

COEFFICIENT BOUND FOR A NEW CLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

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ABSTRACT. In the present investigation we consider a new class of bi-univalent functions in the unit disk Δ using subordination and obtain estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

1. INTRODUCTION AND PRELIMINARIES

Let \mathcal{A} denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1)$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and \mathcal{S} denote the subclass of class \mathcal{A} consisting functions in \mathcal{A} which are also univalent in Δ . A domain $D \subset \mathbb{C}$ is convex if the line segment joining any two points in D lies entirely in D , while a domain is starlike with respect to a point $w_0 \in D$ if the line segment joining any point of D to w_0 lies inside D . A function $f \in \mathcal{A}$ is starlike if $f(\Delta)$ is a starlike domain with respect to origin, and convex if $f(\Delta)$ is convex. Analytically, $f \in \mathcal{A}$ is starlike if and only if $\Re\left(\frac{zf'(z)}{f(z)}\right) > 0$ in Δ , whereas $f \in \mathcal{A}$ is convex if and only if $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. The classes consisting of starlike and convex functions are denoted by \mathcal{S}^* and \mathcal{K} respectively. The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α , $0 \leq \alpha < 1$, are respectively characterized by $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ and $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$ in Δ , $f \in \mathcal{A}$. Also let \mathcal{P} denote the family of analytic functions $p(z)$ in Δ such that $p(0) = 1$ and $\Re(p(z)) > 0$ in Δ .

An analytic function f is subordinate to an analytic function g , written as $f(z) \prec g(z)$ ($z \in \mathbb{U}$), if there is an analytic function w defined on Δ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \Delta$ such that $f(z) = g(w(z))$. In particular, if g is univalent in Δ then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

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It is well known by the Koebe one quarter theorem [5] that the image of Δ under every function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, ($z \in \Delta$) and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f), r_0(f) \geq 1/4).$$

The inverse of $f(z)$ has a series expansion in some disk about the origin of the form

$$f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \dots \quad (2)$$

It was shown early [11, 14] that the inverse of the Koebe function provides the best bound for all $|\gamma_k|$. New proofs of the latter along with unexpected and unusual behavior of the coefficients γ_k for various subclasses of \mathcal{S} have generated further interest in this problem [7, 8, 9, 16].

A function $f(z)$ univalent in a neighborhood of the origin and its inverse satisfy the condition $f(f^{-1}(w)) = w$. Using (1), we have

$$w = f^{-1}(w) + a_2 (f^{-1}(w))^2 + a_3 (f^{-1}(w))^3 + \dots \quad (3)$$

Now using (2) we get the following result

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 + \dots \quad (4)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ . Let Σ denote the class of all bi-univalent functions defined in the unit disk Δ given by the Taylor-Maclaurin series expansion (1). One of the best examples of functions in the class Σ is

$$f(z) = \log \left(\frac{1+z}{1-z} \right)$$

which maps the unit disk univally onto strip $|Im w| < \pi/2$, which in turns contains the unit disk. However, the familiar Koebe function is not a member of Σ because it maps unit disk univally onto entire complex plane minus slit along $-1/4$ to $-\infty$. Hence the image domain does not contain unit disk.

Open problem. Can we find a subclass of the class \mathcal{A} which map unit disk into a domain which consists at least unit disk.

Lewin [10] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [13], on the other hand, showed that $\max_{f \in \Sigma} |a_2| = 4/3$. The coefficient estimate problem i.e. bound of $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$) for each $f \in \Sigma$ given by (1) is still an open problem. Several authors have subsequently studied similar problems in this direction. In [3] (see also [4, 18, 19]), certain subclasses of the bi-univalent function class Σ were introduced, and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ were found. In the present investigation, estimates on the initial coefficients of a new class of bi-univalent functions are obtained. Several related classes are also considered and a connection to earlier known result are made. The classes introduced in this paper are motivated by the corresponding classes investigated in [6, 12, 15]

Let φ be an analytic function with positive real part on Δ , satisfying $\varphi(0) =$

1, $\varphi'(0) > 0$, and $\varphi(\Delta)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots (B_1 > 0). \quad (5)$$

We now introduce the following class of functions:

Definition 1.1. Let $0 \leq \gamma \leq 1$, $\tau \in \mathbb{C} \setminus \{0\}$. A function $f \in \Sigma$ is in the class $\Sigma S_\gamma^\tau(\varphi)$, if the following subordinations hold:

$$1 + \frac{1}{\tau} \left[(1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) - 1 \right] \prec \varphi(z) \quad (z \in \Delta) \quad (6)$$

and

$$1 + \frac{1}{\tau} \left[(1 - \gamma) \frac{g(w)}{w} + \gamma g'(w) - 1 \right] \prec \varphi(w) \quad (w \in \Delta), \quad (7)$$

where $g(w) = f^{-1}(w)$.

