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Zeta distributions and boundary values of Poisson transforms

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Abstract

Let G be the conformal group of a non-Euclidean Jordan algebra and let P be the maximal parabolic subgroup canonically associated to G. Standard intertwining operators between spherical degenerate principal series induced from P determine Zeta distributions. In this article, we obtain a functional equations for Zeta distributions by considering boundary values of Poisson transforms. We relate the constant occurring in the Zeta functional equation to that occurring in the functional equation of Wallach's Generalized Jacquet functionals.

1. Introduction

When s is a complex number with $Re(s) > -1$, then the formula

$$Z(f,s) = \int_{\mathbf{R}} f(x)|x|^s \, dx \tag{1.1}$$

defines a tempered distribution. We call this distribution Tate's Zeta distribution. This Zeta distribution can be regularized so that if Γ denotes the gamma function on $\mathbf{C}$, then $\frac{|x|^s}{\Gamma\left(\frac{s+1}{2}\right)}$ is a well-defined distribution holomorphic on s (see [9]). The Fourier

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transform of Tate's distribution is given by the following formula:

$$Z(f, s-1) = \frac{\pi^{-s+\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} Z(\hat{f}, -s), \tag{1.2}$$

where f is an arbitrary Schwartz function and $\hat{f}$ is the Fourier transform of the function f.

This family of distributions is intimately related to intertwining operators, $\tilde{A}(s)$, between spherical principal series of $SL(2, \mathbf{R})$. Thus, it is not surprising that Eq. (1.2) encodes representation theoretical information. For example, when s is a real number the image of $\tilde{A}(s)$ admits a G-invariant hermitian form $\langle , \rangle_s$. If the image of $\tilde{A}(s)$ is unitary, i.e if $\langle , \rangle_s$ is positive definite, then $\frac{|x|^{s-1}}{\Gamma\left(\frac{s}{2}\right)}$ is a distribution of positive type. By Bochner–Schwartz theorem $\frac{|x|^{s-1}}{\Gamma\left(\frac{s}{2}\right)}$ is of positive type if and only if $\frac{|x|^{-s}}{\Gamma\left(\frac{-s+1}{2}\right)}$ is a positive measure, [9, Vol. 4, p. 157]. Thus, by identity (1.2) and Bochner's theorem, unitarity can only occur when $\frac{|x|^{-s}}{\Gamma\left(\frac{-s+1}{2}\right)}$ is a positive measure. The constant $\frac{\pi^{-s+\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}$ in Eq. (1.2) also has representation theoretical meaning. It coincides with the constant in the functional equation satisfied by Jacquet integrals, see [22, p. 428] and Section 2 of this paper.

In 1972, Godement and Jacquet [10] generalized Tate's results obtaining functional equations for Zeta distributions on matrices $M_{n\times n}(F)$, where F is either the real field $\mathbf{R}$, the complex field $\mathbf{C}$ or the division algebra of quaternionic numbers $\mathbf{H}$. The study of Zeta distributions and their functional equations in the setting of prehomogeneous vector spaces has a long history. See for example [3,4,15,16,18]. Faraut and Korányi in [8] and Clerc in [6] obtained functional equations for Zeta distributions associated to representations of a Jordan algebra by computing boundary values of Poisson transforms.

Motivated by the applications to representation theory, in this paper we adapt the beautiful ideas in [8, Chapter XVI, Section 4] to give functional equations for Zeta distributions associated to "conformal groups" of non-Euclidean Jordan algebras. For this class of groups, Generalized Jacquet integrals, see [21], satisfy a functional equation. As in the $SL(2, \mathbf{R})$ example, we show that the constant occurring in both functional equations, the one for the Zeta distributions and that for the Jacquet Integrals, are the same.

I give an outline of the paper. Section 2 reviews Faraut–Korányi's techniques, to derive the functional equation for Zeta distributions on $M_{n\times n}(\mathbf{R})$. This is our motivating example. The results of this section are not new and they are included here for expository purposes. Section 3 reviews notation and particular characteristics of conformal groups of non-Euclidean Jordan algebras. In Section 4 the Zeta distributions to be studied in this paper are introduced. Section 5 concerns Jacquet integrals. Section 6 contains results on Poisson transforms and their boundary values. In Section 7, I prove the main theorem of the paper.

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2. The $GL(2n, \mathbf{R})$ case

In this section, we derive a functional equation for a Zeta distribution associated to $GL(2n, \mathbf{R})$. The arguments in this section extend readily to the case $G = GL(2n, F)$ with $F = \mathbf{C}$ or $\mathbf{H}$. The techniques and results of this section are essentially contained in [8, pp. 356–362] and they constitute our motivating example.

Let $G = GL(2n, \mathbf{R})$ and let $P = LN$ the maximal middle parabolic subgroup with $L \equiv GL(n, \mathbf{R}) \times GL(n, \mathbf{R})$ and $\mathfrak{n} = Lie(N) \equiv M_{n\times n}(\mathbf{R})$ the set of $n \times n$ matrices over $\mathbf{R}$. Denote by $\bar{P} = L\bar{N}$ the opposite parabolic subgroup. There is a family of degenerate principal series of interest to us. This family consists of normalized induced representations from one dimensional characters of P and left $GL(2n, \mathbf{R})$ action. More precisely, for s a complex number, we denote by χ_s the character

$$\chi_s\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = |\det(a)|^s.$$

The induced representations in question are

$$Ind_P^G(s) = \{\varphi: G \to \mathbf{C} \mid \varphi \text{ is smooth and } \varphi(gp) = \chi_{s+n}(p^{-1})\varphi(g),\ p \in P\}.$$

By the Gelfand–Naimark decomposition, functions in $Ind_P^G(s)$ are determined by their restriction to $\bar{N} \equiv M_{n\times n}(\mathbf{R})$. Thus, $Ind_P^G(s)$ can be identified with

$$I(s) = \{f \in C^\infty(\overline{\mathfrak{n}}): f(Y) = \varphi(\bar{n}_Y), \quad \text{for some } \varphi \in\ Ind_P^G(s)\}.$$

Here $\bar{n}_Y = \exp(Y)$. The group action can be explicitly described in this picture. In particular, for $\ell \in L$ and $\bar{n}_Y \in \bar{N}$

$$(\ell \cdot f)(Y_1) = \chi_{s+n}(\ell) f(Ad(\ell^{-1}) \cdot Y_1),$$

$$(\bar{n}_Y \cdot f)(Y_1) = f(Y_1 - Y).$$

Let w_0 be an element in $SO(2n)$ for which $Ad(w_0)\mathfrak{n} = \overline{\mathfrak{n}}$. We define functions on dense open subsets of $\overline{\mathfrak{n}}$ and $\mathfrak{n}$ by

$$\overline{\nabla}(Y) \equiv \chi_1(\ell(w_0\bar{n}_Y)),\ Y \in \overline{\mathfrak{n}}$$

and

$$\nabla(X) \equiv \overline{\nabla}(\theta(X)),\ X \in \mathfrak{n},$$

where $(\ell(w_0\bar{n}_Y))$ is the L-component in the Gelfand–Naimark decomposition of $w_0\bar{n}_Y$ and θ is the Cartan involution. If we identify $Y\in\bar{\mathfrak{n}}$ with a matrix $y\in M_{n\times n}(\mathbf{R})$, then $\overline{\nabla}(Y)=|\det(y)|$.

For each s with $Re(s)>(n-1)$ there is a G-intertwining operator

$$\tilde{A}_s : I(s)\to I(-s),$$

which is given by a convergent integral. The form of the operator that we will use, and that can be easily derived from ([11, pp. 183, 200]), is

$$(\tilde{A}_s f)(Y)=\int_{\bar{\mathfrak{n}}}\overline{\nabla}(Y_1)^{s-n} f(Y+Y_1)\,dY_1. \tag{2.1}$$

The integral converges for $Re(s)>(n-1)$.

Lemma 2.2. *For $s\in\mathbf{C}$ with $Re(s)>-1$ the function $\nabla(X)^s$ (resp. $\overline{\nabla}(Y)^s$) is a locally L_1 function on $\mathfrak{n}$ (resp. $\bar{\mathfrak{n}}$) and defines a tempered distribution.*

Proof. See for example [1, Lemma 3.14]. □

By Lemma 2.2, there are tempered distributions defined by the integrals

$$\mathbf{Z}(h,s)=\int_{\mathfrak{n}} h(X)\nabla(X)^s\,dX \quad \text{for } h\in\mathcal{S}(\mathfrak{n}) \tag{2.3}$$

and

$$\overline{\mathbf{Z}}(f,s)=\int_{\bar{\mathfrak{n}}} f(Y)\overline{\nabla}(Y)^s\,dY \quad \text{for } f\in\mathcal{S}(\bar{\mathfrak{n}}).$$

Here $\mathcal{S}(\mathfrak{n})$ (resp. $\mathcal{S}(\bar{\mathfrak{n}})$) denotes the space of Schwartz functions on $\mathfrak{n}$ (resp. $\bar{\mathfrak{n}}$). These are our Zeta distributions. If $f\in\mathcal{S}(\bar{\mathfrak{n}})$ and $Re(s)>n-1$ we have

$$(\tilde{A}_s f)(Y)=\overline{\mathbf{Z}}(\bar{n}_Y f, s-n). \tag{2.4}$$

Since intertwining operators admit meromorphic continuations to all of $\mathbf{C}$, so do the Zeta distributions.

