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A lower bound for the norm of the second fundamental form of minimal hypersurfaces of $\mathbb{S}^{n+1}$

By

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Abstract. The aim of this paper is to give an estimate for the squared norm S of the second fundamental form A of a compact minimal hypersurface $M^n \subset \mathbb{S}^{n+1}$ in terms of the gap $n - \lambda_1$, where λ_1 stands for the first eigenvalue of the Laplacian of M. More precisely we will show that there exists a constant $k \geqq \frac{n}{n-1}$ such that $S \geqq k \frac{n-1}{n}(n - \lambda_1)$.

1. Introduction. Let M^n be a closed and orientable Riemannian manifold, i.e. compact without boundary and let us denote by $\mathbb{S}^m$ a unit Euclidean sphere. If $\varphi : M^n \to \mathbb{S}^{n+p}$ is a minimal immersion then $\Delta\varphi + n\varphi = 0$, where Δ stands for the Laplacian of M with its induced metric, see e.g. [13]. Hence n is an upper bound for the first eigenvalue λ_1 of Δ. In 1983 Leung [8] have shown that the gap $n - \lambda_1$ is a lower bound for $S = |A|^2$, where A stands for the second fundamental form of φ, provided S is constant. Among the purposes of this subject one of them is to answer a interesting question posed by Chern [4] for hypersurfaces concerning the gap of S under the assumption of S constant. In a recent paper Barros [1] have improved Leung's gap for compact minimal hypersurfaces $M^n \subset \mathbb{S}^{n+1}$ by showing that $S \geqq c\,(n,k)\,\frac{(n-1)}{n}\,(n - \lambda_1)$, where $c\,(n,k) = \frac{(n-k)}{(n-k-1)}$ and k depends on the dimension of the kernel of A. The main purpose of this paper is to improve the above result for compact minimal hypersurface $M^n \subset \mathbb{S}^{n+1}$ by showing that there is a rational constant $k \in [\frac{n}{n-1}, n]$ depending either on A or on the first eigenfunction of Δ such that $S \geqq k\frac{(n-1)}{n}\,(n - \lambda_1)$.

We point out that the first contribution to such problem was given in 1968 by Simons [12] who showed that if S satisfies $0 \leqq S \leqq \frac{n}{2-\frac{1}{p}}$, then either $S = 0$, and M is totally geodesic, or else $S = \frac{n}{2-\frac{1}{p}}$. In 1969 was shown by Lawson [7] and independently, in 1970, by Chern et al [5], if $S = \frac{n}{2-\frac{1}{p}}$ hence $S = n$ and M^n is a Clifford torus in $\mathbb{S}^{n+1}$. In 1998 Chen-Yang [3] have showed for hypersurfaces if $S > n$ then $S \geqq \frac{4}{3}\,n$. On the other

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hand it was conjectured by Yau [14] that for any embedded compact minimal hypersurface $M^n \subset \mathbb{S}^{n+1}$ the first eigenvalue of Δ satisfies $\lambda_1 = n$. The first general contribution to Yau problem was given by Choi-Wang [6] where they have proved that $\lambda_1 \geqq \frac{n}{2}$. Taking into account our inequality we derive $\lambda_1 \geqq n(1 - \frac{S}{k(n-1)})$. Hence we get an improvement of the Choi-Wang bound provided S is constant and $k\,(n-1) \geqq 2S$. We also point out that Yau problem was solved for a certain class of isoparametric minimal hypersurfaces by Muto [9]. But in such case our gap is zero since $n - \lambda_1 = 0$. It should be noted that according to the results of Simons, Chern-do Carmo-Kobayashi and Lawson quoted above our bound turns out better then their one provided $k\frac{(n-1)}{n}\,(n-\lambda_1) \geqq n$. Now we will announce our result according to the next theorems.