We list few particular cases of this class discussed in the literature

[1] If we set $\gamma = 1$ and $\tau = 1$ in $\Sigma S_\gamma^\tau(\varphi)$ we obtain the class introduced in [1].

[2] If we set $\gamma = 1$ and $\tau = 1$ and $\varphi(z) = \frac{1+(1-2\beta)z}{1-z}$ ($0 \leq \beta < 1$) we obtain the class introduced in [17, p. 1191].

[3] If we set $\gamma = 1$ and $\tau = 1$ and $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ ($0 < \alpha \leq 1$) we obtain the class introduced in [17, p. 1190].

For more details about these classes see the corresponding references.

Further if we set $\tau = 1$, $\gamma = 0$ and $\varphi(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$; $z \in \Delta$) in Definition 1.1, we obtain a new class $\Sigma S(A, B)$ defined in the following way. A function $f \in \Sigma$ is in the class $\Sigma S(A, B)$, if the following subordinations hold:

$$\frac{f(z)}{z} \prec \frac{1+Az}{1+Bz} \quad \text{and} \quad \frac{g(w)}{w} \prec \frac{1+Aw}{1+Bw} \quad (z, w \in \Delta), \quad (8)$$

where $g(w) = f^{-1}(w)$.

To prove our main result we need following Lemma:

Lemma 1.1 (see [5]). *Let the function $p \in \mathcal{P}$ be given by the series*

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots \quad (z \in \Delta), \quad (9)$$

then, the sharp estimate

$$|c_n| \leq 2 \quad (n \in \mathbb{N}), \quad (10)$$

holds.

2. MAIN RESULTS

For functions in the class $\Sigma S_\gamma^\tau(\varphi)$, the following result is obtained.

Theorem 2.1. *Let $f(z) \in \Sigma S_\gamma^\tau(\varphi)$ is of the form (1), then*

$$|a_2| \leq \frac{|\tau| B_1^{3/2}}{\sqrt{|\tau B_1^2(1+2\gamma) + (1+\gamma)^2(B_1 - B_2)|}} \quad (11)$$

and

$$|a_3| \leq B_1 |\tau| \left(\frac{1}{1+2\gamma} + \frac{B_1 |\tau|}{(1+\gamma)^2} \right). \tag{12}$$

Proof. Let $f \in \Sigma S_\gamma^\tau(\varphi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \Delta \rightarrow \Delta$, with $u(0) = v(0) = 0$, satisfying

$$1 + \frac{1}{\tau} \left[(1-\gamma) \frac{f(z)}{z} + \gamma f'(z) - 1 \right] = \varphi(u(z)) \quad (z \in \Delta) \tag{13}$$

and

$$1 + \frac{1}{\tau} \left[(1-\gamma) \frac{g(w)}{w} + \gamma g'(w) - 1 \right] = \varphi(v(w)) \quad (w \in \Delta). \tag{14}$$

Define the functions p_1 and p_2 by

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1+c_1z+c_2z^2+\dots \quad \text{and} \quad p_2(z) = \frac{1+v(z)}{1-v(z)} = 1+b_1z+b_2z^2+\dots.$$

Then p_1 and p_2 are analytic in Δ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \Delta \rightarrow \Delta$, the functions p_1 and p_2 have a positive real part in Δ , and in view of Lemma 1.1

$$|b_n| \leq 2 \quad \text{and} \quad |c_n| \leq 2 \quad (n \in \mathbb{N}). \tag{15}$$

Solving for $u(z)$ and $v(z)$ we have

$$u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2} \left(c_1z + \left(c_2 - \frac{c_1^2}{2} \right) z^2 + \dots \right) \quad (z \in \Delta). \tag{16}$$

and

$$v(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2} \left(b_1z + \left(b_2 - \frac{b_1^2}{2} \right) z^2 + \dots \right) \quad (z \in \Delta). \tag{17}$$

