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Topologizations of a set endowed with an action of a monoid

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Dedicated to Mitrofan Choban and Stoyan Nedev on the occasion of their 70th birthdays

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ABSTRACT

Given a set X and a family G of self-maps of X, we study the problem of the existence of a non-discrete Hausdorff topology on X with respect to which all functions $f \in G$ are continuous. A topology on X with this property is called a *G-topology*. The answer is given in terms of the Zariski G-topology ζ_G on X, that is, the topology generated by the subbase consisting of the sets $\{x \in X\colon\ f(x) \neq g(x)\}$ and $\{x \in X\colon\ f(x) \neq c\}$, where $f, g \in G$ and $c \in X$. We prove that, for a countable monoid $G \subset X^X$, X admits a non-discrete Hausdorff G-topology if and only if the Zariski G-topology ζ_G is non-discrete; moreover, in this case, X admits $2^{\mathfrak{c}}$ hereditarily normal G-topologies.

1. Principal problems

In this paper we consider the following general problem.

Problem 1.1. Given a set X and a family G of self-maps of X, determine whether X admits a non-discrete Hausdorff (or normal) topology with respect to which all functions $g \in G$ are continuous.

Since the composition of continuous functions is continuous, we can assume without loss of generality that the family G is a subsemigroup in the semigroup X^X of all functions $X \to X$ endowed with the operation of composition. Also, we can assume that G contains the identity function id_X of X and, hence, is a submonoid of X^X. Thus, it is natural to consider Problem 1.1 in the context of actions of monoids.

Let G be a monoid with two-sided unit 1_G. A *left unitary action* of a monoid G on a set X is a function $\alpha : G \times X \to X$, $\alpha : (g, x) \mapsto g(x)$, with the following two properties:

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- $f(g(x)) = (fg)(x)$ for all $f, g \in G$ and $x \in X$ and
- $1_G(x) = x$ for all $x \in X$.

A set X endowed with an action of a monoid G is called a (left unitary) *G-act*, or an *act over G* (this is a standard term of semigroup theory; see, e.g., [14]). A topology τ on a G-act X is called a *G-topology* if, for every $g \in G$, the shift $g : X \to X$, $g : x \mapsto g(x)$, is continuous. A G-act X is said to be (*normally*) *G-topologizable* if X admits a non-discrete (normal) Hausdorff G-topology. A topology τ on a set X is called *normal* if the topological space (X, τ) is normal, that is, X is a T_1-space such that any two disjoint closed subsets in X have disjoint open neighborhoods. A topology τ on a set X is *hereditarily normal* if each subspace of the topological space (X, τ) is normal.

In this terminology, Problem 1.1 can be rewritten as follows.

Problem 1.2. Find necessary and sufficient conditions for the (normal) G-topologizability of a given G-act X.

For acts over a group G, this problem was considered in [3].

Problem 1.2 is motivated by Markov's celebrated problem on the existence of a non-discrete Hausdorff group topology on an arbitrary infinite group [18], which is closely related to the problem of whether any unconditionally closed subset of a group G (i.e., a set closed in any Hausdorff group topology on G) is algebraic (i.e., is an intersection of finite unions of solution sets of equations in G), which was also posed by Markov in [18]. By the definition of a group topology, any algebraic set must be unconditionally closed; on the other hand, a group admits a non-discrete Hausdorff group topology if and only if the complement to the identity (or any other) element in this group is not unconditionally closed. In [19] (see also [17]), Markov proved the coincidence of unconditionally closed sets with algebraic sets for countable groups; a general (negative) answer was given only in 1979 by Hesse [11], who constructed an example of an uncountable non-topologizable group in which the complement to the identity element is not algebraic. The first countable non-topologizable group was given in 1980 by Ol'shanskii [20]; other examples were constructed in [16] and [15]. In contrast to these negative results, Zelenyuk [26] proved that each infinite group G admits a non-discrete regular topology with continuous shifts and continuous inversion.

The family of algebraic subsets of a group G coincides with the family of closed subsets of a T_1 topology ζ_G on G, called the *Zariski topology*. Markov did not explicitly introduce this topology, although he implicitly considered it in the form of algebraic closures of sets [19,18]. The Zariski topology was explicitly introduced only in 1977 by Bryant [4] under the name verbal topology; the term was introduced by Dikranjan and Shakhmatov [5]. This topology was studied in, e.g., [4,24,7,9,10].

The family of unconditionally closed subsets of G coincides with the family of closed subsets in another T_1 topology on G, namely, the infimum of all Hausdorff group topologies on G. This topology was explicitly introduced in [6] under the name *Markov topology*.

Thus, Markov's theorem on the coincidence of algebraic and unconditionally closed sets in countable groups asserts that, in such groups, the Zariski topology coincides with the Markov topology, or, in other words, that a countable group admits a non-discrete Hausdorff topology if and only if the Zariski topology of this group is non-discrete. In this paper, we obtain a similar result for G-acts (see Theorem 5.5); namely, we prove that an act over a countable monoid G is G-topologizable if and only if the Zariski G-topology on X, which is defined and studied in Section 2, is non-discrete. We also show that, in this case, X admits a non-discrete metrizable G-topology and $2^{\mathfrak{c}}$ non-discrete hereditarily normal G-topologies. This fact can be compared with a result of Dikranjan and Protasov [8] saying that each topologizable countable group G admits $2^{\mathfrak{c}}$ (pairwise transversal) topologies.

The paper is organized as follows. In Section 2, we introduce the Zariski topology on an S-act over a monoid S and consider particular cases of S-acts G, where G is a semigroup or a group and S is a submonoid of G^G. The technical Section 3 describes properties of maximal G-topologies on G-acts in which the so-called discriminate filters (defined in the same section) converge. Section 4 contains conditions for the existence of discriminate filters on G-acts, given in terms of the Zariski topology ζ_G. The last section contains the proofs of the main results on the topologizability of G-acts.

2. The Zariski G-topology on a G-act

In this section we define the Zariski G-topology on a G-act X and study this topology for particular examples of G-acts.

Definition 2.1. For an act X over a monoid G, the *Zariski G-topology* ζ_G on X is the topology generated by the subbase $\tilde{\zeta}_G$ consisting of the sets $\{x \in X\colon\ f(x) \neq g(x)\}$ and $\{x \in X\colon\ f(x) \neq c\}$, where $f, g \in G$ and $c \in X$.

The following easy fact follows directly from the definition.

Proposition 2.2. *For any G-act X, the Zariski G-topology ζ_G satisfies the separation axiom T_1 and is contained in any Hausdorff G-topology on X.*

An immediate corollary of this observation is that ζ_G is always non-discrete for a G-topologizable G-act. In the case of countable G-acts, the non-discreteness of ζ_G turns out to be sufficient for G-topologizability (see Theorem 5.5).