Now, we consider the Fourier transform of the Zeta distributions. Before defining the Fourier transform, we need to specify the pairing between $\mathfrak{n}$ and $\bar{\mathfrak{n}}$. Let B denote the Killing form on $\mathfrak{g}=\mathfrak{gl}(2n,\mathbf{R})$. Set

$$\langle\,,\rangle\equiv\frac{1}{4n}B,$$

giving a non-degenerate pairing between $\mathfrak{n}$ and $\bar{\mathfrak{n}}$. Define the Fourier transform of a function by

$$\hat{h}(Y)=\int h(X)e^{-2\pi i\langle Y,X\rangle}\,dX \quad \text{for } h\in\mathcal{S}(\mathfrak{n})$$

and

$$\hat{f}(X) = \int f(Y) e^{-2\pi i \langle Y, X \rangle} \, dY \quad \text{for } f \in \mathcal{S}(\bar{\mathfrak{n}}).$$

The functional equation of the Zeta distribution is given in the following theorem.

Theorem 2.5. *Let* $s \in \mathbf{C}$ *and* $f \in \mathcal{S}(\bar{\mathfrak{n}})$. *As meromorphic functions*

$$\frac{\pi^{\frac{ns}{2}}}{\Gamma_n(s)} \mathbf{Z}(\hat{f}, s-n) = \frac{\pi^{\frac{n}{2}(-s+n)}}{\Gamma_n(-s+n)} \overline{\mathbf{Z}}(f, -s), \tag{2.6}$$

where

$$\Gamma_n(s) \equiv \prod_{j=0}^{n-1} \Gamma\left(\frac{s-j}{2}\right).$$

This theorem is known, [10]. Also, compare (2.5) with the equation in [8, p. 359]. As mentioned before, we are interested in the techniques used in [8] to derive similar formulas. The rest of this section sketches the arguments in [8] that will be generalized later in the paper.

Step 1: We start by expressing the intertwining operators $\tilde{A}(s)$ as boundary values of Poisson transforms. If $\varphi \in Ind_P^G(s)$ and $g \in GL(2n, \mathbf{R})$, then the Poisson transform is given by

$$\mathcal{P}_s(\varphi)(g) = \int_K \varphi(gk) \, dk.$$

We convert the integral over K into an integral over $\bar{\mathfrak{n}}$ using [11, Eq. (5.25)]. Thus, we have

$$\mathcal{P}_s(\varphi)(x) = \int_{\bar{\mathfrak{n}}} \chi_{\frac{s-n}{2}}(\ell(\bar{n}_Y^t (x^{-1})^t x^{-1} \bar{n}_Y)) \varphi(\bar{n}_Y) \, dY,$$

where $(x^{-1})^t$ is the transpose of the matrix x^{-1}.

Let a_t be the block diagonal matrix with diagonal blocks $e^t I_{n\times n}$ and $e^{-t} I_{n\times n}$ Set $x = \bar{n}_{Y_0} a_t$. Following for example [11, p. 199], we have the following known result.

Lemma 2.7. *If* $\varphi \in I(s)$ *and* $Re(s) > (n-1)$, *then*

$$\lim_{t\to+\infty} e^{(-ns+n^2)t} \mathcal{P}_s(\varphi)(\bar{n}_{Y_0} a_t) = \tilde{A}_s(\varphi)(Y_0).$$

In particular, if $\varphi(\bar{n}_Y) = \hat{f}(Y)$ with $\hat{f} \in \mathscr{S}(\bar{\mathfrak{n}})$ and if $Re(s) > (n-1)$, then

$$\overline{\mathbf{Z}}(\hat{f}, s-n) = \lim_{t \to +\infty} e^{(-ns+n^2)t} \int_{\bar{\mathfrak{n}}} \chi_{\frac{s-n}{2}}(\ell(\bar{n}_Y^t a_t^{-2} \bar{n}_Y)) \hat{f}(Y)\, dY.$$

Remark 2.8. If we identify $Y \in \bar{\mathfrak{n}}$ with a matrix $y \in M_{n\times n}(\mathbf{R})$, then the Poisson Kernel in matrix notation is given by

$$\chi_{\frac{s-n}{2}}(\ell(\bar{n}_Y^t a_t^{-2} \bar{n}_Y)) = e^{t(sn-n^2)} |\det[e^{-4t} I_{n\times n} + yy^t]|^{\frac{s-n}{2}}.$$

Step 2. Next, we compute the Fourier transform of the Poisson Kernel in the sense of distribution. See [8, Proposition XVI.3.2] and the first part of the proof of Theorem XVI.4.3 in [8]. Such Fourier transform is given in terms of Generalized Bessel functions $K_s(\cdot,\cdot)$ defined in [8, p. 355].

For $X \in \mathfrak{n}$, define for $Re(s) < -(n-1)$

$$\mathscr{J}_{-s,X}(\tau_{a_t}\mathbf{1}) = \frac{e^{t(sn-n^2)} \pi^{\frac{n^2}{2}}}{\Gamma_n(n-s)} K_{\frac{-s}{2}}(e^{-4t} I_{n\times n},\ \pi^2 x x^t),$$

where $x \in M_{n\times n}(\mathbf{R})$ corresponds to X via the identification $\mathfrak{n} \equiv M_{n\times n}(\mathbf{R})$. The function $\mathscr{J}_{-s,X}(\tau_{a_t}\mathbf{1})$ extends to an entire function on s. It is useful to note that for s real, $K_s(\cdot,\cdot)$ is convex.

Proposition 2.9 (Faraut and Koranyi [8, p. 356]). *If $f \in \mathscr{S}(\mathfrak{n})$ and $Re(s) < -(n-1)$, then*

$$e^{(-ns+n^2)t} \mathscr{P}_s(\hat{f})(a_t) = e^{t(-sn+n^2)} \int_{\mathfrak{n}} \mathscr{J}_{-s,X}(\tau_{a_t}\mathbf{1}) f(X)\, dX.$$

Theorem 2.10. *Let a be a real number so that $a > (n-1)$, and let $f \in \mathscr{S}(\mathfrak{n})$. Then, there exists $k > 0$ so that on $\{s \in \mathbf{C} : Re(s) < a\}$*

$$e^{(-ns+n^2)t} \mathscr{P}_s(\widehat{\nabla^{2k}()f})(a_t) = e^{t(-sn+n^2)} \int_{\mathfrak{n}} \mathscr{J}_{-s,X}(\tau_{a_t}\mathbf{1}) \nabla(X)^{2k} f(X)\, dX.$$

Proof. Using estimates for the generalized Bessel function in [8, p. 355], we can verify that $\mathscr{J}_{-s,X}(\tau_{a_t}\mathbf{1}) f(X)$ is integrable when $f \in \mathscr{S}(\mathfrak{n})$ and $Re(s) < -(n-1)$. Moreover, the results in [8, p. 355] also lead to the estimate (uniform on t)

$$e^{t(-sn+n^2)} |\mathscr{J}_{-s,X}(\tau_{a_t}\mathbf{1})| \leqslant \frac{\Gamma_n(Re(s))}{\Gamma_n(n-Re(s))} \nabla(X)^{-Re(s)} \pi^{-nRe(s)+\frac{n^2}{2}}$$

valid when $Re(s) > n-1$. We choose $k > 0$ big enough so that

$$\int_{\mathfrak{n}} \nabla(X)^{-Re(s)+2k}(X) f(X)\, dX$$

converges for all s on the strip $(n-1) < Re(s) < a$. For such a k and for $f \in \mathcal{S}(\mathfrak{n})$, the function $\mathcal{J}_{-s,X}(\tau_{a_t}\mathbf{1})\nabla(X)^{2k} f(X)$ is integrable when s is in $\{s \in \mathbf{C} : (n-1) < Re(s) < a\} \cup \{s \in \mathbf{C} : Re(s) < -(n-1)\}$. Next, we use convexity of the Bessel function to conclude integrability on the entire region $Re(s) < a$. On the other hand, an easy application of Morera's theorem shows that

$$\int_{\mathfrak{n}} \mathcal{J}_{-s,X}(\tau_{a_t}\mathbf{1})\nabla(X)^{2k} f(X)\, dX$$

is holomorphic on the region $Re(s) < a$. Now the theorem follows from Proposition 2.9. □

A combination of the result in Step 1 and Theorem 2.10 yield

$$\overline{\mathbf{Z}}(\widehat{\nabla^{2k}()f}, s-n) = \lim_{t \to +\infty} e^{t(-sn+n^2)} \int_{\mathfrak{n}} \mathcal{J}_{-s,X}(\tau_{a_t}\mathbf{1})\nabla(X)^{2k} f(X)\, dX. \tag{2.11}$$

Step 3. For $s \in \mathbf{C}$ with $Re(s) > n-1$ and away from the poles of $\Gamma_n(n-s)$, the explicit formula for the Bessel function $K_{\frac{-s}{2}}(\cdot,\cdot)$ in [8, p. 355] yield

$$\lim_{t \to +\infty} e^{t(-sn+n^2)} \mathcal{J}_{-s,X}(\tau_{a_t}\mathbf{1}) = \frac{\Gamma_n(s)}{\Gamma_n(n-s)} \nabla(X)^{-s} \pi^{-ns+\frac{n^2}{2}}.$$

Using dominated convergence theorem, we compute the limit in (2.11) to obtain

$$\overline{\mathbf{Z}}(\widehat{\nabla^{2k}()f}, s-n) = \frac{\Gamma_n(s)}{\Gamma_n(n-s)} \pi^{-ns+\frac{n^2}{2}} Z(f, -s+2k).$$

Step 4. The fourth step consists on studying the effect of the change $f \to \nabla(X)^{2k} f$ on the left-hand side of (2.11).

