{e}^{-\pi\sqrt{n}/3})$.

(vi) Theorem 4.24 implies that if we know B_n, then we can compute B_{9n}, $B_{n/9}$, B_{25n}, $B_{n/25}$, B_{49n}, or $B_{n/49}$, that is, by Theorem 4.21, if we know $G(-\mathrm{e}^{-\pi\sqrt{n/3}})$, then we can also evaluate $G(-\mathrm{e}^{-\pi\sqrt{3n}})$, $G(-\mathrm{e}^{-\pi\sqrt{n/27}})$, $G(-\mathrm{e}^{-5\pi\sqrt{3n}})$, $G(-\mathrm{e}^{-\pi\sqrt{n/75}})$, $G(-\mathrm{e}^{-7\pi\sqrt{3n}})$, $G(-\mathrm{e}^{-\pi\sqrt{n/147}})$, or $G(-\mathrm{e}^{-\pi\sqrt{n/147}})$.

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References

[1] C. Adiga, T. Kim, M.S. Mahadeva Naika, H.S. Madhusudhan, On Ramanujan's cubic continued fraction and explicit evaluations of theta-functions, preprint.
[2] C. Adiga, K.R. Vasuki, M.S. Mahadeva Naika, Some new explicit evaluations for Ramanujan's cubic continued fraction, New Zealand J. Math. 31 (2002).
[3] N.D. Baruah, Modular equations for Ramanujan's cubic continued fraction, J. Math. Anal. Appl. 268 (2002) 244–255.
[4] B.C. Berndt, Ramanujan's Notebooks, Part III, Springer, New York, 1991.
[5] B.C. Berndt, Ramanujan's Notebooks, Part IV, Springer, New York, 1994.
[6] B.C. Berndt, Ramanujan's Notebooks, Part V, Springer, New York, 1998.
[7] B.C. Berndt, H.H. Chan, S.-Y. Kang, L.-C. Kang, A certain quotient of eta-functions found in Ramanujan's lost notebook, Pacific J. Math. 202 (2) (2002) 267–304.

[8] B.C. Berndt, H.H. Chan, L.-C. Zhang, Ramanujan's class invariants and cubic continued fraction, Acta Arith. 73 (1995) 67–85.
[9] B.C. Berndt, R.A. Rankin, Ramanujan: Letters and Commentary, Affiliated East-West Press, New Delhi, 1995.
[10] H.H. Chan, On Ramanujan's cubic continued fraction, Acta Arith. 73 (1995) 343–355.
[11] H.H. Chan, W.-C. Liaw, On Russell-type modular equations, Canad. J. Math. 52 (2000) 31–46.
[12] W.-C. Liaw, Contributions to Ramanujan's theories of modular equations, Ramanujan type series for $1/\pi$, and partitions, Ph.D. Thesis, University of Illinois at Urbana-Champaign, 1999.
[13] K.G. Ramanathan, On Ramanujan's continued fraction, Acta Arith. 43 (1984) 209–226.
[14] K.G. Ramanathan, On Some Theorems Stated by Ramanujan, Number Theory and Related Topics, Tata Institute of Fundamental Research, Oxford University Press, Bombay, 1989, pp. 151–160.
[15] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.
[16] K.R. Vasuki, K. Shivashankara, On Ramanujan's continued fractions, Ganita 53 (2002) 81–88.
[17] J. Yi, Construction and Application of Modular Equations, Ph. D. Thesis, University of Illinois, USA, 2000.
[18] J. Yi, Evaluations of the Rogers–Ramanujan continued fraction $R(q)$ by modular equations, Acta Arith. 97 (2001) 103–127.