The subbase $\tilde{\zeta}_G$ of the Zariski G-topology ζ_G has a natural cardinal invariant $\psi(x, \tilde{\zeta}_G)$, called the pseudocharacter of $\tilde{\zeta}_G$ at a point $x \in X$. In fact, $\psi(x, \mathcal{F})$ can be defined for any family $\mathcal{F}$ of subsets of X. Given a point $x \in X$, we set

$$\mathcal{F}_x = \{X\} \cup \{F \in \mathcal{F}\colon\ x \in F\}$$

and define the *pseudocharacter* of $\mathcal{F}$ at x as

$$\psi(x, \mathcal{F}) = \min\Bigl\{|\mathcal{U}|\colon\ \mathcal{U} \subset \mathcal{F}_x \text{ and } \bigcap \mathcal{U} = \bigcap \mathcal{F}_x\Bigr\}.$$

If τ is the topology on X generated by a subbase $\mathcal{F}$, then τ_x is the family of all open neighborhoods of x, and $\psi(x, \tau)$ is the usual pseudocharacter of the topological space (X, τ) at the point x. It is easy to see that $\psi(x, \tau) = \psi(x, \mathcal{F})$ for any non-isolated point x in (X, τ). If x is isolated in (X, τ), then $\psi(x, \tau) = 1$, while $1 \leqslant \psi(x, \mathcal{F}) < \aleph_0$, so the pseudocharacter $\psi(x, \tilde{\zeta}_G)$ carries more information than $\psi(x, \zeta_G)$ for an isolated point x in (X, ζ_G). If x is non-isolated, then $\psi(x, \tilde{\zeta}_G) = \psi(x, \zeta_G)$.

In the algebraic language, the pseudocharacter $\psi(x, \tilde{\zeta}_G)$ equals the least number of inequalities of the form

$$f(x) \neq g(x) \quad \text{or} \quad f(x) \neq c, \quad \text{where } f, g \in G \text{ and } c \in X,$$

in a system of inequalities whose unique solution is x. Note that the pseudocharacter of the Zariski topology is tightly related to the notion of being ungebunden, which was introduced for general algebraic systems by Podewski [22] and corrected and generalized by Hesse [11]. Namely, an algebraic system is said to be *κ-ungebunden* if the pseudocharacter of the naturally defined Zariski topology on this system is at least κ at each point. Podewski proved that any $|G|$-ungebunden algebraic system G admits $2^{2^{|G|}}$ Hausdorff topologies consistent with the algebraic structure, and Hesse investigated κ-ungebunden systems and constructed examples showing that Podewski's sufficient topologizability condition cannot be weakened in the sense that, for any cardinals κ and λ such that $\mathrm{cf}(\lambda) > \kappa = \mathrm{cf}(\kappa) \geqslant \aleph_0$, there exists a group G with $|G| = \lambda$ which is κ-ungebunden (that is, $\psi(\zeta_G) = \kappa$) and admits no Hausdorff group (and even semigroup) topology.

Now, let us consider the Zariski G-topology for some particular G-acts.

Example 2.3. Let X be an infinite set endowed with the natural action of the group G of all bijective functions $f : X \to X$ that have finite support

$$\mathrm{supp}(f) = \{x \in X:\ f(x) \neq x\}.$$

It is easy to see that $\psi(x, \zeta_G) = 1$ and $\psi(x, \tilde{\zeta}_G) = 2$ for any point $x \in X$. Consequently, the Zariski G-topology ζ_G on X is discrete, and the G-act X is not G-topologizable.

Each semigroup G can be considered as an S-act for many natural actions of various submonoids S of the monoid G^G. We define five such natural submonoids of G^G:

- G_l is the smallest submonoid of G^G containing all left shifts $l_a : x \mapsto ax$ of G for $a \in G$;
- G_r is the smallest submonoid of G^G containing all right shifts $r_a : x \mapsto xa$ of G for $a \in G$;
- G_s is the smallest submonoid of G^G containing all two-sided shifts $s_{a,b} : x \mapsto axb$ of G for $a, b \in G$;
- G_m, where $m \in \mathbb{N}$, is the smallest submonoid of G^G containing G_s and the mth power map $x \mapsto x^m$;
- G_p^+ is the smallest submonoid of G^G containing G_s and such that the product $f \cdot g : x \mapsto f(x) \cdot g(x)$ of any two functions $f, g \in G_p^+$ belongs to G_p^+.

Clearly,

$$G_l \cup G_r \subset G_s \subset G_m \subset G_p^+,$$

and hence

$$\zeta_{G_l} \cup \zeta_{G_r} \subset \zeta_{G_s} \subset \zeta_{G_m} \subset \zeta_{G_p^+}.$$

For a group G, we augment the above list of submonoids of G^G by some other submonoids:

- G_q is the subgroup of G^G containing all bijections of the form $f : x \mapsto ax^\varepsilon b$, where $a, b \in G$ and $\varepsilon \in \{1, -1\}$;
- G_m, where $m \in \mathbb{Z}$, is the smallest submonoid of G^G containing G_s and the mth power map $x \mapsto x^m$;
- $G_{[s]}$ is the smallest submonoid of G^G containing the subgroup G_s and the maps $\gamma_a : G \to G$, $\gamma_a : x \mapsto xax^{-1}$, for all $a \in G$;
- $G_{[q]}$ is the smallest submonoid of G^G containing the subgroup G_q and the maps $\gamma_a : G \to G$, $\gamma_a : x \mapsto xax^{-1}$, for all $a \in G$;
- G_p is the smallest submonoid containing the subgroup G_q and such that the product $f \cdot g : x \mapsto f(x) \cdot g(x)$ of any two functions $f, g \in G_p$ belongs to G_p.

We refer to functions from the family G_p as *polynomials* on the group G.

G_l- and G_r-topologies on a group G are known as left- and right-invariant topologies; a group G endowed with a G_l-topology (G_r-topology, G_s-topology, G_q-topology) is said to be *left-topological* (respectively, *right-topological*, *semi-topological*, *quasi-topological*). Following [2], we refer to a group G endowed with a $G_{[s]}$-topology ($G_{[q]}$-topology) as a [*semi*]-*topological* (respectively, [*quasi*]-*topological*) group.

Now, consider the structure of the Zariski S-topologies on a group G for $S \in \{G_l, G_r, G_s, G_{[s]}, G_q, G_{[q]}, G_p^+, G_p\} \cup \{G_m:\ m \in \mathbb{Z}\}$. It should be mentioned that $G_0 = G_1 = G_s$ and $G_{-1} = G_q$.

By the *cofinite topology* on a set X we understand the topology

$$\tau_1 = \{\emptyset\} \cup \{X \setminus F\colon\ F \text{ is a finite subset of } X\}.$$

The following assertion follows easily from the definitions.

Remark 2.4. For any group G, the Zariski topologies ζ_{G_l} and ζ_{G_r} on G coincide with the cofinite topology on G. If G is infinite, then the topologies ζ_{G_l} and ζ_{G_r} are not Hausdorff.

Remark 2.5. For any infinite group G, the Zariski topologies ζ_{G_s} and ζ_{G_q} are not discrete. This follows from a deep result of Zelenyuk [25,26], who proved that each infinite group G admits a non-discrete Hausdorff topology with continuous two-sided shifts and continuous inversion.

Remark 2.6. For a group G endowed with the natural action of the monoid G_p of all polynomial functions on G, the Zariski G_p-topology ζ_{G_p} coincides with the usual Zariski topology $\mathfrak{Z}_G$ on the group G. In particular, for countable non-topologizable groups, the Zariski G_p-topology ζ_{G_p} is discrete, while for Hesse's example of an uncountable non-topologizable group G mentioned above (in which the complement to the identity element is not algebraic), the Zariski topology ζ_{G_p} is non-discrete.

Thus, for infinite groups G, the Zariski G_q-topology ζ_{G_q} is always non-discrete, while the topology ζ_{G_p} may be discrete (e.g., for countable non-topologizable groups).

If a group G is Abelian, then the Zariski topology ζ_{G_s} on G coincides with the topologies ζ_{G_l} and ζ_{G_r} and is therefore cofinite. However, for non-Abelian groups G, the topology ζ_{G_s} may have rather unexpected properties.