Theorem 1. *Let M^n be a compact orientable Riemannian manifold. We consider $\varphi : M^n \to \mathbb{S}^{n+1}$ a minimal immersion. Let f be a first eigenfunction of the Laplacian of M^n associated to λ_1. Let $l(p)$ denotes the number of nonnull components of ∇f with respect to a principal referential $E_p = \{e_1(p), \dots, e_n(p)\}$ in $p \in M$. If $l_0 = \min_{p\in M}\{l(p) \mid \nabla f(p) \neq 0\}$, $k_0 = \frac{n}{n-1}$ if $l_0 = 1$ and $k_0 = l_0$ for $l_0 \geqq 2$, then*

$$\int_M S|\nabla f|^2 \geqq \frac{k_0(n-1)(n-\lambda_1)}{n} \int_M |\nabla f|^2.$$

In particular, if S is constant, we have $S \geqq \frac{k_0(n-1)(n-\lambda_1)}{n}$.

Theorem 2. *Let M^n be a compact orientable Riemannian manifold. We consider $\varphi : M^n \to \mathbb{S}^{n+1}$ a minimal immersion. If f denotes a first eigenfunction of the Laplacian of M^n associated to λ_1 and $k = \max \dim \ker A$, then*

$$\int_M S|\nabla f|^2 \geqq \frac{(n-n_0)(n-1)(n-\lambda_1)}{n} \int_M |\nabla f|^2\,,$$

where

$$n_0 = \begin{cases} k\,, & if \quad k \leqq n-2 \\ n-2\,, & if \quad k = n-1 \;\; or \;\; k = n. \end{cases}$$

In particular, if S is constant, we have $S \geqq \frac{(n-n_0)(n-1)(n-\lambda_1)}{n}$.

2. Preliminaries. One of the basic tools in our work is the Bochner-Lichnerowicz formula which states that for a differentiable function $f : M \to R$

$$(2.1) \qquad \frac{1}{2}\Delta(|\nabla f|^2) = \mathrm{Ric}(\nabla f, \nabla f) + \langle \nabla f, \nabla(\Delta f)\rangle + |\,\mathrm{Hess}\, f|^2\,,$$

where Hess and Ric denote, respectively, the Hessian form and the Ricci tensor of M, and the norm of an operator T considered here is the Euclidean, which is given by $|T|^2 = \mathrm{tr}(TT^*)$. The proof of this formula can be found in [2] or [11].

If $\triangle f + \lambda_1 f = 0$, according to formula (2.6) of Barros [1] we have

$$\int_M |\operatorname{Hess} f|^2 = \int_M |\operatorname{Hess} f + \frac{\lambda_1}{n} fI|^2 + \frac{\lambda_1}{n}\int_M |\nabla f|^2. \tag{2.2}$$

Therefore,

$$\int_M |\operatorname{Hess} f|^2 \geqq \frac{\lambda_1}{n}\int_M |\nabla f|^2 . \tag{2.3}$$

Moreover, the equality holds if and only if M is isometric to the sphere $S^n(\sqrt{\frac{\lambda_1}{n}})$. See Theorem A of Obata [10].

Another ingredients to aid our proofs are the next two lemmas of linear algebra. The first one states the following:

Lemma 1. *Let V be a vector space of finite dimension n. Let $T : V \to V$ be a traceless symmetric linear operator and let $\{e_1, \dots, e_n\}$ be an orthonormal referential such that $Te_i = \mu_i e_i$, $i = 1, \dots, n$. For $v = \sum_{i=1}^{n} v_i e_i$ in V let l be the number of nonnull components v_i of v and we set $k_0 = \frac{n}{n-1}$ if $l = 1$ and $k_0 = l$, otherwise. Then we have*

$$\frac{1}{k_0}|T|^2|v|^2 \geqq \sum_{i=1}^{n} \mu_i^2 v_i^2 .$$

Proof. In order to derive the lemma we will use the Lagrange multipliers method to find the maximum of the function

$$F : (x_1, \dots, x_n, y_1, \dots, y_n) \longmapsto \sum_{i=1}^{n} x_i^2 y_i^2 ,$$

with constraints

$$\sum_{i=1}^{n} x_i^2 = |T|^2, \quad \sum_{i=1}^{n} y_i^2 = |v|^2, \quad \sum_{i=1}^{n} x_i = 0,$$

$$y_1, \dots, y_l \neq 0 \quad \text{and} \quad y_{l+1} = \dots = y_n = 0 .$$

Then, using Lagrange multipliers we obtain the following system

$$\begin{cases} x_i y_i^2 = \alpha x_i + \gamma \\ x_i^2 y_i = \beta y_i \end{cases}, \qquad i = 1, \dots, n. \tag{2.4}$$

