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Rigidity of Minimal Submanifolds in Space Forms

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1. Introduction

(1.1) Let c be a real number. Represent by $\bar{M}^n(c)$ a n-dimensional space form of curvature c. Let M^N be a N-dimensional connected Riemannian manifold. The question that served as the starting point for this paper was to find simple conditions on the metric of M so that, if $f:M^n \to \bar{M}^{n+p}(c)$, $p \geqq 1$, is an isometric minimal immersion then f is rigid in the following sense: given another minimal immersion $g:M^n \to \bar{M}^{n+q}(c)$, $q \geqq p$, then there exists a rigid motion T of $\bar{M}^{n+q}(c)$ such that $g = T \circ f$, ($\bar{M}^{n+p}(c)$ being considered as a totally geodesic submanifold of $\bar{M}^{n+q}(c)$).

In what follows we present a satisfactory answer to this question when $n \geqq 3$ and $p=1$ by imposing a restriction on the possible values of the nullity of the curvature tensor of M.

(1.2) Let's recall that the nullity of the curvature tensor of M is a function $\mu: M \to \mathbb{Z}$ that associates to each $p \in M$ the dimension of $\ker_p R$, the kernel of the curvature tensor R of M at p. That is,

$$\mu(p) = \dim\{X \in T_pM;\, R(X,Y) = c\langle Y, \cdot\rangle X - c\langle X, \cdot\rangle Y \quad \text{for all} \quad Y \in T_pM\}$$

where $\langle \cdot, \cdot \rangle$ stands for the metric of M. The result we obtain can be stated as follows:

(1.3) **Theorem.** *Let $M^n (n \geqq 3)$ be a connected Riemannian manifold and p be a point of M. Assume $\mu(p) \leqq n-3$. If $f: M^n \to \bar{M}^{n+1}(c)$ and $g: M^n \to \bar{M}^{n+k}(c)$ are isometric minimal immersions then there is a rigid motion T of $\bar{M}^{n+k}(c)$ such that $g = T \circ f$, ($\bar{M}^{n+1}(c)$ being considered as a totally geoderic submanifold of $\bar{M}^{n+k}(c)$).*

(1.4) The hypothesis of this theorem can not be weakened as shown by the following examples:

Let $f: M^2 \to R^3$ and $g: M^n \to R^{n+k}$ be isometric minimal immersions. Denote by f_* the conjugate immersion of f. As in [5] we can define

$$f_\theta: M^2 \to R^6 = R^3 \oplus R^3$$

by

$$f_\theta = \cos\theta f \oplus \sin\theta f_*$$

It is well known that f_θ is also an isometric minimal immersion and so are

$$f \oplus g : M^2 \oplus M^n \to R^{n+k+3}$$

and

$$f_\theta \oplus g : M^2 \oplus M^n \to R^{n+k+6}$$

By an appropriate choice of g (and k) one obtains the examples.

(1.5) In [2] do Carmo and Dajczer have exhibited sufficient conditions on the metric of M^n for the existence of a minimal isometric immersion $f: M^n \to \bar{M}^{n+1}(c)$. Under this set of conditions the hypothesis of Theorem (1.3) are fulfilled and so, the examples of do Carmo and Dajczer are rigid in the sense of (1.1).

(1.6) Assume we have a minimal immersion $f: M^n \to \bar{M}^{n+l}(c)$, $l > 1$, which is *substantial* in the sense that we can not reduce the codimension. Then, if there is a point $p \in M$ such that $\mu(p) \leqq n-3$, (1.3) implies the nonexistence of a minimal isometric immersion of M^n into $\bar{M}^{n+1}(c)$. The next theorem shows that if we strengthen the hypothesis on μ then we obtain, in fact, an obstruction to the existence of isometric immersions of M^n into $\bar{M}^{n+1}(c)$.

(1.7) **Theorem.** *Let $f: M^n \to \bar{M}^{n+l}(c)$, $2 \leqq l \leqq n-2$, be a minimal and substantial isometric immersion. Assume there is a point $p \in M$ such that $\mu(p) \leqq n-l-2$, then M^n can not be isometrically immersed into $\bar{M}^{n+1}(c)$.*

(1.8) By taking $k=0$ (and consequently $\mu = n$) in the examples presented in (1.4) we see that the above theorem is false without the hypothesis about μ. The theorem is also false if we maintain the hypothesis about μ but give up the restrictions on the possible values for the codimension l, as one can see by taking two immersions of the sphere $S^2_{k(s)}$, of curvature $k(s) = 2/s(s+1)$, into the spheres S^3_1 and S^{2s}_1: the first given by a standard isometric immersion as an umbilic hypersurface; the second, one of the minimal immersions described by do Carmo and Wallach in [4].