Example 2.7 *(Dikranjan–Toller).* Let H be a finite discrete topological group with trivial center (for example, let $H = \Sigma_3$ be the group of bijections of a 3-element set). For any cardinal κ, the Zariski topologies ζ_{G_s} and ζ_{G_p} on the group $G = H^\kappa$ coincide with the Tychonoff product topology τ on $G = H^\kappa$; hence these topologies are compact and Hausdorff and have pseudocharacter $\psi(x, \tau) = \kappa < 2^\kappa = |G_s| = |G_p| = |G|$ at each point $x \in G$.

Proof. Observe that the Tychonoff product topology τ on $G = H^\kappa$ turns the group G into a compact topological group. Thus, each polynomial map on G is continuous, and each set $U \in \tilde{\zeta}_{G_p}$ is open in G with this topology. Consequently, $\zeta_{G_s} \subset \zeta_{G_p} \subset \tau$. The Tychonoff product topology τ is generated by the subbase consisting of the sets

$$U_{\alpha,h} = \bigl\{x \in G\colon\ \mathrm{pr}_\alpha(x) = h\bigr\},$$

where $\alpha \in \kappa$, $h \in H$, and $\mathrm{pr}_\alpha : H^\kappa \to H$ denotes the αth coordinate projection. To prove that $\zeta_{G_s} = \zeta_{G_p} = \tau$, it suffices to check that each set $U_{\alpha,h}$ belongs to the topology ζ_{G_s}.

Consider the embedding $i_\alpha : H \to H^\kappa$ which takes each $x \in H$ to the point $i_\alpha(x) \in H^\kappa$ such that $\mathrm{pr}_\alpha \circ i_\alpha(x) = x$ and $\mathrm{pr}_\beta \circ i_\alpha(x) = 1_H$ for all $\beta \neq \alpha$.

Given a point $h \in H$, consider the finite set $A_h = \{(a, b) \in H \times H\colon\ ah \neq hb\}$ and observe that $\{h\} = \bigcap_{(a,b)\in A_h} \{x \in H\colon\ x^{-1}ax \neq b\}$. Indeed, it follows from the triviality of the center of H that, for any $x \in H \setminus \{h\}$, there is an element $a \in H$ such that $(xh^{-1})a \neq a(xh^{-1})$ and, therefore, $h^{-1}ah \neq x^{-1}ax$. For $b = x^{-1}ax$, we have $h^{-1}ah \neq b$, whence $(a, b) \in A_h$.

For each pair $(a, b) \in A_h$, consider the left and right shifts $l_a : x \mapsto i_\alpha(a) \cdot x$ and $r_b : x \mapsto x \cdot i_\alpha(b)$ of the group $G = H^\kappa$. These shifts generate the subbase set

$$U_{a,b} = \bigl\{x \in G\colon\ i_\alpha(a) \cdot x \neq x \cdot i_\alpha(b)\bigr\} = \bigl\{x \in G\colon\ l_a(x) \neq r_b(x)\bigr\} \in \tilde{\zeta}_{G_s}.$$

It remains to note that

$$\mathrm{pr}_\alpha^{-1}(h) = \bigcap_{(a,b)\in A_h} U_{a,b} \in \zeta_{G_s};$$

therefore, $\tau = \zeta_{G_p} = \zeta_{G_s}$. The equality $\zeta_{G_s} = \zeta_{G_p}$ implies $\zeta_{G_s} = \zeta_{G_{[s]}} = \zeta_{G_q} = \zeta_{G_{[q]}} = \zeta_{G_p^+} = \zeta_{G_p}$. □

Next, we consider the Zariski topologies on permutation groups.

Remark 2.8. Given a set X of cardinality $|X| \geqslant 3$, let $S(X)$ be the group of all bijective transformations of X, and let $S_\omega(X)$ be the normal subgroup of $S(X)$ consisting of all bijective transformations $f : X \to X$ with finite support $\mathrm{supp}(f) = \{x \in X\colon\ f(x) \neq x\}$. According to [2], for any group G with $S_\omega(X) \subset G \subset S(X)$ and any monoid $S \in \{G_{[s]}, G_{[q]}, G_p\}$, the Zariski S-topology ζ_S on G coincides with the topology of pointwise convergence $\mathcal{T}_p$ and therefore is completely regular. If X is infinite, then the Zariski topologies ζ_{G_s} and $\zeta_{G_{[s]}}$ are distinct, as shown in (the proof of) Lemma 2.9 in [2].

Remark 2.9. According to [2, 6.3], for any set X, the topology ζ_{G_s} on the permutation group $G = S_\omega(X)$ is σ-discrete (i.e., G can be represented as a countable union of discrete subspaces of (G, ζ_G)). Consequently, for any submonoid $S \subset G^G$ containing G_s, the S-topology ζ_S on $G = S_\omega(X)$ is σ-discrete. Permutation groups $S_\omega(X)$ belong to the class of perfectly supportable semigroups, introduced in [1]. According to [1, 3.5], for each perfectly supportable semigroup G, the topological space (G, ζ_{G_s}) is σ-discrete.

By Zelenyuk's result [26], mentioned in Remark 2.5, for every infinite group G the Zariski topologies $\zeta_{G_0} = \zeta_{G_1} = \zeta_{G_s}$ and $\zeta_{G_{-1}} = \zeta_{G_q}$ are not discrete. On the other hand, we have:

Proposition 2.10. *For any non-zero number $m \in \mathbb{Z} \setminus \{-2^n, 2^n\colon\ n \in \omega\}$ there is a countable infinite group G with the discrete Zariski topology ζ_{G_m}.*

Proof. By assumption, the number m has a prime divisor $p \geqslant 3$. Let us write m as $m = l \cdot p^\alpha$, where α is a positive integer and l is an integer not divisible by p, and choose a positive integer β such that $p^{\alpha\beta} \geqslant 665$. According to [20] and [21], there is a countable infinite group G containing a cyclic subgroup C of order $|C| = p^{\alpha\beta}$ such that

$$G \setminus \{1_G\} = \big(C \setminus \{1_G\}\big) \cup \bigcup_{c\in C\setminus\{1_G\}} \big\{x \in G\colon\ x^{p^{\alpha\beta}} = c\big\}.$$

Since l^β is coprime to $p^{\alpha\beta}$, it follows that, given any element $c \in C$, we have $c \neq 1_G$ if and only if $c^{l^\beta} \neq 1_G$. This implies

$$G \setminus \{1_G\} = \big(C \setminus \{1_G\}\big) \cup \bigcup_{c\in C\setminus\{1_G\}} \big\{x \in G\colon\ \big(x^{p^{\alpha\beta}}\big)^{l^\beta} = c\big\}.$$

The continuity of the map $x \mapsto x^m$ (with respect to the topology ζ_{G_m}) implies the continuity of the map $f : G \to G$, $f : x \mapsto x^{m^\beta} = x^{p^{\alpha\beta} l^\beta}$. Therefore, the set $G \setminus \{1_G\} = (C \setminus \{1_G\}) \cup f^{-1}(C \setminus \{1_G\})$ is closed in the topology ζ_{G_m}, and its complement $\{1_G\}$ is open, which implies the discreteness of the topology ζ_{G_m}. □

Thus, for every infinite group G, the Zariski topologies ζ_{G_m}, $m \in \{-1, 0, 1\}$, are not discrete, while for any odd $m \notin \{-1, 1\}$, there is an infinite group G with the discrete Zariski topology ζ_{G_m}.