If $P(X) \equiv \nabla(X)^2$, then let $P(\partial_X)$ be the constant coefficient differential operator characterized by $P(\partial_X)e^{\langle X,Y\rangle} = \overline{\nabla}(Y)^2 e^{\langle X,Y\rangle}$. Then, compare with [8, Proposition XVI.4.1], we have

$$\bar{P}(\partial_X)^k \overline{\nabla}(X)^t = (-1)^{nk} 2^{2nk} \frac{\Gamma_n(-t+2k)\Gamma_n(t+n)}{\Gamma_n(-t)\Gamma_n(t+n-2k)} \overline{\nabla}(X)^{t-2k}. \tag{2.12}$$

On the other hand,

$$\widehat{\nabla()^{2k} f}(X) = (-1)^{kn}(2\pi)^{-2kn} \bar{P}(\partial_X)^k \hat{f}(X). \tag{2.13}$$

From (2.12) and (2.13), it follows that

$$\overline{\mathbf{Z}}(\widehat{\nabla()^{2k}f}, s-n) = \pi^{-2kn}\frac{\Gamma_n(-s+2k+n)\Gamma_n(s)}{\Gamma_n(-s+n)\Gamma_n(s-2k)}\overline{\mathbf{Z}}(\hat{f}, s-2k-n).$$

Step 5. Combining steps 1–4, at least on some open set, we have

$$\overline{\mathbf{Z}}(\hat{f}, (s-2k)-n) = \pi^{\frac{n^2}{2}-n(s-2k)}\frac{\Gamma_n(s-2k)}{\Gamma_n(-s+2k+n)}Z(f, -s+2k).$$

Now the change of variables $\sigma = s-2k$ followed by analytic continuation gives the result in Theorem 2.5.

3. Structure

Let G be a real reductive Lie group with Cartan involution θ and maximal compact subgroup $K = G^\theta$. As is customary we write the Lie algebra of G (resp. K) as $\mathfrak{g}$ (resp. $\mathfrak{k}$). The Cartan involution determines a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$. As mentioned in the introduction we will only be considering a class of groups associated to non-Euclidean Jordan algebras. Each simple non-Euclidean Jordan algebra occurs as the abelian nilradical of a maximal parabolic subalgebra of a reductive Lie algebra $\mathfrak{g}$. There is a reductive group G, with Lie algebra $\mathfrak{g}$, satisfying the following conditions.

Assumption 3.1. *G contains a parabolic subgroup $P = LN$ (a Levi decomposition) such that*

(1) *P and its opposite parabolic $\bar{P} = L\bar{N}$ are G-conjugate,*
(2) *N is abelian, and*
(3) *L has only one open orbit in $\mathfrak{n}$.*

For a given simple non-Euclidean Jordan algebra the choice of G as above is not unique. The list of groups we work with is given in [7] and in Table 1 in Section 8 of this paper. Note that our tables associate $GL(2n, \mathbf{R})$ to the Jordan algebra $M_{n\times n}(\mathbf{R})$ and $O(2n, 2n)$ to $Skew_{2n\times 2n}(\mathbf{R})$. The groups $SL(2n, \mathbf{R})$ and $SO(2n, 2n)_0$ do not satisfy assumption (3).

Following [13] there is an abelian subalgebra $\mathfrak{b}$ of $\mathfrak{l} \cap \mathfrak{s}$ with the following properties:

(1) There are commuting copies of $\mathfrak{sl}(2, \mathbf{R})$ in $\mathfrak{g}$ spanned by $\{F_j, H_j, E_j\}$, a standard basis in the sense that

$$\theta(E_j) = -F_j \quad \text{and} \quad \theta(H_j) = -H_j,$$

$$[E_j, F_j] = H_j, \;\; [H_j, E_j] = 2E_j \quad \text{and} \quad [H_j, F_j] = -2F_j$$

with $E_j \in \mathfrak{n}$, $F_j \in \bar{\mathfrak{n}}$ and $\mathfrak{b} = \sum_{j=1}^{n} \mathbf{R}H_j$.

(2) For $\varepsilon_k\left(\sum_{j=1}^n a_j H_j\right) \equiv a_k$, the $\mathfrak{b}$-roots in $\mathfrak{g}$, $\mathfrak{l}$ and $\mathfrak{n}$ are

$$\Sigma(\mathfrak{g},\mathfrak{b}) = \{\pm(\varepsilon_j - \varepsilon_k) \colon 1 \leqslant j < k \leqslant n\} \cup \{\pm(\varepsilon_j + \varepsilon_k) \colon 1 \leqslant j, k \leqslant n\},$$

$$\Sigma(\mathfrak{l},\mathfrak{b}) = \{\pm(\varepsilon_j - \varepsilon_k) \colon 1 \leqslant j < k \leqslant n\} \quad \text{and}$$

$$\Sigma(\mathfrak{n},\mathfrak{b}) = \{\varepsilon_j + \varepsilon_k \colon 1 \leqslant j,\ k \leqslant n\}.$$

For each G the roots in $\mathfrak{n}$ have just two multiplicities. We denote the multiplicity of the short roots by $2d$ and that of the long roots by $e+1$. We choose our positive root system to be $\Sigma^+(\mathfrak{g},\mathfrak{b}) = \{\varepsilon_j - \varepsilon_k \colon 1 \leqslant j < k \leqslant n\} \cup \Sigma(\mathfrak{n},\mathfrak{b})$.

Definition 3.2. Taking $n = \dim(\mathfrak{b})$ as above, we make the following definitions:

(1) The rank of $\mathfrak{n}$ is n.
(2) $\Lambda_0 \equiv \sum_{j=1}^n \varepsilon_j$.
(3) χ is the positive character of L with differential $2d\Lambda_0$.
(4) $m \equiv \dim(\mathfrak{n})$.

Lemma 3.3. *Set* $\rho(\mathfrak{n}) \equiv \frac{1}{2}\sum_{\alpha \in \Sigma(\mathfrak{n},\mathfrak{b})} \alpha$. *Then*

(1) $m = n(d(n-1) + (e+1))$,
(2) $\rho(\mathfrak{n}) = \frac{m}{n}\Lambda_0$,
(3) $|\det(Ad(\ell)|_{\mathfrak{n}})| = \chi(\ell)^{\frac{m}{dn}}$.

Let B denote the killing form of $\mathfrak{g}$. Set

$$\langle\, , \rangle \equiv \frac{n}{4m} B,$$

giving a non-degenerate pairing between $\mathfrak{n}$ and $\bar{\mathfrak{n}}$.

Our assumption that L acts on $\mathfrak{n}$ with only one open orbit plays an important role in the paper. If we set

$$X_n \equiv E_1 + \cdots + E_n,$$

then $ad(X_n) : \mathfrak{l} \to \mathfrak{n}$ is onto and the L-orbit $\mathcal{O}_n = L(X_n)$ is open in $\mathfrak{n}$. We write $\mathcal{O}_n = L/S_n$, where S_n is the stabilizer of X_n. The orbit $\mathcal{O}_n$ is a semisimple symmetric space, see for example [1]. Similarly, L acts on $\bar{\mathfrak{n}}$ with only one open orbit. A base point for such orbit is $Y_n = F_1 + \cdots + F_n$.

Example. If $G = O(2n,2n)$, then $L \equiv GL(2n,\mathbf{R})$ and $\mathfrak{n} \equiv Skew_{2n\times 2n}(\mathbf{R})$. The open orbit $\mathcal{O}_n$ consists of skew symmetric matrices of maximal rank and $S_n = Sp_n(\mathbf{R})$.

Let $P_0 = M_0A_0N_0$ be a minimal parabolic subgroup of G contained in P. Let $\Sigma(P_0, A_0)$ be the set of positive restricted roots of A_0. Denote by $W_G(A_0)$ the Weyl group of G with respect to A_0. For each $w \in W_G(A_0)$, write w^* its representative in K. Also, let w_M be the element in $W_L(A_0)$ such that $w_M\Sigma(P_0 \cap L, A_0) = -\Sigma(P_0 \cap L, A_0)$ and choose w_M^* in $L \cap K$.

Proposition 3.4. *Take G satisfying the hypotheses of Assumption* (3.1).

(1) *If $X \in \mathcal{O}_n$, then $ad(X): \bar{\mathfrak{n}} \to \mathfrak{g}_{\mathbf{C}}$ is injective.*
(2) *If $X \in \mathcal{O}_n$, then the character $\eta_X(Y) = e^{2\pi i\langle X,Y\rangle}$ is non-degenerate in the sense of* [21, p. 127].
(3) *If $X \in \mathcal{O}_n$ and $<Ad(w^{*-1})Ad(w_M^*)\mathfrak{n}_0 \cap \bar{\mathfrak{n}}, X> = 0$, then $w = 1$.*

Proof. See [21, p. 137, 19, p. 26]. Parts (1) and (2) are easy to prove. Assume that $Y \in \bar{\mathfrak{n}}$ is so that If $ad(X_n)Y = 0$. Then, $0 = [Y_n, [X_n, Y]] = [\Sigma H_i, Y] = -2Y$. Part (3) is [23, Lemma 8]. □

4. Zeta distributions

There is a diffeomorphism of $\bar{\mathfrak{n}} \times L \times \mathfrak{n}$ onto a dense open set in G given by $(Y, \ell, X) \to \bar{n}_Y \ell n_X$, where $\bar{n}_Y = \exp(Y)$ and $n_X = \exp(X)$. Therefore, on a dense open subset of G, there is a decomposition $g = \bar{n}_Y \ell n_X$. Furthermore, $L = MA$ where $A = \exp(\mathfrak{a})$, $\mathfrak{a} \equiv \bigcap_{j<k} ker(\varepsilon_j - \varepsilon_k)$. In particular, the L part of the decomposition has a component in A. We define $a(g) \in A$ by

$$g \in \bar{N}Ma(g)N. \tag{4.1}$$

We may define functions on dense open subsets of $\bar{\mathfrak{n}}$ and $\mathfrak{n}$ by

$$\overline{\nabla}(Y) \equiv e^{\Lambda_0(\log(a(w_0\bar{n}_Y)))}, \quad Y \in \bar{\mathfrak{n}}$$

and

$$\nabla(X) \equiv \overline{\nabla}(\theta(X)), \quad X \in \mathfrak{n}. \tag{4.2}$$

With our choice of X_n, we have $\nabla(X_n) = 1$, see [1, Lemma 3.12].