2. Proof of Theorem (1.3)

(2.1) Let M be a connected n-dimensional Riemannian manifold and R be its curvature tensor. Let $f: M \to \bar{M}^N(c)$ be an isometric immersion and let α be its second fundamental form. The kernels of R and α at a point p of M are defined, respectively, by:

$$\ker_p R = \{X \in T_pM;\, R(X,Y) = c\langle Y,\cdot\rangle X - c\langle X,\cdot\rangle Y, \forall Y \in T_pM\}$$

$$\ker_p \alpha = \{X \in T_pM;\, \alpha(X,Y) = 0, \forall Y \in T_pM\}$$

where $\langle \cdot,\cdot \rangle$ stands for the metric of $\bar{M}^N(c)$.

(2.2) **Lemma.** *If* $f: M^n \to \bar{M}^N(c)$ *is a minimal isometric immersion then* $\ker_p \alpha = \ker_p R$ *for each p in M.*

Proof. The Gauss equation clearly shows that

$$\ker_p \alpha \subset \ker_p R \tag{2.3}$$

for each $p \in M$. To prove the converse one takes the Ricci tensor of M and observes that for an othonormal basis $X, e_2, \ldots, e_n$ for T_pM we have that:

$$(n-1)\,\mathrm{Ric}(X) = (n-1)c - |\alpha(X,X)|^2 - \sum_{i=2}^{n} |\alpha(e_i, X)|^2 . \tag{2.4}$$

Here we have already used the minimality of f. Since $\mathrm{Ric}(X) = c$ when $X \in \ker_p R$ then the converse of (2.3) clearly follows from (2.4).

Now let $g: M^n \to \bar{M}^N$ (c) be another isometric minimal immersion of M with β as its second fundamental form. Then the following proposition holds:

(2.5) **Proposition.** *Assume* $\dim(\ker_p \alpha) \leqq n-3$. *If* $e_1, \ldots, e_n$ *is an orthonormal bases of* T_pM *that diagonalizes* α *then it also diagonalzes* β.

Proof. Let $e_1, \ldots, e_n$ be an orthonormal basis of T_pM that diagonalizes α. Set $\alpha_{ij} = \alpha(e_i, e_j)$ and $\beta_{ij} = \beta(e_i, e_j)$, $1 \leqq i \leqq n$. From (2.2) it follows that $\ker_p \alpha = \ker_p \beta$. Hence we can assume both kernels to be trivial, otherwise we can work on their orthogonal complement. The hypothesis on $\ker_p \alpha$ now just means $n \geqq 3$. From (2.4) one obtains:

$$|\alpha_{ii}|^2 = -(n-1)(\mathrm{Ric}(e_i) - c) = \sum_{j=1}^{n} |\beta_{ij}|^2 . \tag{2.6}$$

On the other hand, since codimension of f is one, the Gauss equation gives us

$$|\alpha_{ii}|^2 |\alpha_{jj}|^2 = [\langle \beta_{ii}, \beta_{ij} \rangle - |\beta_{ij}|^2]^2 \qquad i \neq j \tag{2.7}$$

From (2.6) and (2.7) we now obtain

$$\begin{aligned} [\langle \beta_{ii}, \beta_{jj} \rangle - |\beta_{ij}|^2]^2 &= \sum_k |\beta_{ki}|^2 \sum_k |\beta_{kj}|^2 \\ &\geqq (|\beta_{ii}|^2 + |\beta_{ij}|^2)(|\beta_{jj}|^2 + |\beta_{ij}|^2) \end{aligned} \tag{2.8}$$

Schwarz inequality applied to the right hand side of (2.8) yields:

$$[\langle \beta_{ii}, \beta_{jj} \rangle - |\beta_{ij}|^2]^2 \geqq [\langle \beta_{ii}, \beta_{jj} \rangle + |\beta_{ij}|^2]^2 . \tag{2.9}$$

Therefore if $\beta_{ij} \neq 0$ we have

$$\langle \beta_{ii}, \beta_{jj} \rangle \leqq 0, \qquad 1 \leqq i,j \leqq n, \qquad i \neq j . \tag{2.10}$$

Let K_{ij} denote the sectional curvature of M of the plane determined by e_i, e_j. From Gauss equation and (2.10) we obtain

$$K_{ij} - c = \langle \beta_{ii}, \beta_{jj} \rangle - |\beta_{ij}|^2 \leqq -|\beta_{ij}|^2 . \tag{2.11}$$

On the other hand we also have

$$K_{ij} - c = \langle \alpha_{ii}, \alpha_{jj} \rangle = a_i a_j$$

where a_i are the eigenvalues of the matrix of α. Since $n \geqq 3$, the products $a_i a_j$ can not all be negative, therefore there are indices i j, $i \neq j$ for which $\beta_{ij} = 0$. For this particular set of indices (2.8) tells us that

$$\langle \beta_{ii}, \beta_{jj} \rangle \geqq |\beta_{ii}|^2 |\beta_{jj}|^2 \tag{2.12}$$

where equality holds if and only if

$$\beta_{ki} = \beta_{kj} = 0 \quad \text{for all} \quad k \neq i, \quad k \neq j .$$