From where we obtain

$$\begin{cases} x_i^2 y_i^2 = \alpha x_i^2 + \gamma x_i \\ x_i^2 y_i^2 = \beta y_i^2 \end{cases}, \qquad i = 1, \dots, n . \tag{2.5}$$

Summing up the above equations one obtains

$$F = \alpha |T|^2 = \beta |v|^2. \tag{2.6}$$

Let us assume that $\alpha \neq 0$ and $\beta \neq 0$, otherwise $F = 0$ by (2.6). If $l = n$, it follows from the equations $x_i^2 y_i = \beta y_i$, $i = 1, \ldots, n$, of (2.4) that $\beta = x_1^2 = \ldots = x_n^2$, $|T|^2 = n x_1^2 = n\beta$. Consequently we obtain

$$F = \frac{1}{n} |T|^2 |v|^2.$$

If $l < n$, by system (2.4), we infer that

$$\beta = x_1^2 = \ldots = x_l^2$$

and

$$x_{l+1} = \ldots = x_n = -\frac{\gamma}{\alpha}.$$

When $l = 1$, we have $\beta = x_1^2$ and the constraint $\sum_{i=1}^{n} x_i = 0$ yields

$$x_2 = \ldots = x_n = -\frac{1}{n-1} x_1.$$

If $\gamma = 0$, then $x_n = 0$ implies $x_1 = 0$ and $F = 0$. Hence we may assume that $\gamma \neq 0$ to obtain

$$|T|^2 = x_1^2 + (n-1)x_n^2 = x_1^2 + \frac{1}{n-1} x_1^2 = \frac{n}{n-1} \beta.$$

Therefore,

$$F = \beta |v|^2 = \frac{n-1}{n} |T|^2 |v|^2.$$

Let us suppose now $2 \leqq l < n$. In this case, $|T|^2 = l x_1^2 + (n-l) x_n^2$. Thus, $|T|^2 \geqq l x_1^2 = l\beta$ and consequently we have

$$\frac{1}{l} |T|^2 |v|^2 \geqq F$$

which finishes the proof of the lemma. □

Lemma 2. *Let V be a vector space of finite dimension n and let $T : V \to V$ be a traceless symmetric nontrivial linear operator. Let also $\{e_1, \ldots, e_n\}$ be an orthonormal referential such that $T e_i = \mu_i e_i$, $i = 1, \ldots, n$. If $k = \dim \ker T$ then given a nonnull vector $v = \sum_{i=1}^{n} v_i e_i$, we have*

$$\frac{1}{n-k} |T|^2 |v|^2 \geqq \sum_{i=1}^{n} \mu_i^2 v_i^2.$$

Proof. Without loss of generality we suppose that $\mu_1 = \ldots = \mu_k = 0$ and $\mu_{k+1}, \ldots, \mu_n \neq 0$. As in the previous lemma, we will use also Lagrange multipliers method. Now we should to find the maximum of the function

$$G : (x_1, \ldots, x_n, y_1, \ldots, y_n) \longmapsto \sum_{i=1}^{n} x_i^2 y_i^2 ,$$

with constraints

$$\sum_{i=1}^{n} y_i^2 = |v|^2, \quad \sum_{i=1}^{n} x_i^2 = |T|^2, \quad \sum_{i=1}^{n} x_i = 0,$$

$$x_1 = \ldots = x_k = 0 \qquad \text{and} \qquad x_{k+1}, \ldots, x_n \neq 0.$$

Then, we will find solutions of the system

$$\text{(2.7)} \qquad \begin{cases} x_i y_i^2 &= \alpha x_i + \gamma \\ x_i^2 y_i &= \beta y_i \end{cases}, \qquad i = 1, \ldots, n.$$