Lemma 4.3 ([1, Lemma 2.10]). $\overline{\nabla}(Y)$ (*respectively* $\nabla(X)$) *extends to a well-defined function on* $\bar{\mathfrak{n}}$ (*respectively*, $\mathfrak{n}$) *and the following hold*:

(1) $\overline{\nabla}(Y)^2$ (*respectively*, $\nabla(X)^2$) *is a homogeneous polynomial on* $\bar{\mathfrak{n}}$ (*respectively* $\mathfrak{n}$) *of degree* $2n$.
(2) $\overline{\nabla}(\ell \cdot Y) = |\det(Ad(\ell)|_{\bar{\mathfrak{n}}})|^{\frac{n}{m}}\overline{\nabla}(Y) = \chi(\ell)^{-\frac{1}{d}}\overline{\nabla}(Y)$ *for* $\ell \in L$ *and* $Y \in \bar{\mathfrak{n}}$, *and* $\nabla(\ell \cdot X) = |\det(Ad(\ell)|_{\mathfrak{n}})|^{\frac{n}{m}}\nabla(X) = \chi(\ell)^{\frac{1}{d}}\nabla(X)$ *for* $\ell \in L$ *and* $X \in \mathfrak{n}$.

Examples. The functions ∇ are closely related to the Jordan algebra determinant functions, see Table 2 in Section 8. I should also mention that if Q is the basic relative invariant of the prehomogeneous vector space $(L, \mathfrak{n})$, then $\nabla = |Q|$ when $e = 0$ and ∇^2 is a multiple of Q in other cases.

The functions $\nabla(X)^s$ and $\overline{\nabla}(Y)^s$ are locally L_1 functions provided $Re(s) > -(e+1)$, [1, Lemma 3.14]. Therefore, tempered distributions are defined by the integrals

$$\mathbf{Z}(h,s) = \int_{\mathfrak{n}} h(X)\nabla(X)^s\, dX \quad \text{for } h \in \mathscr{S}(\mathfrak{n})$$

and

$$\overline{\mathbf{Z}}(f,s) = \int_{\overline{\mathfrak{n}}} f(X)\overline{\nabla}(X)^s\, dX \quad \text{for } f \in \mathscr{S}(\overline{\mathfrak{n}}).$$

Note that in the range $Re(s) > -(e+1)$ both expressions are complex analytic functions of s. We will see that there is a meromorphic continuation to all of $\mathbf{C}$ and a functional equation relating the two distributions via the Fourier transform. The fact that there is a meromorphic continuation is well known [18]. We include some details of the proof since we will need several formulas that arise.

By Lemma 4.3, $P(X) \equiv \nabla(X)^2$ and $\bar{P}(Y) \equiv \overline{\nabla}(Y)^2$ are polynomials. They define constant coefficient differential operators characterized by

$$P(\partial_X)e^{\langle X,Y\rangle} = \bar{P}(Y)e^{\langle X,Y\rangle} \quad \text{and} \quad \bar{P}(\partial_Y)e^{\langle X,Y\rangle} = P(X)e^{\langle X,Y\rangle}.$$

There is a polynomial $b(s)$ [2] so that

$$P(\partial_X)\nabla(X)^s = b(s)\nabla(X)^{s-2} \quad \text{and} \quad \bar{P}(\partial_Y)\overline{\nabla}(Y)^s = b(s)\overline{\nabla}(Y)^{s-2}.$$

In particular, for $b_k(s) \equiv b(s)b(s-2)b(s-4)\cdots b(s-2(k-1))$,

$$P(\partial_X)^k\nabla(X)^s = b_k(s)\nabla(X)^{s-2k} \quad \text{and} \quad \bar{P}(\partial_Y)^k\overline{\nabla}(Y)^s = b_k(s)\overline{\nabla}(y)^{s-2k}.$$

It follows that,

$$\mathbf{Z}(P(\partial_Y)^k h, s) = b_k(s)\mathbf{Z}(h, s-2k) \tag{4.4}$$

and

$$\overline{\mathbf{Z}}(\bar{P}(\partial_Y)^k f, s) = b_k(s)\overline{\mathbf{Z}}(f, s-2k)$$

for $Res \gg 0$. Since the left hand side is analytic for $Re(s) > -(e+1)$, $\mathbf{Z}(h,s)$ and $\overline{\mathbf{Z}}(f,s)$ continue to meromorphic functions on $Re(s) > -(e+1) - 2k$ for any k.

5. Jacquet integrals

For each $s\in\mathbf{C}$ consider the normalized principal series representation

$$Ind_P^G(s)=\{\varphi\colon G\to\mathbf{C}\mid\varphi \text{ is smooth and}$$
$$\varphi(gman)=e^{-(s+\frac{m}{n})\Lambda_0(\log(a))}\varphi(g),\ man\in P=MAN\},$$

where G acts by left translation. We endow the space $Ind_P^G(s)$ with the C^∞ topology. We note that the restriction $f\to f|_K$ defines an isomorphism of topological vector spaces between $Ind_P^G(s)$ and the space $Ind_{K\cap L}^K(1)$ consisting on smooth functions from $K\to\mathbf{C}$ so that $f(km)=f(k)$ for $m\in M\cap K$ with the smooth topology. The inverse map associates to $f\in Ind_{K\cap L}^K(1)$ the function $f_{P,s}(kman)=e^{-(s+\frac{m}{n})\Lambda_0(\log(a))}f(k)$ where $k\in K$, $ma\in L$ and $n\in N$. Under the identification $Ind_P^G(s)\equiv Ind_{K\cap L}^K(1)$ the action of G is given by $(gf)(k)=f_{P,s}(g^{-1}k)$.

This principal series representation may be realized in the non-compact picture as smooth functions on $\bar{\mathfrak{n}}$ as follows. Write $\bar{n}_Y\equiv\exp(Y)$ for $Y\in\bar{\mathfrak{n}}$. For $\varphi\in Ind_P^G(s)$ set $f(Y)=\varphi(\bar{n}_Y)$, $Y\in\bar{\mathfrak{n}}$. Then $Ind_P^G(s)$ may be identified with

$$I(s)=\{f\in C^\infty(\bar{\mathfrak{n}}):f(Y)=\varphi(\bar{n}_Y)\quad\text{for some } \varphi\in Ind_P^G(s)\}.$$

Since $\bar{N}P$ is dense in G and any $g\in\bar{N}P$ has a unique decomposition as

$$g=\bar{n}(g)m(g)a(g)n(g)\in\bar{N}MAN,$$

the G-action is given by

$$(g\cdot f)(Y)=e^{-(s+\frac{m}{n})\Lambda_0(\log a(g^{-1}\bar{n}_Y))}f(\log(\bar{n}(g^{-1}\bar{n}_Y))).$$

Proposition 5.1. *Let η_X be a non-degenerate character in the sense of* [21, p. 127]. *Then, the integral*

$$\mathscr{J}_{X,s}(f)=\int_{\bar{\mathfrak{n}}}\eta_X(\bar{n}_Y)f_{P,s}(\bar{n}_Y)\,dY \tag{5.2}$$

converges absolutely and uniformly in compacta of $\{s\in\mathbf{C}: Res>d(n-1)\}$ for all $f\in Ind_{K\cap L}^K(1)$.

Proof. It is enough to note that if $f_{P,s}\in Ind_P^G(s)$ and $Y\in\bar{\mathfrak{n}}$, then

$$|f_{P,s}(\bar{n}_Y)|\leqslant Ce^{-(Re(s)+\frac{m}{n})\Lambda_0(H(\bar{n}_Y))},$$

where $\bar{n}_Y=\kappa(\bar{n}_Y)e^{H(\bar{n}_Y)}m(\bar{n}_Y)n(\bar{n}_Y)\in KMAN$. Thus, smooth functions in $Ind_P^G(s)$ are integrable if the function $e^{-(Re(s)+\frac{m}{n})\Lambda_0(H(\bar{n}_Y))}$ is integrable. By [1, Lemma 3.15] this is the case when $Re(s)>d(n-1)$. $\square$

If follows from Proposition 5.1 that $\mathscr{J}_{X,s}$ defines a continuous linear functional on $Ind_{K\cap L}^{K}(1)$. Denote by $(Ind_P^G(s))'$ the continuous dual of $Ind_P^G(s)$. Let the space of Whittaker vectors be

$$W_X(Ind_P^G(s)) = \{T\in(Ind_P^G(s))': T(\bar{n}f) = \eta_X(\bar{n})T(f) \quad \text{for } \bar{n}\in\bar{N} \text{ and } f\in Ind_P^G(s)\}. \tag{5.3}$$

Under the identification $Ind_P^G(s) \equiv Ind_{K\cap L}^{K}(1)$, for $Re(s) > d(n-1)$, the generalized Jacquet functional $\mathscr{J}_{X,s}$ belongs to $W_X(Ind_P^G(s))$.

The space $W_X(Ind_P^G(s))$ has been studied, in more generality, by Wallach in [21,23]. In particular, we have

Lemma 5.4 (Wallach [23, Lemma 10]). *For* $s\in\mathbf{C}$, $\dim[W_X(Ind_P^G(s))] = 1$.

Moreover, Wallach proved that $s\to\mathscr{J}_{X,s}$ has a weakly holomorphic extension to $\mathfrak{a}_{\mathbf{C}}^*$ and that $W_X(Ind_P^G(s)) = \mathbf{C}\mathscr{J}_{X,s}$. Wallach result is

Theorem 5.5 (Wallach [21, Theorem 7.2]). *Assume that G is a Lie group satisfying the conditions listed in Assumptions* (3.1). *Assume that $X\in\mathcal{O}_n$. The Jacquet functional initially defined for $Re(s) > d(n-1)$ extends to a weakly holomorphic map from $\mathfrak{a}_{\mathbf{C}}^*$ into $(Ind_P^G(s))'$. Moreover, $W_X(Ind_P^G(s)) = \mathbf{C}\mathscr{J}_{X,s}$.*

The proof of Theorem 5.5 uses an explicit method of construction of Whittaker vectors in tensor products with finite-dimensional representations. Some aspects of this construction will be needed later in this paper. We summarize some of these results.