Problem 2.11. Does there exist an infinite group G with the discrete Zariski topology ζ_{G_2}? Equivalently, is there an infinite group G admitting no non-discrete shift-invariant Hausdorff topology with continuous map $x \mapsto x^2$?

By the result of Zelenyuk mentioned in Remark 2.5, each infinite group G is G_s-topologizable. The following simple example shows that this result cannot be generalized to semigroups.

Example 2.12. For the monoid $G = (\mathbb{N}, \min)$, the Zariski topology ζ_{G_s} (which coincides with ζ_{G_l} and ζ_{G_r}) is discrete. Indeed, for each $n \in \mathbb{N}$, the singleton $\{n\}$ belongs to the topology ζ_{G_s}, because

$$\{n\} = \bigl\{x \in \mathbb{N}\colon \ \min\{x, n\} \neq n\bigr\} \cap \bigcap_{k<n} \{x \in \mathbb{N}\colon \ x \neq k\}.$$

This implies that the monoid $G = (\mathbb{N}, \min)$ is not G_s-topologizable.

Example 2.13. For the monoid $G = (\mathbb{N}, \max)$, the Zariski topology ζ_{G_s} (which coincides with ζ_{G_l} and ζ_{G_r}) is discrete. Indeed, for each $n \in \mathbb{N}$, the singleton $\{n\}$ belongs to the topology ζ_{G_s}, because

$$\{n\} = \bigl\{x \in \mathbb{N}\colon \ \max\{x, n\} \neq x\bigr\} \cap \bigcap_{k<n} \{x \in \mathbb{N}\colon \ x \neq k\}.$$

This implies that the monoid $G = (\mathbb{N}, \max)$ is not G_s-topologizable.

3. *G*-topologies on *G*-acts generated by discriminate filters

In this section we describe and study G-topologies on G-acts generated by filters of a special form.

A *filter* on a set X is a family φ of subsets of X such that

- $\emptyset \notin \varphi$;
- $A \cap B \in \varphi$ for any sets $A, B \in \varphi$;
- $A \cup B \in \varphi$ for any sets $A \in \varphi$ and $B \subset X$.

By the *pseudocharacter* $\psi(\varphi)$ *of a filter* φ we understand the smallest cardinality $|\mathcal{F}|$ of a subfamily $\mathcal{F} \subset \varphi$ such that $\bigcap \mathcal{F} = \bigcap \varphi$. The *character* $\chi(\varphi)$ of a filter φ equals the smallest cardinality of a subfamily $\mathcal{F} \subset \varphi$ such that each set $\Phi \in \varphi$ contains some set $F \in \mathcal{F}$. Note that the *character* $\chi(x, \tau)$ of a topological space (X, τ) at a point x can be defined as the character $\chi(\tau_x)$ of the neighborhood filter $\tau_x = \{U \in \tau\colon \ x \in U\}$.

Given a filter φ on X, consider the family

$$\varphi^+ = \{E \subset X\colon \ \forall F \in \varphi \ F \cap E \neq \emptyset\}$$

equal to the union of all filters on X that contain φ. It is easy to check that, for each $A \subset X$ with $A \notin \varphi$, we have $X \setminus A \in \varphi^+$.

A filter φ on a topological space X is said to *converge to a point* x_0 if each neighborhood $U \subset X$ of x_0 belongs to φ.

Now, suppose that G is a monoid, X is a G-act, and φ is a filter on X such that $\bigcap \varphi = \{x_0\}$ for some point x_0. Then we can consider the largest G-topology τ_φ on X in which the filter φ converges to x_0. This topology admits the following simple description.

Proposition 3.1. *The topology τ_φ consists of all sets $U \subset X$ such that, for any $g \in G$ with $x_0 \in g^{-1}(U)$, the preimage $g^{-1}(U)$ belongs to the filter φ.*

Our strategy is to find some class of filters φ on X which generate G-topologies τ_φ on X.

Definition 3.2. Let κ be a cardinal. We say that an injective transfinite sequence $(x_\alpha)_{\alpha<\kappa}$ of points of a G-act X is *discriminate* if there is a (not necessarily bijective) enumeration $G = \{g_\alpha\}_{\alpha<\kappa}$ of the monoid G such that, for all ordinals $\alpha < \kappa$ and $\beta, \gamma, \delta < \alpha$, the following conditions hold:

(1) if $g_\beta(x_0) \neq g_\gamma(x_0)$, then $g_\beta(x_\alpha) \neq g_\gamma(x_\alpha)$;
(2) if $g_\beta(x_0) \neq g_\gamma(x_\delta)$, then $g_\beta(x_\alpha) \neq g_\gamma(x_\delta)$.

Definition 3.3. A filter φ on a G-act X is said to be *discriminate* if there is a cardinal κ and a discriminate sequence $(x_\alpha)_{\alpha<\kappa}$ in X such that $\bigcap\varphi = \{x_0\}$ and $\{x_0\} \cup \{x_\beta\colon \beta > \alpha\} \in \varphi$ for all ordinals $\alpha < \kappa$. In this case, the set $X_0 = \{x_\alpha\}_{\alpha<\kappa}$ is called the *discriminate support* of φ.

For a discriminate filter φ on a G-act X, the G-topology τ_φ has many nice properties.

Theorem 3.4. *For any discriminate filter φ on a G-act X with discriminate support X_0 and intersection $\bigcap\varphi = \{x_0\}$, the G-topology τ_φ has the following properties:*

(1) *the topological space (X, τ_φ) is hereditarily normal;*
(2) *for any $F \in \varphi$, the set $G(F) = \{g(x)\colon g \in G, x \in F\}$ is open and closed in (X, τ_φ) and $X \setminus G(F)$ is discrete in (X, τ_φ);*
(3) $\{F \cap X_0\colon F \in \varphi\} = \{U \cap X_0\colon x_0 \in U \in \tau_\varphi\}$;
(4) *$\psi(x_0, \tau_\varphi) = \psi(\varphi)$ and $\chi(x_0, \tau_\varphi) \geqslant \chi(\varphi)$.*

Proof. By definition, the discriminate support X_0 of φ admits an enumeration $X_0 = \{x_\alpha\}_{\alpha<\kappa}$ such that conditions (1) and (2) in Definition 3.2 hold for some enumeration $G = \{g_\alpha\}_{\alpha<\kappa}$ of the monoid G. If κ is finite, then φ consists of all subsets of X containing x_0, so that the topology τ_φ is discrete and, therefore, automatically possesses the required properties. Thus, hereafter, we assume κ to be infinite.

For every ordinal $\alpha < \kappa$, consider the set $X_{>\alpha} = \{x_\beta\colon \alpha < \beta < \kappa\}$ and observe that $\{x_0\} \cup X_{>\alpha} \in \varphi$ (by Definition 3.3). Now, we shall prove the required properties of the G-topology τ_φ as separate claims.

Claim 3.5. *The topology τ_φ satisfies the separation axiom T_1.*

Proof. Given any point $x \in X$, we must show that $X \setminus \{x\} \in \tau_\varphi$. Since the discriminate filter φ contains all sets $\{x_0\} \cup X_{>\alpha}$ for $\alpha \in \kappa$, it suffices, given any map $g \in G$ with $g(x_0) \in X \setminus \{x\}$, to find $\alpha < \kappa$ such that $g(X_{>\alpha}) \subset X \setminus \{x\}$. If $x \notin G(X_0)$, then $g(X_{>0}) \subset G(X_0) \subset X \setminus \{x\}$, and we are done.

