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Stability of minimal surfaces in a 3-dimensional hyperbolic space

By

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1. Introduction.

(1.1). Let $x\colon M^n \to \bar{M}^{n+1}$ be a minimal immersion of a n-dimensional orientable manifold into a $n+1$-dimensional Riemannian manifold $\bar{M}$. Let D be a domain in M with compact closure $\bar{D}$ and piecewise smooth boundary ∂D. D is then a critical point for the area function of the induced metric, for all variations of $\bar{D}$ keeping ∂D fixed. We say that D is *stable* when such a critical point is a relative minimum. The geometrical characterization of stable domains is the main problem on the subject. For references on this we mention [1] and [3].

In this paper we prove the following theorem that is an improvement of Theor. (1.3) of [1]. We do not know whether the result is sharp. Let K denote the Gaussian curvature of M in the induced metric.

(1.2) Theorem. *Let $x\colon M^2 \to H^3(b)$ be an isometric immersion of M into the hyperbolic space $H^3(b)$ with constant curvature $b < 0$. If $D \subset M$ is simply connected and*

$$\int_{\bar{D}} (|K| + \tfrac{2}{3} b)\, dM < 2\pi$$

then D is stable.

2. General observations.

(2.1). The question whether D is stable or not is naturally studied by looking at the sign of the second variation formula. This is a quadratic form I_D that acts on normal vector fields to M that vanish on ∂D. Simons [5] has given an expression for I_D in a quite general set up. In the case of codimension one, by choosing a unit normal vector field N, I_D can be thought as an operator that acts in the space of C^∞ real functions on D vanishing on ∂D, given by

$$(2.2)\qquad \int_{\bar{D}} (-u \Delta u - (\tilde{R} + \|A\|^2) u^2)\, dM,$$

where $\|A\|^2$ is the square of the norm of the second fundamental form of x and $\tilde{R}$ is the Ricci curvature in the normal direction. Let $U \subset M$ denote an open set. A solution of the equation

$$\Delta u + (\tilde{R} + \| A \|^2) u = 0 \tag{2.3}$$

in U is called a *Jacobi field* in U. It is known that D is stable if and only if there is no Jacobi defined in $U \subset \bar{D}$ that is zero on ∂U. Let $\lambda_1(D)$ be the first eigenvalue of the Laplacian on D.

(2.4) Proposition. *Let $x: M^n \to \bar{M}^{n+1}$ be a minimal immersion. Assume $D \subset M$ is a domain with smooth boundary and that the sectional curvatures R_σ of $\bar{M}$ satisfy $b_1 \leqq R_\sigma \leqq b_2$. Now we have:*

(a) *if $\| A \|^2 + n b_1 > \lambda_1(D)$ then D is unstable.*

(b) *if $\| A \|^2 + n b_2 < \lambda_1(D)$ then D is stable.*

(2.5). As a consequence of this proposition it follows that if $b_2 \leqq 0$ and $\| A \|^2 \leqq -n b_2$ then M is globally stable. In particular, if $b_2 \leqq 0$ any totally geodesic submanifold is globally stable. It also follows that as much as b_2 becomes negative as much as one can expect to find non trivial examples of minimal submanifolds of $\bar{M}$ that are globally stable.

(2.6). The proof of the above proposition follows from the observation that if $b_1 \leqq R_\sigma \leqq b_2$ then $n b_1 \leqq \tilde{R} \leqq n b_2$ and

$$-\int_{\bar{D}} u \Delta u \, dM \geqq \lambda_1(D) \int_{\bar{D}} u^2 \, dM$$

for all piecewise smooth u that are zero on ∂D. Equality occurs if and only if u is a first eigenfunction for Δ.

3. A curvature estimate.

(3.1). Let $x: M^2 \to \bar{M}^3$ be a minimal immersion. Denote by ds^2 the metric on M induced by x and by A its second fundamental form. Let e_1, e_2, e_3 be an adapted frame to the immersion x, let w_1, w_2 be the dual coframe to e_1, e_2 and let $h_{ij}, i, j = 1, 2$, be the coefficients of the second fundamental form of x in the frame e_1, e_2. Then

$$\| A \|^2 = \sum h_{ij}^2 .$$

Let R_{ABCD}, $A, B, C, D = 1, 2, 3$, be the coefficients of the curvature form of $\bar{M}$. If σ is the plane generated by e_A, e_B then $R_\sigma = R_{ABAB}$. Set $u = \frac{1}{2}\| A \|^2 + a$, $a > 0$, and $d\sigma^2 = u\, ds^2$. We are interested in finding an upper bound for the Gauss curvature $\hat{K}$ of $(M, d\sigma^2)$.