Let $\{Z_1, \dots, Z_m\}$ be a basis of $\bar{\mathfrak{n}}$ such that $(Z_i, Z_k) = \delta_{i,k}$ and let $W_k = [\theta X, \theta Z_k]$. Define $Q = \Sigma_i W_i(Z_i - d\eta_X(Z_i))$ and $T_j = Q + jI$. Let F be a finite-dimensional representation of G so that the space

$$F^{\bar{\mathfrak{n}}} = \{v\in F: yv = 0 \text{ for } y\in\bar{\mathfrak{n}}\}$$

is one dimensional. The Lie algebra $\mathfrak{a}$ acts on $F^{\bar{\mathfrak{n}}}$ by a linear functional $-k\Lambda_0$ where $k>0$. Denote by F^* the contragredient representation and let

$$F^{*^j} = \{v\in F^*: av = (-k+j)\Lambda_0(\log(a))v \text{ for } a\in A\}.$$

Set for $\gamma\in W_X(Ind_P^G(s))$ and $f\in F^{*^j}$

$$\Gamma_j(\gamma\otimes f) = \gamma\otimes f \quad \text{when } j=0 \text{ or } j=1,$$

$$\Gamma_j(\gamma\otimes f) = T_1\circ\cdots\circ T_r(\gamma\otimes f) \quad \text{when } j=2r \text{ or } j=2r+1.$$

Define Γ: $W_X(Ind_P^G(s)) \otimes F^* \to (Ind_P^G(s))' \otimes F^*$, by means of $\Gamma(\gamma \otimes \Sigma_j f^j) = \Sigma_j \Gamma_j(\gamma \otimes f^j)$ where $f^j \in F^{*^j}$.

The linear map Γ is defined in [21, p. 132] in more generality. We note that in this paper the roles of $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ have been reversed. We also remark that the operator Γ only depends on X and F but not on s.

Theorem 5.6. *Let $\{\xi_1, \dots, \xi_r\}$ be a basis of F^*. For $s \in \mathbf{C}$ there exists an open neighborhood U of s and functions $a_j(s)$, holomorphic on U so that on U*

$$\mathscr{I}_{X,s-k} = \Sigma_j a_j(s) \Gamma(\mathscr{I}_{X,s} \otimes \xi_j).$$

Proof. This is an immediate consequence of Wallach's proof of [23, Theorem 2, part III]. This result is also contained in the Wallach's argument in [21, pp. 143–144].

For W an $\bar{\mathfrak{n}}$-module, set $W^{\bar{\mathfrak{n}}} = \{w \in W : \bar{\mathfrak{n}} w = 0\}$. The operator Γ induces an isomorphism between $W_X(Ind_P^G(s)) \otimes F^*$ and $[(Ind_P^G(s))' \otimes F^* \otimes \mathbf{C}_{-\eta_X}]^{\bar{\mathfrak{n}}}$. See [21, Theorem 3.4] and [23, p. 12]. Thus, $\{\Gamma(\mathscr{I}_{X,s} \otimes \xi_j)\}$ form a basis of $[(Ind_P^G(s))' \otimes F^* \otimes \mathbf{C}_{-\eta_X}]^{\bar{\mathfrak{n}}}$. Denote by Z_s the kernel of the natural map T_s: $Ind_P^G(s) \otimes F^* \to Ind_P^G(s-k)$. Wallach verifies that $\dim(\Sigma_j \mathbf{C} \cdot \Gamma(\mathscr{I}_{X,s} \otimes \xi_j)|_{Z_s}) = r-1$. Hence, after relabeling, one can assume that

$$\Gamma(\mathscr{I}_{X,s} \otimes \xi_1)|_{Z_s} = \Sigma_{j \neq 1} b_j(s) \Gamma(\mathscr{I}_{X,s} \otimes \xi_j)|_{Z_s},$$

where $b_j(s)$ are function holomorphic on a neighborhood of s. The function

$$\phi(s) = \Gamma(\mathscr{I}_{X,s} \otimes \xi_1) - \Sigma_{j \neq 1} b_j(s) \Gamma(\mathscr{I}_{X,s} \otimes \xi_j)$$

vanishes on Z_s. Now, $[Ind_{P \cap K}^K(1) \otimes F^*]/Z_s \equiv Ind_{P \cap K}^K(1)$ is an isomorphism of topological spaces. Hence, ϕ is an holomorphic function on a neighborhood of s with values in $(Ind_{P \cap K}^K(1))'$. Then, Wallach shows that

(1) $\phi(s)(\bar{n}f) = \eta_X(\bar{n})\phi(s)(f)$ when $f \in Ind_{P \cap K}^K(1)$,
(2) for $f \in C_c^\infty(\bar{N})$ and $f_{P,s-k}(\bar{n}man) = e^{-(s-k+\frac{m}{n})\Lambda_0(\log(a))} f(\bar{n})$, there is a function $\alpha(s)$ holomorphic on s so that $\alpha(s)\phi(s)(f) = \int_{\bar{N}} \eta_X(\bar{n}) f(\bar{n})\, d\bar{n}$.

The proof is completed by applying Theorem 3.15 in [12]. $\square$

Let $\tilde{A}_s$ denote the standard G-intertwining operator

$$\tilde{A}_s: Ind_P^G(s) \to Ind_P^G(-s).$$

When $Re(s) > d(n-1)$, the operator is given by the following convergent integral

$$(\tilde{A}_s \varphi)(g) = \int_{\bar{\mathfrak{n}}} \varphi(g w_0 \bar{n}_{Y_1})\, dY_1. \tag{5.7}$$

In the non-compact picture, if we identify $f(Y) = \varphi(\bar{n}_Y)$, $\tilde{A}_s$ can be written as

$$(\tilde{A}_s f)(Y) = \int_{\bar{\mathfrak{n}}} \overline{\nabla}(Y_1)^{s-\frac{m}{n}} f(Y + Y_1)\, dY_1. \tag{5.8}$$

This formula can be derived by copying the argument in [11, p. 183 and 200].

Proposition 5.9. *There exists a meromorphic function $M_X(s)$ such that*

$$\mathscr{I}_{X,-s} \circ \tilde{A}_s = M_X(s) \mathscr{I}_{X,s}.$$

Proof. It follows from Theorem 5.5. □

If $X \in \mathfrak{n}$ has $\nabla(X) \neq 0$, then $X \in \mathcal{O}_n$ and we can write $X = \ell \cdot X_n$ for some $\ell \in L$. Set η_X as in Proposition 3.4. By using the integral formula (5.2) and a change of variable it is possible to relate $\mathscr{I}_{X,s}(f)$ to $\mathscr{I}_{X_n,s}(f)$. We state the result in the following Lemma.

Lemma 5.10. *If $X = \ell \cdot X_n$ and $s \in \mathbf{C}$ with $Re(s) > d(n-1)$, then*

$$\mathscr{I}_{\ell \cdot X_n, s}(f) = e^{(s-\frac{m}{n})\Lambda_0(H(\ell))} \mathscr{I}_{X_n,s}(\ell^{-1} \cdot f).$$

Proposition 5.11. *Let $b_k(s)$ be the Bernstein polynomial introduced in Section* 4. *Then,*

$$M_{X_n}(s) = (-1)^{kn} (2\pi)^{-2kn} b_k\left(s - \frac{m}{n}\right) M_{X_n}(s - 2k).$$

Proof. Let $f \in \mathscr{S}(\bar{\mathfrak{n}})$ and assume that $Re(s-2k) > d(n-1)$. Since the Schwartz space $\mathscr{S}(\bar{\mathfrak{n}})$ is contained in $I(s)$ for all s, we can assume that $\bar{P}(\partial_Y)^k f \in I(s)$. By (5.8) and (4.4), we have

$$\begin{aligned} b_k\left(s - \frac{m}{n}\right) \tilde{A}_{s-2k}(f)(Y) &= b_k\left(s - \frac{m}{n}\right) \overline{\mathbf{Z}}\left(\bar{n}_Y f, s - 2k - \frac{m}{n}\right) \\ &= \overline{\mathbf{Z}}\left(\bar{n}_Y \bar{P}(\partial_Y)^k f, s - \frac{m}{n}\right) \\ &= \tilde{A}_s(\bar{P}(\partial_Y)^k f)(Y). \end{aligned} \tag{5.12}$$

Hence, $\tilde{A}_s(\bar{P}(\partial_Y)^k f)(Y) = b_k\left(s - \frac{m}{n}\right) \tilde{A}_{s-2k}(f)(Y)$ as meromorphic functions. Assume now that $Re(s) < -d(n-1)$ so that $\tilde{A}_s(\bar{P}(\partial_Y)^k f) \in L^1(\bar{\mathfrak{n}}, dY)$, then by (5.12) we have

$$\int_{\bar{\mathfrak{n}}} \eta_{X_n}(Y) \tilde{A}_s(\bar{P}(\partial_Y)^k f)(Y)\, dY = b_k\left(s - \frac{m}{n}\right) \int_{\bar{\mathfrak{n}}} \eta_{X_n}(Y) \tilde{A}_{s-2k}(f)(Y)\, dY.$$

Since, both sides of the previous identity admit meromorphic continuations, as meromorphic functions

$$\mathscr{I}_{X_n,-s}(\tilde{A}_s(\bar{P}(\partial_Y)^k f)) = b_k\left(s - \frac{m}{n}\right)\mathscr{I}_{X_n,-s+2k}(\tilde{A}_{s-2k}(f)). \tag{5.13}$$

We complete the proof of the proposition by using the functional equation in Theorem 5.9 in both sides of (5.13). □

Examples. In Section 2, we computed $\mathscr{I}_{X,s}$ in terms of generalized Bessel functions when $G = GL(2n, \mathbf{R})$. Shimura has also studied these functions in the setting of classical groups associated to non-Euclidean Jordan algebras. See [19].