Suppose that $x \in G(X_0)$. Let us find ordinals $\gamma, \delta < \kappa$ such that $x = g_\gamma(x_\delta)$ and, in addition, an ordinal $\beta < \kappa$ for which $g_\beta = g$. Since $g_\beta(x_0) = g(x_0) \neq x = g_\gamma(x_\delta)$, it follows from condition (2) in Definition 3.2 that $g(x_\alpha) = g_\beta(x_\alpha) \neq g_\gamma(x_\delta) = x$ for all $\alpha > \max\{\beta, \gamma, \delta\}$. Consequently, for the ordinal $\alpha = \max\{\beta, \gamma, \delta\}$, we have the required inclusion $g(X_{>\alpha}) \subset X \setminus \{x\}$. □

Claim 3.6. *The topology τ_φ is hereditarily normal.*

Proof. To prove the hereditary normality of the topology τ_φ, take any subspace Y of the topological space (X, τ_φ) and choose two closed disjoint sets A, B in Y. Let $A_0 = \bar{A} \setminus \bar{B}$ and $B_0 = \bar{B} \setminus \bar{A}$, where $\bar{A}$, $\bar{B}$ are the closures of the sets A, B in (X, τ_φ). The equalities $A = \bar{A} \cap Y$ and $B = \bar{B} \cap Y$ imply that $A \subset A_0$ and $B \subset B_0$.

Consider the sequences of sets $(A_n)_{n\in\omega}$ and $(B_n)_{n\in\omega}$ defined by recursion as

$$A_{n+1} = A_n \cup \{g_\alpha(x_\gamma)\colon\ \alpha < \gamma < \kappa,\ g_\alpha(x_0) \in A_n,\ g_\alpha(x_\gamma) \notin B_n \cup \bar{B}\}$$

and

$$B_{n+1} = B_n \cup \{g_\beta(x_\delta)\colon\ \beta < \delta < \kappa,\ g_\beta(x_0) \in B_n,\ g_\beta(x_\delta) \notin A_n \cup \bar{A}\}.$$

We claim that the sets $A_\omega = \bigcup_{n\in\omega} A_n$ and $B_\omega = \bigcup_{n\in\omega} B_n$ are open disjoint neighborhoods of A_0 and B_0 in (X, τ_φ). First, we check that these sets are disjoint. Assuming the opposite, we can find numbers $n, m \in \omega$ such that $A_{n+1} \cap B_{m+1} \neq \emptyset$ but $A_n \cap B_{m+1} = \emptyset = A_{n+1} \cap B_m$. Choose any point $c \in A_{n+1} \cap B_{m+1}$. By the definition of the sets A_{n+1} and B_{m+1}, the point c is of the form $g_\alpha(x_\gamma) = c = g_\beta(x_\delta)$ for some ordinals $\alpha < \gamma < \kappa$ and $\beta < \delta < \kappa$ such that $g_\alpha(x_0) \in A_n$ and $g_\beta(x_0) \in B_m$. It follows from $A_n \cap B_m = \emptyset$ that $g_\alpha(x_0) \neq g_\beta(x_0)$. Condition (1) in Definition 3.2 guarantees that $\gamma \neq \delta$. Without loss of generality, we can assume that $\delta > \gamma$. Since $g_\beta(x_0) \neq g_\alpha(x_\gamma)$, it follows from condition (2) in Definition 3.2 that $g_\beta(x_\delta) \neq g_\alpha(x_\gamma)$; this contradiction shows that $A_\omega \cap B_\omega = \emptyset$.

Now, let us show that the set A_ω is open in (X, τ_φ). Given an ordinal $\alpha < \kappa$ for which $g_\alpha(x_0) \in A_\omega$, we must find a set $F \in \varphi$ such that $g_\alpha(F) \subset A_\omega$. Let $n \in \omega$ be the smallest number for which $g_\alpha(x_0) \in A_n$. We claim that the set $F = \{x_0\} \cup \{x \in X_{>\alpha}\colon\ g_\alpha(x) \notin B_n \cup \bar{B}\}$ belongs to the filter φ. Assuming that $F \notin \varphi$, we conclude that the set $X_{>\alpha} \setminus F$ belongs to the family φ^+, and the set $E_k = \{x \in X_{>\alpha}\colon\ g_\alpha(x) \in B_k \cup \bar{B}\}$ with $k = n$ belongs to φ^+ as well.

Let $k \leqslant n$ be the smallest number for which $E_k \in \varphi^+$. We claim that $k > 0$. Indeed, since $\bar{B}$ is a closed subset in (X, τ_φ), its complement $X \setminus \bar{B}$ is an open neighborhood of the point $g_\alpha(x_0) \in A_n$. By the definition of the topology τ_φ, there is a set $F_0 \in \varphi$ such that $g_\alpha(F_0) \subset X \setminus \bar{B}$. Since F_0 intersects E_k and is disjoint with $E_0 = \{x \in X_{>\alpha}\colon\ g_\alpha(x) \in \bar{B}\}$, we conclude that $k > 0$.

Since $\varphi^+ \not\ni E_{k-1} \subset E_k \in \varphi^+$, the set $E_k \setminus E_{k-1}$ is non-empty and, hence, contains some point x_γ with $\gamma > \alpha$. It follows that $g_\alpha(x_\gamma) \in B_k \setminus B_{k-1}$; therefore, $g_\alpha(x_\gamma) = g_\beta(x_\delta)$ for some ordinals $\beta < \delta < \kappa$ such that $g_\beta(x_0) \in B_{k-1}$.

By condition (1) in Definition 3.2, $\delta \neq \gamma$ (because $g_\alpha(x_\gamma) = g_\beta(x_\delta)$ and $g_\alpha(x_0) \neq g_\beta(x_0)$). If $\delta > \gamma$, then the equality $g_\alpha(x_\gamma) = g_\beta(x_\delta)$ is excluded by condition (2) in Definition 3.2, since $B_{k-1} \not\ni g_\alpha(x_\gamma) \neq g_\beta(x_0) \in B_{k-1}$. If $\gamma > \delta$, then the equality $g_\alpha(x_\gamma) = g_\beta(x_\delta)$ is also excluded by (2), because $A_n \ni g_\alpha(x_0) \neq g_\beta(x_\delta) = g_\alpha(x_\gamma) \in B_k$. This contradiction shows that $F \in \varphi$ and $g_\alpha(F) \subset A_{n+1} \subset A_\omega$, which means the openness of A_ω.

Similarly, the set B_ω is open in (X, τ_φ). Thus, $A_\omega \cap Y$ and $B_\omega \cap Y$ are disjoint open neighborhoods of the sets $A = A_0 \cap Y$ and $B = B_0 \cap Y$ in Y, which proves the normality of the topological T_1-space Y and hereditary normality of (X, τ_φ). □

The definition of the topology τ_φ implies that, for every $F \in \varphi$, the set $G(F) = \{g(x)\colon\ g \in G, x \in F\}$ is open and closed in (X, τ_φ) and $X \setminus G(F)$ is discrete in (X, τ_φ).

Claim 3.7. $\{F \cap X_0\colon\ F \in \varphi\} = \{U \cap X_0\colon\ x_0 \in U \in \tau_\varphi\}$ *and, therefore,* $\chi(\varphi) \leqslant \chi(x_0, \tau_\varphi)$.