In view of (5) and (13)-(17), clearly

$$\begin{aligned} & 1 + \frac{1}{\tau} \left[(1-\gamma) \frac{f(z)}{z} + \gamma f'(z) - 1 \right] \\ &= \varphi \left(\frac{p_1(z) - 1}{p_1(z) + 1} \right) \\ &= 1 + \frac{1}{2} B_1 c_1 z + \left(\frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \right) z^2 + \dots \end{aligned} \tag{18}$$

and

$$\begin{aligned} & 1 + \frac{1}{\tau} \left[(1-\gamma) \frac{g(w)}{w} + \gamma g'(w) - 1 \right] \\ &= \varphi \left(\frac{p_2(w) - 1}{p_2(w) + 1} \right) \\ &= 1 + \frac{1}{2} B_1 b_1 w + \left(\frac{1}{2} B_1 \left(b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2 \right) w^2 + \dots \end{aligned} \tag{19}$$

Since $f \in \Sigma$ has the Maclaurin series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 + \dots \tag{20}$$

Using (1) and (20) in (18) and (19), we obtain

$$\frac{(1+\gamma)a_2}{\tau} = \frac{B_1 c_1}{2}, \tag{21}$$

$$\frac{(1+2\gamma)a_3}{\tau} = \frac{1}{2}B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}B_2c_1^2, \quad (22)$$

$$\frac{-(1+\gamma)a_2}{\tau} = \frac{B_1b_1}{2} \quad (23)$$

and

$$\frac{(1+2\gamma)(2a_2^2 - a_3)}{\tau} = \frac{1}{2}B_1 \left(b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4}B_2b_1^2. \quad (24)$$

From (21) and (23), it follows that $c_1 = -b_1$. Further computation gives

$$a_2^2 = \frac{\tau^2 B_1^3 (b_2 + c_2)}{4 \left[\tau B_1^2 (1+2\gamma) + (1+\gamma)^2 (B_1 - B_2) \right]}. \quad (25)$$

and

$$a_3 = \frac{B_1^2 \tau^2 b_1^2}{4(1+\gamma)^2} + \frac{B_1 \tau}{4(1+2\gamma)} (c_2 - b_2). \quad (26)$$

In view of Lemma 1.1 we get the desired result (11) and (12).

Remark 2.1. If we set $\gamma = 1$ and $\tau = 1$ in Theorem 2.1 we get Theorem 2.1 of [1].

If we set

$$\varphi(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1; z \in \Delta) \quad (27)$$

in Theorem 2.1, we get the following Corollary:

Corollary 2.1. Let $f(z) \in \Sigma S_\gamma^\tau \left(\frac{1+Az}{1+Bz} \right)$ is of the form (1), then

$$|a_2| \leq \frac{|\tau|(A-B)}{\sqrt{|\tau(A-B)(1+2\gamma) + (1+\gamma)^2(1+B)|}} \quad (28)$$

and

$$|a_3| \leq (A-B)|\tau| \left(\frac{1}{1+2\gamma} + \frac{(A-B)|\tau|}{(1+\gamma)^2} \right). \quad (29)$$

Remark 2.2. If we set $\gamma = 1$, $\tau = 1$, $A = 1 - 2\beta$ ($0 \leq \beta < 1$) and $B = -1$ in Corollary 2.1 we get the Theorem 2 of [17].

Corollary 2.2. Let $f(z) \in \Sigma S_\gamma^\tau \left(\left(\frac{1+z}{1-z} \right)^\alpha \right)$ is of the form (1), then

$$|a_2| \leq \frac{2|\tau|\alpha}{\sqrt{|2\alpha\tau(1+2\gamma) + (1+\gamma)^2(1-\alpha)|}} \quad (30)$$

and

$$|a_3| \leq 2\alpha|\tau| \left(\frac{1}{1+2\gamma} + \frac{2\alpha|\tau|}{(1+\gamma)^2} \right). \quad (31)$$

Remark 2.3. Further if we set $\gamma = 1$, $\tau = 1$ in Corollary 2.2, we get Theorem 1 of [17].

Finally setting $\tau = 1$, $\gamma = 0$ in Corollary 2.1, we get the following new result:

Corollary 2.3. *Let $f(z) \in \Sigma S(A, B)$ is of the form (1), then*

$$|a_2| \leq \frac{A-B}{\sqrt{A+1}} \quad \text{and} \quad |a_3| \leq (A-B+1)(A-B). \quad (32)$$

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