Some boundary behavior of $\mathscr{I}_{X_n,s}(f)$ will be needed. Write $L = AM$ with $A = \exp(\mathfrak{a})$. Observe that $\mathfrak{a}$ is one-dimensional, choose $H = \Sigma_1^n H_j$ and set $a_t = \exp(tH)$ with $t \in \mathbf{R}$. Denote by $\mathbf{1}_{P,s} = e^{-(s+\frac{m}{n})\Lambda_0 H(x)}$, the function in the induced representation $I(s)$ that corresponds to the trivial K-type. We compute the limit as t tends to infinity of $a_t^{(-s+\frac{m}{n})\Lambda_0}\mathscr{I}_{X_n,s}(a_t\ell^{-1}\cdot\mathbf{1})$. Before stating the result we introduce one more definition.

Definition 5.14. Assume that $s \in \mathbf{C}$ has $Re(s) > d(n-1)$. Define,

$$c_P(s) = \int_{\bar{\mathfrak{n}}} e^{-(s+\frac{m}{n})\Lambda_0(H(\exp(Y)))}\, dY.$$

It is known that $c_P(s)$ admits a meromorphic continuation to $\mathbf{C}$.

Proposition 5.15. *Assume that $s \in \mathbf{C}$ has $Re(s) > d(n-1)$ and that $\ell \in L$. Then, with the notation just introduced, we have*

$$\lim_{t\to+\infty} a_t^{(-s+\frac{m}{n})\Lambda_0}\mathscr{I}_{X_n,s}(a_t\ell^{-1}\cdot\mathbf{1}) = c_P(s)\; e^{(-s+\frac{m}{n})\Lambda_0(H(\ell))}.$$

Proof. If $Re(s) > d(n-1)$, then

$$\begin{aligned}
&\lim_{t\to+\infty} a_t^{(-s+\frac{m}{n})\Lambda_0}\mathscr{I}_{X_n,s}(a_t\ell^{-1}\cdot\mathbf{1}) \\
&= \lim_{t\to+\infty} a_t^{(-s+\frac{m}{n})\Lambda_0}\int_{Y\in\bar{\mathfrak{n}}} e^{2\pi i\langle X_n, Y\rangle} e^{-(s+\frac{m}{n})\Lambda_0(H(\ell a_t^{-1}\exp(Y)))}\, dY \\
&= \lim_{t\to+\infty} a_t^{2\frac{m}{n}\Lambda_0}\int_{Y\in\bar{\mathfrak{n}}} e^{2\pi i\langle X_n, Y\rangle} e^{-(s+\frac{m}{n})\Lambda_0(H(\ell a_t^{-1}\exp(Y)a_t))}\, dY \\
&= \lim_{t\to+\infty} \int_{Y\in\bar{\mathfrak{n}}} e^{2\pi i\langle X_n, Ad(a_t)Y\rangle} e^{-(s+\frac{m}{n})\Lambda_0(H(\ell\exp(Y)))}\, dY
\end{aligned}$$

$$= \int_{\bar{\mathfrak{n}}} e^{-(s+\frac{m}{n})\Lambda_0(H(\ell \exp(Y)))}\, dY$$

$$= e^{-(s+\frac{m}{n})\Lambda_0(H(\ell))} \int_{\bar{\mathfrak{n}}} e^{-(s+\frac{m}{n})\Lambda_0(H(\ell \exp(Y)\ell^{-1}))}\, dY$$

$$= c_P(s) e^{(-s+\frac{m}{n})\Lambda_0(H(\ell))}. \qquad \square$$

Lemma 5.16. *Assume that $X = \ell \cdot X_n$ for some $\ell \in L$. If $Re(s) > d(n-1)$ and $f \in I(s)$, then $a_t^{(-s+\frac{m}{n})\Lambda_0} \mathscr{I}_{X,s}(a_t f)$ is uniformly bounded on t. Moreover, there is a constant C independent of s so that*

$$|e^{-(s+\frac{m}{n})\Lambda_0 H(\ell)} a_t^{(-s+\frac{m}{n})\Lambda_0} \mathscr{I}_{X_n,s}(a_t \ell^{-1} \cdot f)| < C\ c_P(Re(s)) \nabla(\ell \cdot X_n)^{-Re(s)}$$

uniformly on t.

Proof. The first assertion follows easily from the computations in the proof of Lemma 5.15. The estimate is obtained from the uniform boundedness statement and Lemma 5.1. $\square$

Proposition 5.17. *Let a be a real number so that $a > d(n-1)$ and let $f \in \mathscr{S}(\mathfrak{n})$. Let $\mathcal{O}_n = L/S_n$ be the open L-orbit in $\mathfrak{n}$. Then, there exists $k > 0$ such that, for $s \in \{s \in \mathbf{C} : Re(s) < a\}$,*

$$e^{-(s+\frac{m}{n})\Lambda_0 H(\ell)} \mathscr{I}_{X_n,s}(a_t \ell^{-1} \cdot \mathbf{1}) f(\ell \cdot X_n) \nabla^{2k}(\ell \cdot X_n)$$

belongs to $L^1(L/S_n, \nabla^{\frac{m}{n}}(\ell \cdot X_n)\, d\ell)$.

Proof. First, observe that for $Re(s) < -d(n-1)$, because of the functional equation in Proposition 5.9

$$\begin{aligned} &e^{-(s+\frac{m}{n})\Lambda_0 H(\ell)} \mathscr{I}_{X_n,s}(\ell^{-1} \cdot \mathbf{1}) f(\ell \cdot X_n) \\ &\quad = \frac{c_P(s)}{M_{X_n}(s)} e^{-(s+\frac{m}{n})\Lambda_0 H(\ell)} \mathscr{I}_{X_n,-s}(\ell^{-1} \cdot \mathbf{1}) f(\ell \cdot X_n). \end{aligned} \tag{5.18}$$

Lemma 5.16 guarantees integrability of the right-hand side of Eq. (5.18). Second, choose $k = \frac{m}{n} + \eta$ with η big enough so that for s on the strip $d(n-1) < Re(s) < a$

$$\int_{\mathfrak{n}} |f(X)| \nabla(X)^{-Re(s)+2k}\, dX < \infty.$$

The estimate in Lemma 5.16 and the choice of k makes the function

$$e^{-(s+\frac{m}{n})\Lambda_0 H(\ell)} \mathscr{I}_{X_n,s}(\ell^{-1} \cdot \mathbf{1}) f(\ell \cdot X_n) \nabla^{2k}(\ell \cdot X_n)$$

integrable on $\{s\in\mathbf{C} : d(n-1) < Re(s) < a\} \cup \{s\in\mathbf{C} : -d(n-1) > Re(s)\}$. If s is on the strip $-d(n-1) \leqslant Re(s) \leqslant d(n-1)$, we take $l = 2\frac{m}{n}$, and F the finite dimensional G span of $\bigwedge^m \mathfrak{n}$ inside $\bigwedge^m \mathfrak{g}$. Then, $Re(s) + l > d(n-1)$ and on a neighborhood U of s

$$\mathscr{I}_{X,s} = \Sigma_j a_j(s) \Gamma(\mathscr{I}_{X,s+l} \otimes \xi_j)$$

with Γ, $a_j(s)$ as in Theorem 5.6 and ξ_j a basis of F^*. The functions $a_j(s)$ are holomorphic on U. The operator Γ is independent of s and it is given explicitly in terms of differential operators of the form $\Sigma_{i,j}(X, X_i')[\theta X_i', \theta Z_j] Z_j - d\eta_X(Z_j)$ with$\{Z_j\}$ a basis of $\overline{\mathfrak{n}}$ and $\{X_i'\}$ a basis of $\mathfrak{n}$. Thus,

$$\begin{aligned} &e^{-(s+\frac{m}{n})\Lambda_0 H(\ell)} f(\ell \cdot X_n) \mathscr{I}_{X_n,s}(\ell^{-1} \cdot \mathbf{1}) \nabla^{2k}(\ell \cdot X_n) \\ &\quad = \nabla^{-Re(s)+2k}(X) f(X) \mathscr{I}_{X,s}(\mathbf{1}) = \nabla^{-Re(s)+2k}(X) f(X) \Sigma_j a_j(s) \Gamma(\mathscr{I}_{X,s+l}(\mathbf{1}) \otimes \xi_j). \end{aligned} \tag{5.19}$$

Now integrability of the right-hand side of equality (5.19) follows from the explicit formula for Γ and the boundedness result in Lemma 5.16. $\square$

Proposition 5.20. *Let a be a real number so that* $a > d(n-1)$ *and let* $f \in \mathscr{S}(\mathfrak{n})$. *Let* $\mathcal{O}_n = L/S_n$ *be the open L-orbit in* $\mathfrak{n}$. *If k is as in Proposition* 5.17, *then*

$$\int_{L/S_n} \mathscr{I}_{X_n,s}(a_t \ell^{-1} \cdot \mathbf{1}) e^{-(s+\frac{m}{n})\Lambda_0(H(\ell))} f(\ell \cdot X_n) \nabla(\ell \cdot X_n)^{2k+\frac{m}{n}} \, d\ell$$

is holomorphic on the region $\{s\in\mathbf{C} : Re(s) < a\}$.