Proof. By the definition of the topology τ_φ, we have $\{U \in \tau_\varphi\colon\ x_0 \in U\} \subset \varphi$; hence $\{U \cap X_0\colon\ x_0 \in U \in \tau_\varphi\} \subset \{F \cap X_0\colon\ F \in \varphi\} = \{F \in \varphi\colon\ F \subset X_0\}$.

To prove the reverse inclusion, fix any subset $F \in \varphi$ with $F \subset X_0$ and consider the set $U = F \cup (X \setminus X_0)$. We claim that $U \in \tau_\varphi$. To show this, given any ordinal $\alpha < \kappa$ for which $g_\alpha(x_0) \in U$, we must find a set $E \in \varphi$ such that $g_\alpha(E) \subset U$. Take $\beta < \kappa$ for which $g_\beta = \mathrm{id}_X$ and consider the set

$$E = \{x_0\} \cup \{x_\gamma \in F\colon\ \max\{\alpha, \beta\} < \gamma < \kappa\} \in \varphi.$$

We have $g_\alpha(E) \subset U$. Indeed, assuming the converse, we can find an ordinal $\gamma > \max\{\alpha, \beta\}$ such that $x_\gamma \in F$ and $g_\alpha(x_\gamma) \in X_0 \setminus F$, which implies $g_\alpha(x_\gamma) = x_\delta = \mathrm{id}_X(x_\delta) = g_\beta(x_\delta)$ for some ordinal $\delta < \kappa$. Since $x_\gamma \in F$ and $x_\delta \notin F$, it follows that the ordinals γ and δ are distinct.

If $\gamma < \delta$, then the inequality $g_\beta(x_0) = x_0 \neq x_\delta = g_\alpha(x_\gamma)$ and condition (2) in Definition 3.2 imply $g_\beta(x_\delta) \neq g_\alpha(x_\gamma)$, which is a contradiction.

If $\gamma > \delta$, then the inequality $g_\alpha(x_0) \neq x_\delta = g_\beta(x_\delta)$ and condition (2) in Definition 3.2 imply $g_\alpha(x_\gamma) \neq g_\beta(x_\delta) = x_\delta$, which again leads to a contradiction. □

Claim 3.8. *The topology τ_φ has pseudocharacter $\psi(x_0, \tau_\varphi) = \psi(\varphi)$ at the point x_0.*

Proof. The inequality $\psi(\varphi) \leqslant \psi(x_0, \tau_\varphi)$ follows from Claim 3.7. To show that $\psi(x_0, \tau_\varphi) \leqslant \psi(\varphi)$, fix a family $\mathcal{F} \subset \varphi$ such that $|\mathcal{F}| = \psi(\varphi)$ and $\bigcap \mathcal{F} = \{x_0\}$. For every $F \in \mathcal{F}$, we take the open neighborhood $U^F \in \tau_\varphi$ of x_0 being the union $U^F = \bigcup_{n\in\omega} U_n^F$ of the sequence of sets $(U_n^F)_{n\in\omega}$ defined by the recursion as $U_0^F = \{x_0\}$ and

$$U_{n+1}^F = U_n^F \cup \{g_\alpha(x_\beta)\colon\ \alpha < \beta < \kappa,\ x_\beta \in F,\ g_\alpha(x_0) \in U_n^F\} \quad \text{for every } n \in \omega.$$

The definition of the topology τ_φ implies that $U^F = \bigcup_{n\in\omega} U_n^F$ is an open neighborhood of x_0 in X.

Let us show that $\bigcap_{F\in\mathcal{F}} U^F = \{x_0\}$. Assume that, on the contrary, this intersection contains a point x distinct from x_0. For every $F \in \mathcal{F}$, find the smallest number $n_F \in \omega$ such that $x \in U_{n_F}^F$. Since $U_0^F = \{x_0\} \neq \{x\}$, we conclude that $n_F > 0$; hence $x \notin U_{n_F-1}^F$. By the definition of the set $U_{n_F}^F$, there are ordinals $\alpha_F < \beta_F < \kappa$ such that $x_{\beta_F} \in F$ and $x = g_{\alpha_F}(x_{\beta_F}) \neq g_{\alpha_F}(x_0) \in U_{n_F-1}^F$.

We claim that there are two sets $F, E \in \varphi$ with $\beta_F \neq \beta_E$. Fix any set $F \in \mathcal{F}$. Since $x_{\beta_F} \in F$ and $x_{\beta_F} \notin \{x_0\} = \bigcap \mathcal{F}$, there is a set $E \in \mathcal{F}$ such that $x_{\beta_F} \notin E$. Then $\beta_F \neq \beta_E$. So, $F, E \in \varphi$ are two sets with $\beta_F \neq \beta_E$. We lose no generality assuming that $\beta_F < \beta_E$. Since $\beta_E > \max\{\alpha_E, \beta_F, \alpha_F\}$ and $g_{\alpha_E}(x_0) \neq x = g_{\alpha_F}(x_{\beta_F})$, the condition (2) of Definition 3.2 guarantees that $x = g_{\alpha_E}(x_{\beta_E}) \neq g_{\alpha_F}(x_{\beta_F}) = x$, which is the desired contradiction that proves the equality $\bigcap_{F\in\mathcal{F}} U^F = \{x_0\}$ and the upper bound $\psi(x_0, \tau_\varphi) \leqslant \psi(\varphi)$. □

□

4. Zariski G-topology and the existence of discriminate filters

In light of Theorem 3.4, it is of interest to describe G-acts possessing discriminate sequences and discriminate filters.

Proposition 4.1. *Suppose that X is a G-act over a monoid G, $x_0 \in X$, and κ is an infinite cardinal.*

(1) *If $|G| \leqslant \kappa \leqslant \psi(x_0, \zeta_G)$, then the G-act X contains a discriminate sequence $(x_\alpha)_{\alpha<\kappa}$.*
(2) *If the G-act X contains a discriminate sequence $(x_\alpha)_{\alpha<\kappa}$, then $|G| \leqslant \kappa$ and $\mathrm{cf}(\kappa) \leqslant \psi(x_0, \zeta_G)$.*

Proof. 1. Suppose that $|G| \leqslant \kappa \leqslant \psi(x_0, \zeta_G)$ and $G = \{g_\alpha\colon\ \alpha < \kappa\}$ is an enumeration of the monoid G such that $g_0 = 1_G$. Let us construct an injective transfinite sequence $(x_\alpha)_{\alpha<\kappa}$ of points of the set X such that, for any $\alpha < \kappa$ and $\beta, \gamma, \delta < \alpha$,

(a) if $g_\beta(x_0) \neq g_\gamma(x_0)$, then $g_\beta(x_\alpha) \neq g_\gamma(x_\alpha)$;
(b) if $g_\beta(x_0) \neq g_\gamma(x_\delta)$, then $g_\beta(x_\alpha) \neq g_\gamma(x_\delta)$.

We construct this sequence by transfinite induction.

Assume that, for some ordinal $\alpha < \kappa$, the points x_β with $\beta < \alpha$ have already been constructed. For any ordinals $\beta, \gamma, \delta < \alpha$, consider the open neighborhoods

$$U_{\beta,\gamma} = \bigl\{x \in X\colon\ g_\beta(x_0) \neq g_\gamma(x_0) \Rightarrow g_\beta(x) \neq g_\gamma(x)\bigr\}$$

and

$$V_{\beta,\gamma,\delta} = \bigl\{x \in X\colon\ g_\beta(x_0) \neq g_\gamma(x_\delta) \Rightarrow g_\beta(x) \neq g_\gamma(x_\delta)\bigr\}$$

of x_0 in the Zariski G-topology ζ_G. Since $\psi(x_0, \zeta_G) \geqslant \kappa$, the intersection $\bigcap_{\beta,\gamma,\delta<\alpha} U_{\beta,\gamma} \cap V_{\beta,\gamma,\delta}$ has cardinality $\geqslant \kappa$ and hence contains some point $x_\alpha \in X \setminus \{x_\beta\colon\ \beta < \alpha\}$. Clearly, this point x_α satisfies conditions (a) and (b).