Using a similar argument as that one of the previous lemma, we multiply the n first equations of (2.7) by x_i, the n last ones by y_i and summing up we obtain

$$\text{(2.8)} \qquad G = \alpha |T|^2 = \beta |v|^2.$$

We will suppose that $\alpha \neq 0$ and $\beta \neq 0$. In another way, by (2.8) we have $G = 0$. Since $x_i = 0$, for $i = 1, \ldots, k$, it follows from (2.7) that $y_1 = \ldots = y_k = 0$. Taking into account this on the first k-equations of (2.7) we derive that $\gamma = 0$. Hence we have $x_i y_i^2 = \alpha x_i$, for $i = k+1, \ldots, n$. Therefore, $y_{k+1}, \ldots, y_k \neq 0$ and by the equations $x_i^2 y_i = \beta y_i$, $i = k+1, \ldots, n$, we infer that

$$\beta = x_{k+1}^2 = \ldots = x_n^2 .$$

Thus, $|T|^2 = (n-k) x_n^2 = (n-k)\beta$ and we conclude that

$$\frac{1}{n-k} |T|^2 |v|^2 \geqq G$$

which finishes the proof of the lemma. □

3. Proof of Theorems.

Proof of Theorem 1. Given $p \in M$, let k_i be the principal curvatures of M in p, relationed with the referential E_p, i.e., $Ae_i = k_i e_i$, $i = 1, \ldots, n$, in p. Making use of Gauss equation we derive

$$\mathrm{Ric}(e_i, e_j) = (n - 1 - k_i^2)\delta_{ij} .$$

Now for a differentiable function f defined on M^n, writing $\nabla f = \sum_{i=1}^{n} f_i e_i$ in p we have

$$\mathrm{Ric}(\nabla f, \nabla f) = (n-1)|\nabla f|^2 - \sum_{i=1}^{n} k_i^2 f_i^2.$$

We may apply Lemma 1 in each point of M to obtain the inequality

$$\frac{1}{k_0} S|\nabla f|^2 \geqq \sum_{i=1}^{n} k_i^2 f_i^2,$$

where k_0 is given according to Theorem 1. Consequently we derive

$$\mathrm{Ric}(\nabla f, \nabla f) \geqq (n-1)|\nabla f|^2 - \frac{1}{k_0} S|\nabla f|^2. \tag{3.1}$$

If, in addition $\triangle f = -\lambda_1 f$, then the Bochner-Lichnerowicz formula (2.1) yields

$$\frac{1}{2}\triangle|\nabla f|^2 = |\operatorname{Hess} f|^2 + \mathrm{Ric}(\nabla f, \nabla f) - \lambda_1 |\nabla f|^2. \tag{3.2}$$

Hence integrating (3.2) and using the inequalities (2.3) and (3.1), we get

$$0 \geqq \frac{\lambda_1}{n} \int_M |\nabla f|^2 + (n-1) \int_M |\nabla f|^2 - \frac{1}{k_0} \int_M S|\nabla f|^2 - \lambda_1 \int_M |\nabla f|^2.$$

Therefore, we obtain

$$\int_M S|\nabla f|^2 \geqq \frac{k_0(n-1)(n-\lambda_1)}{n} \int_M |\nabla f|^2$$

which concludes the proof of the theorem. □

Proof of Theorem 2. The proof of this theorem is similar to that one of Theorem 1. First we choose a local orthonormal referential $\{e_1, \ldots, e_n\}$ such that $Ae_i = k_i e_i$, $i = 1, \ldots, n$ to derive

$$\mathrm{Ric}(e_i, e_j) = (n - 1 - k_i^2)\delta_{ij}.$$

Second we choose also an eigenfunction f associated to the Laplacian of M and write $\nabla f = \sum_{i=1}^{n} f_i e_i$. Hence, we use Lemma 2 to show in this case that

$$\frac{1}{(n-n_0)} S|\nabla f|^2 \geqq \sum_{i=1}^{n} k_i^2 f_i^2.$$

However, we note that $\dim\ker A \geqq n-1$ implies $A \equiv 0$, because M is minimal. In this way, we can guarantee that $2 \leqq n - n_0$. Therefore,

$$\mathrm{Ric}(\nabla f, \nabla f) \geqq (n-1)|\nabla f|^2 - \frac{1}{n-n_0} S|\nabla f|^2 \tag{3.3}$$

and the proof follows as that one of the previous theorem after integrating (3.2) and using (3.3). □

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