Proof. By Theorem 5.6 it is enough to show, for $f_{P,s} \in Ind_P^G(s)$, that

$$\int_{L/S_n} \mathscr{I}_{X_n,s}(a_t \ell^{-1} \cdot f_{P,s}) e^{-(s+\frac{m}{n})\Lambda_0(H(\ell))} f(\ell \cdot X_n) \nabla(\ell \cdot X_n)^{2k+\frac{m}{n}} \, d\ell$$

is holomorphic on the region $\{s\in\mathbf{C} : |Re(s)| > d(n-1)\}$. We prove that this is the case by using the estimate in Lemma 5.16. Indeed, it follows from the estimate in Lemma 5.16 that the family of functions depending on s that we are integrating is dominated by a L^1 function when s is in compact sets lying on $Re(s) > d(n-1)$. On the other hand, Eq. (5.18) and Lemma 5.16 yield the same conclusion for s in compact sets lying in the region $Re(s) < -d(n-1)$ without the poles of $\frac{c_P(s)}{M_{X_n}(s)}$. $\square$

6. The Poisson transform

The Poisson transform $\mathscr{P}_s\colon Ind_P^G(s)\to C^\infty(G/K)$ is given by the following formula

$$\mathscr{P}_s(\varphi)(g) = \int_K \varphi(gk)\,dk = \int_{\bar{N}} e^{(s-\frac{m}{n})\Lambda_0(H(g^{-1}\bar{n}))}\varphi(\bar{n})\,d\bar{n}.$$

Lemma 6.1. *If $Re(s) > d(n-1)$, then*

$$\lim_{t\to+\infty} a_t^{(-s+\frac{m}{n})\Lambda_0}\mathscr{P}_s(\varphi)(\bar{n}_0 a_t) = \tilde{A}_s(\varphi)(\bar{n}_0).$$

Proof.

$$\begin{aligned}
\mathscr{P}_s(\varphi)(\bar{n}_0 a_t) &= \mathscr{P}_s(\varphi)(\bar{n}_0 w_0 a_t^{-1})\\
&= \int_K \varphi(\bar{n}_0 w_0 a_t^{-1} k)\,dk\\
&= \int_{\bar{N}} e^{(s-\frac{m}{n})\Lambda_0(H(\bar{n}))}\varphi(\bar{n}_0 w_0 a_t^{-1}\bar{n})\,d\bar{n}\\
&= a_t^{(s+\frac{m}{n})\Lambda_0}\int_{\bar{N}} e^{(s-\frac{m}{n})\Lambda_0(H(\bar{n}))}\varphi(\bar{n}_0 w_0 a_t^{-1}\bar{n}a_t)\,d\bar{n}\\
&\quad \text{by change of variables } a_t^{-1}\bar{n}a_t\to\bar{n}\\
&= a_t^{(s-\frac{m}{n})\Lambda_0}\int_{\bar{N}} e^{(s-\frac{m}{n})\Lambda_0(H(a_t\bar{n}a_t^{-1}))}\varphi(\bar{n}_0 w_0\bar{n})\,d\bar{n}.
\end{aligned}$$

Thus, we have

$$\lim_{t\to+\infty} a_t^{(-s+\frac{m}{n})\Lambda_0}\mathscr{P}_s(\varphi)(\bar{n}_0 a_t) = \lim_{t\to+\infty}\int_{\bar{N}} e^{(s-\frac{m}{n})\Lambda_0(H(a_t\bar{n}a_t^{-1}))}\varphi(\bar{n}_0 w_0\bar{n})\,d\bar{n}.$$

As t tends to infinity, $a_t\bar{n}a_t^{-1}$ tends to 1. We use dominated convergence to exchange limit and integration and obtain the required result. We verify that $e^{(Re(s)-\frac{m}{n})\Lambda_0(H(a_t\bar{n}a_t^{-1}))}e^{(-Re(s)-\frac{m}{n})\Lambda_0(H(\bar{n}))}$ is dominated by an integrable function as a_t tends to infinity. To do so we can, for example, imitate the argument in [11, p. 199]. We write $Re(s) = d(n-1) + \varepsilon\frac{m}{n} + \lambda$ with λ positive and $0 < \varepsilon \leqslant 1$. Then, our function is

$$e^{-(\frac{m}{n}+d(n-1)+\varepsilon(1+e))\Lambda_0(H(\bar{n}))}\{e^{(-1+\varepsilon)(1+e)\Lambda_0(H(a_t\bar{n}a_t^{-1}))}\}$$

$$\{e^{(\lambda+\varepsilon d(n-1))\Lambda_0(H(a_t\bar{n}a_t^{-1}))}e^{-(\lambda+\varepsilon d(n-1))\Lambda_0(H(\bar{n}))}\}.$$

The expressions in braces are each bounded above by 1 and the first factor is integrable by [1, Lemma 3.15]. □

Proposition 6.2. *We keep the notation just introduced. Let s be a complex number with* $Re(s) < -d(n-1)$. *Let* $\varphi(\bar{n}_Y) \in I(s)$ *be of the form* $\varphi(\bar{n}_Y) = \hat{f}(Y)$, *with* $f \in \mathscr{S}(\mathfrak{n})$. *Then,*

$$\mathscr{P}_s(\varphi)(\bar{n}_{Y_0}a_t) = \int_{\mathfrak{n}} f(X)e^{-2\pi i\langle Y_0, X\rangle} \mathscr{J}_{X,-s}(a_t \cdot \mathbf{1})\, dX.$$

Proof.

$$\begin{aligned}\mathscr{P}_s(\varphi)(\bar{n}_{Y_0}a_t) &= \int_{\bar{N}} e^{(s-\frac{m}{n})\Lambda_0(H(a_t^{-1}\bar{n}))}\varphi(\bar{n}_{Y_0}\bar{n})\, d\bar{n} \\ &= \int_{\bar{\mathfrak{n}}} e^{(s-\frac{m}{n})\Lambda_0(H(a_t^{-1}\exp(Y)))}\hat{f}(Y_0+Y)\, dY. \qquad (6.3)\end{aligned}$$

The function $\mathbf{1}_{P,-s}(x) = e^{(s-\frac{m}{n})\Lambda_0 H(x)}$ is in the induced space $Ind_P^G(-s)$. Since we are assuming that $Re(s) < -d(n-1)$, it follows that $\mathbf{1}_{P,-s}(\exp(Y)) \in L^1(\bar{\mathfrak{n}}) \cap L^2(\bar{\mathfrak{n}})$. We apply Plancherel theorem in the right-hand side of Eq. (6.3) to obtain

$$\mathscr{P}_s(\varphi)(\bar{n}_{Y_0}a_t) = \int_{\mathfrak{n}} \mathscr{J}_{X,-s}(a_t \cdot \mathbf{1})f(X)e^{-2\pi i\langle Y_0, X\rangle}\, dX. \qquad \square$$

Remark 6.4. Compare with Proposition 2.9. When $G = GL(2n, \mathbf{R})$, Faraut and Korányi found an explicit integral formula for $\mathscr{J}_{X,-s}(a_t \cdot \mathbf{1})$ valid for every $s \in \mathbf{C}$. We used that explicit expression in order to find estimates for $\mathscr{J}_{X,-s}(a_t \cdot \mathbf{1})$ in the region $\{s \in \mathbf{C} : Re(s) > (n-1)\}$. For a more general group G, a formula for $\mathscr{J}_{X,s}(a_t \cdot \mathbf{1})$ is only available in the region $\{s \in \mathbf{C}; Re(s) > d(n-1)\}$. Thus, in the next proposition we express $\mathscr{P}_s(\varphi)$ in terms of $\mathscr{J}_{X,s}(a_t \cdot \mathbf{1})$.

Recall that the open L-orbit in $\mathfrak{n}$, $\mathcal{O}_n = L/S_n$, is dense in $\mathfrak{n}$ and write $X \in \mathcal{O}_n$ as $X = \ell \cdot X_n$.

Proposition 6.5. *With the notation just introduced and for* $s \in \mathbf{C}$ *with* $Re(s) < -d(n-1)$, *we have*

$$\mathscr{P}_s(\varphi)(\bar{n}_{Y_0}a_t) = \frac{M_{X_n}(s)}{c_P(s)} \int_{L/S_n} \mathscr{J}_{X_n,s}(a_t\ell^{-1} \cdot \mathbf{1})e^{-(s+\frac{m}{n})\Lambda_0(H(\ell))} f(\ell \cdot X_n)e^{-2\pi i\langle Y_0, \ell\cdot X_n\rangle}\nabla(\ell \cdot X_n)^{\frac{m}{n}}\, d\ell.$$

Proof. By the previous proposition, for $s \in \mathbf{C}$ with $Re(s) < -d(n-1)$ we have

$$\begin{aligned}\mathscr{P}_s(\varphi)(\bar{n}_{Y_0}a_t) &= \int_{\mathfrak{n}} f(X)e^{-2\pi i\langle Y_0, X\rangle}\mathscr{J}_{X,-s}(a_t \cdot \mathbf{1})\, dX \\ &= \int_{L/S_n} f(\ell \cdot X_n)e^{-2\pi i\langle Y_0, \ell\cdot X_n\rangle}\mathscr{J}_{\ell\cdot X_n,-s}(a_t \cdot \mathbf{1})\nabla(\ell \cdot X_n)^{\frac{m}{n}}\, d\ell, \qquad (6.6)\end{aligned}$$

where $\nabla(\ell \cdot X_n)^{\frac{m}{n}} = |\det(\mathrm{Ad}(\ell)|_{\mathfrak{n}})| = e^{2\frac{m}{n}\Lambda_0(H(\ell))}$. Relating $\mathscr{I}_{\ell \cdot X_n,-s}$ to $\mathscr{I}_{X_n,-s}$ as in Lemma 5.10, we have

$$\mathscr{P}_s(\varphi)(\bar{n}_{Y_0}a_t) = \int_{L/S_n} \mathscr{I}_{X_n,-s}(a_t\ell^{-1} \cdot \mathbf{1})e^{-(s+\frac{m}{n})\Lambda_0(H(\ell))}$$

$$f(\ell \cdot X_n)e^{-2\pi i\langle Y_0, \ell \cdot X_n\rangle}\nabla(\ell \cdot X_n)^{\frac{m}{n}}\, d\ell, \tag{6.7}$$

where $\mathscr{I}_{X_n,-s}(a_t\ell^{-1} \cdot \mathbf{1}) = c_P(s)^{-1}\mathscr{I}_{X_n,-s}(a_t\ell^{-1} \cdot \tilde{A}_s(\mathbf{1}))$.