2. Now, suppose that X contains a discriminate sequence $X_0 = \{x_\alpha\}_{\alpha<\kappa}$ for some infinite cardinal κ. Let $G = \{g_\alpha\}_{\alpha<\kappa}$ be an enumeration of the monoid G such that conditions (1) and (2) in Definition 3.2 are satisfied. Then $|G| \leqslant \kappa$.

Let us prove that $\psi(x_0, \zeta_G) \geqslant \mathrm{cf}(\kappa)$. Assume that, on the contrary, there exists a family $\mathcal{U} \subset \tilde{\zeta}_G$ such that $\bigcap \mathcal{U} = \{x_0\}$ and $|\mathcal{U}| < \mathrm{cf}(\kappa)$. For each set $U \in \mathcal{U} \subset \tilde{\zeta}_G$, choose ordinals $\alpha_U, \beta_U < \kappa$ and a point $c_U \in X$ so that U is equal to either $\{x \in X\colon\ g_{\alpha_U}(x) \neq g_{\beta_U}(x)\}$ or $\{x \in X\colon\ g_{\alpha_U}(x) \neq c_U\}$. If $c_U \in G(X_0)$, then we also choose ordinals γ_U, $\delta_U < \kappa$ such that $c_U = g_{\gamma_U}(x_{\delta_U})$; otherwise, we set $\gamma_U = \delta_U = 0$. Since the set $A_U = \{\alpha_U, \beta_U, \gamma_U, \delta_U\colon\ U \in \mathcal{U}\}$ has cardinality $< \mathrm{cf}(\kappa)$, there is an ordinal $\alpha < \kappa$ for which $\alpha > \sup A_U$. We claim that $x_\alpha \in \bigcap \mathcal{U}$. Indeed, take any set $U \in \mathcal{U}$. If $U = \{x \in X\colon\ g_{\alpha_U}(x) \neq g_{\beta_U}(x)\}$, then, by condition (1) in Definition 3.2, we have $x_\alpha \in U$, because $x_0 \in U$. If $U = \{x \in X\colon\ g_{\alpha_U}(x) \neq c_U\}$ and $c_U \in G(X_0)$, then the relations $c_U = g_{\gamma_U}(x_{\delta_U})$ and $x_0 \in U$ and condition (2) in Definition 3.2 imply $x_\alpha \in U$. If $c_U \notin G(X_0)$, then $g_{\alpha_U}(x_\alpha) \neq c_U$, and hence $x_\alpha \in U$. Therefore, $x_\alpha \in \bigcap \mathcal{U} = \{x_0\}$, which is the desired contradiction. □

5. *G*-topologizability of *G*-acts

In this section we apply the results of the preceding sections to prove the following theorem, which is our main result.

Theorem 5.1. *Let X be a G-act over a monoid G such that $|G| \leqslant \psi(x_0, \zeta_G)$ for some point $x_0 \in X$. Then, for every infinite cardinal κ satisfying $|G| \leqslant \kappa \leqslant \psi(x_0, \zeta_G)$ and every infinite cardinal $\lambda \leqslant \mathrm{cf}(\kappa)$, the G-act X admits 2^{2^κ} hereditarily normal G-topologies with pseudocharacter λ at the point x_0.*

Proof. By Proposition 4.1, the space X contains a discriminate sequence $X_0 = \{x_\alpha\}_{\alpha<\kappa}$. Let φ_0 be the filter on X generated by the sets $\{x_0\} \cup \{x_\beta\colon\ \beta > \alpha\}$, where $\alpha < \kappa$, and let $\uparrow\varphi_0$ denote the set of all filters φ on X which contain the filter φ_0 and satisfy the condition $\bigcap \varphi = \{x_0\}$.

Claim 5.2. *For any infinite cardinal $\lambda \leqslant \mathrm{cf}(\kappa)$, the set $\mathcal{F}_\lambda = \{\varphi \in \uparrow\varphi_0\colon\ \psi(\varphi) = \lambda\}$ has cardinality $|\mathcal{F}_\lambda| = 2^{2^\kappa}$.*

Proof. First, observe that the family of all filters on the set X_0 has cardinality $\leqslant 2^{2^\kappa}$. Thus, $|\mathcal{F}_\lambda| \leqslant 2^{2^\kappa}$. To prove the reverse inequality, we consider two cases.

Case 1: $\lambda = \mathrm{cf}(\kappa)$. Let us represent the set X_0 as the disjoint union $X_0 = X_0' \cup X_0''$ of two sets of cardinality $|X_0'| = |X_0''| = \kappa$ such that $x_0 \in X_0'$. On the set X_0' consider the filter $\varphi_0|X_0' = \{F \cap X_0'\colon\ F \in \varphi_0\}$. Pospíšil's Theorem [23] (see also [13]) implies that the family $\mathcal{U}_0$ of all ultrafilters on X_0'' which contain the filter $\varphi_0|X_0'' = \{F \cap X_0''\colon\ F \in \varphi_0\}$ has cardinality 2^{2^κ}. For any ultrafilter $u \in \mathcal{U}_0$, consider the filter $\varphi_u = \{A \subset X\colon\ A \cap X_0'' \in u, A \cap X_0' \in \varphi_0|X_0'\}$ and note that $\psi(\varphi_u) = \psi(\varphi_0|X_0') = \mathrm{cf}(\kappa)$. For distinct ultrafilters $u, v \in \mathcal{U}_0$, the filters φ_u and φ_v are distinct; therefore, $|\mathcal{F}_{\mathrm{cf}(\kappa)}| \geqslant 2^{2^\kappa}$.

Case 2: $\lambda < \mathrm{cf}(\kappa)$. In this case the ordinal κ can be identified with the product $\kappa \times \lambda$ endowed with the lexicographic order, in which $(\alpha, \beta) < (\alpha, \beta')$ if and only if either $\alpha < \alpha'$ or $\alpha = \alpha'$ and $\beta < \beta'$. Let $\xi : \kappa \times \lambda \to \kappa$ be the order isomorphism. Consider the filter φ_λ of cofinite subsets on the cardinal λ. This filter has pseudocharacter $\psi(\varphi_\lambda) = \lambda$. As shown in Case 1, the family $\mathcal{F}_{\mathrm{cf}(\kappa)}$ has cardinality 2^{2^κ}. Given any filter $u \in \mathcal{F}_{\mathrm{cf}(\kappa)}$, consider the filter φ_u on X generated by the sets

$$\Phi_{U,L} = \{x_0\} \cup \{x_{\xi(\alpha,\beta)}\colon\ x_\alpha \in U,\ \beta \in L\} \quad \text{where } U \in u,\ L \in \varphi_\lambda.$$

It can be shown that $\psi(\varphi_u) = \psi(\varphi_\lambda) = \lambda$ and the filters φ_u and φ_v are distinct for any distinct filters $u, v \in \mathcal{F}_{\mathrm{cf}(\kappa)}$. Consequently, $|\mathcal{F}_\lambda| \geqslant |\mathcal{F}_{\mathrm{cf}(\kappa)}| \geqslant 2^{2^\kappa}$. □