But Schwarz inequality tells us that (2.12) must be an equality! The same argument can now be applied to each set of indices i, k to show that

$$\beta_{kl} = 0 \quad \text{for all} \quad k \neq l .$$

Hence β is diagonal and the proof of (2.5) is finished. We can go further and observe that Schwarz inequality applied to (2.12) also tell us that β_{ii} and β_{jj} are linearly dependent. The argument repeated to β_{ik} yields the same conclusion for β_{ii} and β_{kk}. Hence we have proved that:

(2.13) *The first normal space of the immersion g at p has at most dimension one.*

Now, for proving Theorem (1.3) one starts by observing that, if $\mu(p) \leqq n-3$ then there is a neighborhood U of p in M where the same inequality holds. From Proposition (2.5) it follows that the normal bundle of the immersion g in this neighborhood is flat. This together with (2.13) implies (according to [1]) that $g(U)$ lies in some $\bar{M}^{n+1}(c) \subset \bar{M}(c)$. Now a standard rigidy Theorem [6] can be used to show the existence of a rigid motion T of $\bar{M}^n(c)$ such that $g(q) = T(f(q))$ for all $q \in U$. Since M is connected and g and f are real analytic (as solutions of the minimal surface system) the above equality holds everywhere and so $g = T \circ f$.

3. Proof of Theorem (1.7)

(3.1) Let $f: M^n \to \bar{M}^{n+l}(c)$, $2 \leqq l \leqq n-2$ be an isometric substantial minimal immersion and p be a point of M^n such that $\mu(p) = \dim \ker_p \alpha \leqq n-l-2$.

Suppose there exists an isometric immersion $g: M \to \bar{M}^{n+1}(c)$. Denote by α and β respectively the second fundamental forms of f and g. Let $T_p^{\perp}(f)$ and $T_p^{\perp}(g)$ denote the normal spaces at $p \in M$ of f and g, consider the first normal spaces

$$N_f(p) = \operatorname{span}\{\alpha(X, Y); X, Y \in T_pM\},$$

$$N_g(p) = \operatorname{span}\{\beta(X, Y);\ X, Y \in T_pM\}$$

and introduce

$$W = N_f(p) \oplus N_g(\mathrm{p}).$$

Define a symmetric bilinear form $B: T_pM \times T_pM \to W$ by

$$B(X, Y) = \alpha(X, Y) \oplus \beta(X, Y). \tag{3.2}$$

Define also a Lorentz inner product on W by:

$$(\eta \oplus \eta', \xi \oplus \xi') = \langle \eta, \xi \rangle_{T_p^{\perp}(f)} - \langle \eta', \xi' \rangle_{T_p^{\perp}(g)}. \tag{3.3}$$

It follows from the Gauss equations for f and g that

$$(B(X, Y), B(Z, W)) = (B(X, W), B(Y, Z)) \tag{3.4}$$

for all X, Y, Z, W in T_pM. Therefore B is *flat* with respect to $(\cdot, \cdot)$ in the sense of [7]. The *nullity* of B is:

(3.5) $$N(B)=\{Z\in T_pM\,;\, B(X,Z)=0\ \forall X\in T_pM\}\,.$$

It is clear that $N(B)\subset \mathrm{Ker}_p\alpha$ and so

(3.6) $$\dim N(B)\leqq n-l-2\,.$$

Observe that β can not be zero at p otherwise $(\cdot, \cdot)$ would be positive definite in W and hence [7, p. 463, Corollary 1] we would have:

$$\dim N(B)\geqq \dim T_pM-\dim W\geqq n-l-1$$

which would be in contradiction with (3.6).

(3.7) **Lemma.** *Under the hypothesis of Theorem (1.7), there are vectors* $\eta\in T_p^{\perp}(f)$ *and* $\delta\in T_p^{\perp}(g)$ *such that*

$$\langle\alpha(X,Y),\eta\rangle=\langle\beta(X,Y),\delta\rangle$$

for all X, Y *in* T_pM.

This result is a consequence of the theory of flat bilinear forms as developed by Moore [7]. Its proof can be done by minor modifications of the proof of Theorem (1.2) in (3).

Now, if $e_1,\dots,e_n$ is an orthonormal bases for T_pM then, since f is minimal,

$$\sum_{i=1}^{n}\langle\beta(e_i,e_i),\delta\rangle=0\,.$$

Using that $T_p^{\perp}(g)$ has dimension one we conclude that g has zero mean curvature at p. Now observe that the hypothesis $\mu(p)\leqq n-l-2$ is an open condition and so this is true for each point in a neighborhood U of p in M. Therefore g is minimal in U. According to Theoem (1.3) this would imply that $f|_U$ and $g|_U$ differ by a rigid motion of $M^{n+l}(c)$. Since f is analytic then we would reduce its codimension which is in contradiction with the hypothesis that f is substantial and $l\geqq 2$. This contradiction proves the theorem.

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