On the other hand, by Theorem 5.9

$$c_P(s)^{-1}\mathscr{I}_{X_n,-s}(a_t\ell^{-1} \cdot \tilde{A}_s(\mathbf{1}_{P,s})) = \frac{M_{X_n}(s)}{c_P(s)}\mathscr{I}_{X_n,s}(a_t\ell^{-1} \cdot \mathbf{1}_{P,s}). \tag{6.8}$$

We complete the proof of the proposition by substituting (6.8) into (6.7). □

Remark 6.9. The identity in Proposition 6.5 shows that the integral on the right-hand side has a meromorphic continuation as a function of s with singularities possibly at the poles of $\frac{c_P(s)}{M_{X_n}(s)}$.

7. The functional equation

The main result of this section is the following theorem. We let $M_{X_n}(s)$ be the constant occurring in the functional equation in Theorem 5.9.

Theorem 7.1. *Let $s \in \mathbf{C}$ and let $f \in \mathscr{S}(\mathfrak{n})$. As meromorphic functions*

$$\overline{\mathbf{Z}}\left(\hat{f}, s - \frac{m}{n}\right) = M_{X_n}(s)\mathbf{Z}(f, -s).$$

Proof. If $a > d(n-1)$, on the region $\{s \in \mathbf{C} : Re(s) < a\}$ we have proved in Proposition 5.17 that there exists $k > 0$ so that $e^{-(s+\frac{m}{n})\Lambda_0 H(\ell)}\mathscr{I}_{X_n,s}(a_t\ell^{-1} \cdot \mathbf{1})\nabla(\ell \cdot X_n)^{2k}$ belongs to $L^1(L/S_n, \nabla(\ell \cdot X_n)^{\frac{m}{n}}\, d\ell)$. Moreover, by Proposition 5.20

$$\int_{L/S_n} \mathscr{I}_{X_n,s}(a_t\ell^{-1} \cdot \mathbf{1})e^{-(s+\frac{m}{n})\Lambda_0(H(\ell))}f(\ell \cdot X_n)\nabla(\ell \cdot X_n)^{2k+\frac{m}{n}}\, d\ell$$

is holomorphic on the region $Re(s) < a$.

Set $\varphi(\bar{n}_Y) = \widehat{\nabla()^{2k}f}(Y)$. Since $\mathscr{P}_s(\varphi)$ is entire on s, by Propositions 6.2 and 6.5 on an open neighborhood of s we have

$$a_t^{(-s+\frac{m}{n})\Lambda_0} \frac{c_P(s)}{M_{X_n}(s)} \mathscr{P}_s(\varphi)(a_t)$$
$$= a_t^{(-s+\frac{m}{n})\Lambda_0} \int_{L/S_n} \mathscr{I}_{X_n,s}(a_t \ell^{-1} \cdot \mathbf{1}) \quad e^{-(s+\frac{m}{n})\Lambda_0(H(\ell))} f(\ell \cdot X_n) \nabla(\ell \cdot X_n)^{2k+\frac{m}{n}} \, d\ell. \quad (7.2)$$

By Lemma 6.1 the limit as t tends to infinity of the left-hand side of 7.2 is $\frac{c_P(s)}{M_{X_n}(s)} \tilde{A}_s(\varphi)(1)$. To compute the limit as t tends to infinity of the right-hand side of 7.2, we use Proposition 5.15 and Lebesgue Dominated Convergence theorem. We obtain

$$\begin{aligned}\tilde{A}_s(\varphi)(1) &= \lim_{t \to +\infty} a_t^{(-s+\frac{m}{n})\Lambda_0} \mathscr{P}_s(\varphi)(a_t) \\ &= M_{X_n}(s) \int_{L/S_n} f(\ell \cdot X_n) \nabla^{-s}(\ell \cdot X_n) \nabla(\ell \cdot X_n)^{2k+\frac{m}{n}} \, d\ell \\ &= M_{X_n}(s) \int_{\mathfrak{n}} f(X) \nabla^{2k-s}(X) \, dX. \qquad (7.3)\end{aligned}$$

As $\widehat{\nabla^{2k} f} = (-1)^{kn} (2\pi)^{-2kn} \bar{P}^k(\partial_X) \hat{f}$, the left-hand side of (7.3) is equal to $(-1)^{kn} (2\pi)^{2kn} \tilde{A}_s(\bar{P}^k(\partial_X)\hat{f})$. Using Eqs. (5.8) and (4.4) we can conclude

$$\begin{aligned}\tilde{A}_s(\widehat{\nabla^{2k}()f})(1) &= (-1)^{kn} (2\pi)^{-2kn} \tilde{A}_s(\bar{P}^k(\partial_X)\hat{f})(1) \\ &= (-1)^{kn} (2\pi)^{-2kn} b_k\left(s - \frac{m}{n}\right) \overline{\mathbf{Z}}\left(\hat{f}, s - \frac{m}{n} - 2k\right). \qquad (7.4)\end{aligned}$$

Combining Eqs. (7.3) and (7.4), we obtain

$$(-1)^{kn} (2\pi)^{-2kn} b_k\left(s - \frac{m}{n}\right) \overline{\mathbf{Z}}\left(\hat{f}, s - \frac{m}{n} - 2k\right) = M_{X_n}(s) Z(f, -s+2k). \quad (7.5)$$

Next, we express the Bernstein polynomial $b_k(s)$ in terms of $M_{X_n}(s)$ as in Proposition 5.11 to obtain

$$\overline{\mathbf{Z}}\left(\hat{f}, s - \frac{m}{n} - 2k\right) = M_{X_n}(s - 2k) Z(f, -s+2k). \quad (7.6)$$

The change of variables $\sigma = s - 2k$ gives the result, at least on some open set on $\mathbf{C}$. The full result is obtained by analytic continuation. □

8. Tables

Table 1 gives information on the groups satisfying Assumptions 3.1. The Jordian algebras for the groups are shown in Table 2.

Remark 8.1. For Cases 10 and 11 we view the quaternionic matrices as complex matrices of the form $Z = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$. Then $\det_{\mathbf{C}}$ refers to the determinant of the complex matrix.

Table 1
Groups satisfying Assumptions 3.1

	G	$\mathrm{rank}(\mathfrak{n})$	$m = \dim(\mathfrak{n})$	d	e
1.	$GL(2n, \mathbf{R}),\ n \geqslant 2$	n	n^2	1	0
2.	$O(2n, 2n),\ n \geqslant 2$	n	$n(2n-1)$	2	0
3.	$E_7(7)$	3	27	4	0
4.	$O(p,q),\ p,q \geqslant 3$	2	$p+q-2$	$\frac{p+q-4}{2}$	0
5.	$Sp(n, \mathbf{C})$	n	$n(n+1)$	1	1
6.	$GL(2n, \mathbf{C})$	n	$2n^2$	2	1
7.	$SO(4n, \mathbf{C})$	n	$2n(2n-1)$	4	1
8.	$E_{7,\mathbf{C}}$	3	54	8	1
9.	$SO(p, \mathbf{C})$	2	$2(p-2)$	$p-4$	1
10.	$Sp(n,n)$	n	$n(2n+1)$	2	2
11.	$SL(2n, \mathbf{H})$	n	$4n^2$	4	3
12.	$SO(p,1)$	1	p	0	$p-1$

Table 2
Jordan algebras for the groups in Table 1

	$V \simeq \mathfrak{n}$	L	∇
1.	$M(n \times n, \mathbf{R})$	$(GL(n, \mathbf{R}) \times GL(n, \mathbf{R}))$	$\lvert\det\rvert$
2.	$Skew(2n\colon \mathbf{R})$	$GL(2n, \mathbf{R})$	Pfaffian
3.	$Herm(3, \mathbf{O}_{\mathrm{split}})$	$E_6(6) \times \mathbf{R}^{\times}$	degree 3 **R**-poly
4.	$\mathbf{R}^{p-1,q-1}$	$\mathbf{R}^{\times} SO(p-1, q-1)$	(X, X)
5.	$Sym(n, \mathbf{C})$	$GL(n\colon\ \mathbf{C})$	$\lvert\det\rvert$
6.	$M(n \times n, \mathbf{C})$	$S(GL(n, \mathbf{C}) \times GL(n, \mathbf{C}))$	$\lvert\det\rvert$
7.	$Skew(2n, \mathbf{C})$	$GL(2n, \mathbf{C})$	|Pfaffian|
8.	$Herm(3, \mathbf{O})_{\mathbf{C}}$	$E_{6,\mathbf{C}} \mathbf{C}^{\times}$	|degree 3 **C**-poly|
9.	$\mathbf{C}^{p-1}$	$O(p-2\colon\ \mathbf{C}) \times \mathbf{C}^{\times}$	$\lvert(Z, Z)\rvert$
10.	$Sym(2n, \mathbf{C}) \cap M_{n \times n}(\mathbf{H})$	$GL(n, \mathbf{H})$	$\lvert\det_{\mathbf{C}}(Z)\rvert^{\frac{1}{2}}$
11.	$M(n \times n, \mathbf{H})$	$GL(n, \mathbf{H}) \times GL(n, \mathbf{H})$	$\lvert\det_{\mathbf{C}}(Z)\rvert^{\frac{1}{2}}$
12.	$\mathbf{R}^{p,1}$	$O(p-1) \times \mathbf{R}^{\times}$	$\lVert \cdot \rVert$

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Further reading

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