(3.2) Lemma. *Let $\bar{M}$ have constant curvature c and let $a \geqq \max(c, -c/3)$, $a > 0$. Then $\hat{K} \leqq 1$.*

(3.3) Proof. It is easily computed that (see e.g. [2] eq. (2.8)) the Gaussian curvature $\hat{K}$ of $d\sigma^2 = u\, ds^2$ is given by

$$\hat{K} = \frac{K}{u} + \frac{1}{u^2}\left(-\Delta u + \frac{u_1^2 + u_2^2}{u}\right), \tag{3.4}$$

where K is the Gaussian curvature of ds^2, Δu is the Laplacian of u in the metric ds^2 and u_1, u_2 are defined by

$$du = u_1 w_1 + u_2 w_2 . \tag{3.5}$$

Since $u = \frac{1}{2}\| A \|^2 + a = \frac{1}{2} \sum h_{ij}^2 + a$, we obtain

$$u_k = \sum h_{ij} h_{ijk}, \qquad \Delta u = \sum h_{ijk}^2 + \sum h_{ij} \Delta h_{ij} . \tag{3.6}$$

The gauss equation states that

$$R_{1212} - K = \frac{1}{2} \| A \|^2 . \tag{3.7}$$

Substitution of (3.6) and (3.7) in (3.4) yields

$$\hat{K} = \frac{1}{u}(R_{1212} + a - u) + \frac{1}{2u^3}\left(\sum_k (\sum h_{ij} h_{ijk})^2 - \frac{1}{2}\sum h_{rs}^2 h_{ijk}^2\right) + \frac{a}{2u^3}\sum h_{ijk}^2 - \frac{1}{2u^2}\sum h_{ij}\Delta h_{ij} . \tag{3.8}$$

Since $\bar{M}$ has constant curvature, following [4] and using that $1 \leqq i, j \leqq 2$, one obtains

$$\sum h_{ij} \Delta h_{ij} = (4 R_{1212} - \sum R_{3k3k})(\sum h_{ij}^2) - (\sum h_{ij}^2)^2 .$$

Since $R_{ABAB} = c$, we then have

$$\sum h_{ij} \Delta h_{ij} = 2c \| A \|^2 - \| A \|^4 .$$

Furthermore, one can easily check (see [1], proof of Prop. (2.2)) that the second term on the right hand side of (3.8) is zero. It follows that

$$\hat{K} = \frac{1}{u}(c + a - u) - \frac{a}{2u^3}\sum h_{ijk}^2 - \frac{1}{2u^2}(4c(u-a) - 4(u-a)^2) \tag{3.9}$$

and by dropping the second term on the right hand side of (3.9), we obtain

$$\hat{K} \leqq f(u)$$

where

$$f(u) = 1 + \frac{1}{u^2}(2a(a+c) - (3a+c)u) .$$

Now we need to prove that, for $u \geqq a \geqq \max(c, -c/3)$, we have $f(u) < 1$. To do this, one studies separately the case $c > 0$ and $c < 0$ and shows that the graphs of f for $u \geqq a$ are below 1 and approach asymptotically 1 as $u \to \infty$. This proves Lemma (3.2).

4. Proof of Theorem (1.2).

(4.1). The proof of Theorem (1.2) is now completed by following the general method developed in [1]. If u is a piecewise differentiable function in D such that u

restricted to the boundary of D is zero, then

$$I_D(u) = \int_{\bar{D}} (-u\Delta u - (\tilde{R} + \|A\|^2)u^2)\,dM, \tag{4.2}$$

where now $\tilde{R} = 2b < 0$. By the conformal change of metric:

$$d\sigma^2 = (\tfrac{1}{2}\|A\|^2 + a)\,ds^2, \tag{4.3}$$

where $a \geqq \max(b, -b/3)$, and $a > 0$, we obtain

$$I_D(u) = \int_{\bar{D}} \left(-u\Delta_\sigma u - 2\frac{2b + \|A\|^2}{2a + \|A\|^2}u^2\right) dM_\sigma, \tag{4.4}$$

where dM_σ and Δ_σ are respectively the area element and the Laplacian operator in the metric $d\sigma^2$. It follows that

$$I_D(u) \geqq \int_{\bar{D}} (-u\Delta_\sigma u - 2u^2)\,dM_\sigma \geqq (\lambda_1 - 2)\int_{\bar{D}} u^2\,dM_\sigma, \tag{4.5}$$

where λ_1 is the first eigenvalue of Δ_σ on D. According to Lemma (3.2), the Gaussian curvature K_σ of the metric $d\sigma^2$ satisfies $K_\sigma \leqq 1$. From Proposition (3.3) of [1] we have that

$$\lambda_1 \geqq \lambda_1(D^*), \tag{4.6}$$

where D^* is a geodesic disk on the sphere $S^2(1)$ such that $\text{area}_\sigma D = \text{area}\, D^*$. But, by the Gauss equation,

$$\text{area}\, D^* = \int_{\bar{D}} (\tfrac{1}{2}\|A\|^2 + a)\,dM = \int_{\bar{D}} (|K| + \tfrac{2}{3}b)\,dM < 2\pi. \tag{4.7}$$

Hence D^* is contained in a hemisphere and so $\lambda_1(D^*) > 2$. Therefore $I_D(u) \geqq 0$ and D is stable. This completes the proof of Theorem (1.2).

References

[1] J. L. Barbosa and M. do Carmo, Stability of minimal surfaces and eigenvalues of the Laplacian. To appear at the Math. Z.
[2] J. L. Barbosa and M. do Carmo, A necessary condition for a metric to be isometrically and minimally immersed in R^{n+1}. An. Acad. Brasil. Cienc. **50**, 415–454 (1978).
[3] M. do Carmo, Stability of minimal submanifolds. To appear in LNM.
[4] S. S. Chern, Minimal Submanifolds in a Riemannian manifold. University of Kansas, Dept. of Math., Tech. Rep. **19**, Kansas 1968.
[5] J. Simons, Minimal Varieties in Riemannian Manifolds. Ann. Math. 88, 62–105 (1968).

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