According to Theorem 3.4, for any filter $\varphi \in \mathcal{F}_\lambda$, the G-topology τ_φ on X is hereditarily normal and has pseudocharacter $\psi(x_0, \tau_\varphi) = \psi(\varphi) = \lambda$ at x_0. Theorem 3.4(3) implies that distinct filters $u, v \in \mathcal{F}_\lambda$ determine distinct topologies τ_u and τ_v. Therefore, X admits at least $|\mathcal{F}_\lambda| = 2^{2^\kappa}$ hereditarily normal G-topologies with pseudocharacter λ at x_0. □

Remark 5.3. Example 2.7 shows that the condition $|G| \leqslant \kappa \leqslant \psi(x_0, \zeta_G)$ in Theorem 5.1 is sufficient but not necessary for normal G-topologizability: the group $G = H^\kappa$ is normally G_s-topologizable but $\psi(x_0, \zeta_{G_s}) = \kappa < 2^\kappa = |G_s|$ for any point x_0. On the other hand, this condition is necessary for the existence of a Hausdorff G-topology with pseudocharacter $|G|$ (indeed, it follows directly from Proposition 2.2 that $\psi(x_0, \zeta_{G_s}) \geqslant \psi(x_0, \tau)$ for any Hausdorff G-topology τ on X). Thus, for acts over monoids of regular cardinality, Theorem 5.1 has the following corollary: *Let X be a G-act over a monoid G of regular cardinality $\kappa = |G|$, and let $x_0 \in X$. Then $\psi(x_0, \zeta_G) = \kappa$ if and only if X admits a Hausdorff G-topology τ with $\psi(x_0, \tau) = \kappa$.*

Remark 5.4. For each of the S-acts G over $S \in \{G_l, G_r, G_s, G_q, G_{[s]}, G_{[q]}, G_p^+, G_p\} \cup \{G_m\colon\ m \in \mathbb{Z}\}$ defined in Section 2 and any infinite cardinals λ and κ, $\lambda \leqslant \mathrm{cf}(\kappa)$, the condition $|G| \leqslant \kappa \leqslant \psi(x_0, \zeta_S)$ for some point $x_0 \in X$ implies the existence of 2^{2^κ} hereditarily normal S-topologies with pseudocharacter λ at the point x_0. In particular, any group G admits $2^{2^{|G|}}$ hereditarily normal left-invariant and $2^{2^{|G|}}$ hereditarily normal right-invariant topologies.

In the case of countable monoids G, Theorem 5.1 implies the following characterization of G-topologizability, which solves Problem 1.2 in this case.

Theorem 5.5. *For a countable monoid G and a G-act X, the following conditions are equivalent:*

(1) *X admits a non-discrete Hausdorff G-topology;*
(2) *X admits a non-discrete metrizable G-topology;*
(3) *the Zariski G-topology ζ_G on X is non-discrete;*
(4) *X admits $2^{\mathfrak{c}}$ non-discrete hereditarily normal G-topologies.*

Proof. The equivalence (1) ⇔ (3) ⇔ (4) follows from Theorems 3.4 and 5.1. The implication (2) ⇒ (1) is trivial. It remains to prove that (3) ⇒ (2). If the Zariski G-topology ζ_G on X is non-discrete, then some point $x_0 \in X$ has infinite pseudocharacter $\psi(x_0, \zeta_G)$. By Proposition 4.1, the G-act contains a discriminate sequence $(x_n)_{n<\omega}$. Let φ be the discriminate filter on X generated by the base consisting of the sets $\{x_0\} \cup \{x_m\colon\ m \geqslant n\}$, $n \in \omega$, and let $\tau_\varphi = \{U \subset X\colon\ \forall g \in G g^{-1}(U) \in \varphi\}$ be the G-topology on X generated by the filter φ. By Theorem 3.4(2), the orbit $G(X_0)$ of the set $X_0 = \{x_n\}_{n<\omega}$ is open in (X, τ_φ), and its complement $X \setminus G(X_0)$ is an open discrete subspace of X.

By Theorem 3.4(1, 4), the topology τ_φ is non-discrete and hereditary normal, which implies that the open subspace $G(X_0)$ is Tychonoff. Therefore, given any distinct points $x, y \in G(X_0)$, we can find a continuous function $f_{x,y} : G(X_0) \to [0, 1]$ such that $f_{x,y}(x) = 0$ and $f_{x,y}(y) = 1$. Since the monoid G is countable, so is the function family

$$\mathcal{F} = \big\{f_{x,y} \circ g\colon\ g \in G,\ x, y \in G(X_0),\ x \neq y\big\}.$$

Choose any enumeration $\mathcal{F} = \{f_n\}_{n\in\omega}$ of the family $\mathcal{F}$ and consider the continuous metric d on $G(X_0)$ defined by

$$d(x,y) = \max_{n\in\omega} \frac{1}{2^n}\bigl|f_n(x) - f_n(y)\bigr|.$$

We extend the metric d to the whole space X by setting $d(x,x) = 0$ for any $x \in X$ and $d(x,y) = 1$ for any pair $(x,y) \in X^2 \setminus G(X_0)^2$ with $x \neq y$. Since $X \setminus G(X_0)$ is an open-and-closed discrete subspace of X, the metric d thus extended remains continuous. It can be shown that each function $g \in G \subset X^X$ is continuous with respect to the metric d; hence the topology τ_d on X generated by the metric d is a G-topology, which is contained in the G-topology τ_φ and therefore not discrete. Thus, τ_d is the required non-discrete metrizable G-topology on X. □

Note that Theorem 5.5 applies also to finite monoids G. Indeed, it is easy to reduce the finite case to the infinitely countable one by considering the direct product $G \times S$ of a given finite monoid G and any infinite countable monoid S acting trivially on X. Clearly, for the product action of $G \times S$, we have $\zeta_G = \zeta_{G\times S}$, and any G-topology on X is a $(G \times S)$-topology.

We do not know whether this theorem remains valid for arbitrary G-acts.

Problem 5.6. Let G be an uncountable monoid (group), and let X be a G-act for which the Zariski G-topology ζ_G is non-discrete. Is it true that X is G-topologizable?

In solving this problem, results of [12] may be useful.

The general Problem 5.6 is also interesting in special cases, for example, where X is a (semi)group, the action is assumed to satisfy additional conditions (e.g., be transitive, faithful, or free), or the orbit space has certain properties (e.g., is countable).

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References

[1] T. Banakh, I. Guran, Perfectly supportable semigroups are σ-discrete in each Hausdorff shift-invariant topology, Topol. Algebra Appl. 1 (2013) 1–8.
[2] T. Banakh, I. Guran, I. Protasov, Algebraically determined topologies on permutation groups, Topol. Appl. 159 (9) (2012) 2258–2268.
[3] T. Banakh, I. Protasov, Topologizations of G-spaces, Ukr. Math. Bull. 9 (3) (2012) 308–317.
[4] R. Bryant, The verbal topology of a group, J. Algebra 48 (2) (1977) 340–346.
[5] D. Dikranjan, D. Shakhmatov, Selected topics from the structure theory of topological groups, in: E.M. Pearl (Ed.), Open Problems in Topology II, Elsevier Science B.V., Amsterdam, 2007, pp. 389–406.
[6] D. Dikranjan, D. Shakhmatov, Reflection principle characterizing groups in which unconditionally closed sets are algebraic, J. Group Theory 11 (3) (2008) 421–442.
[7] D. Dikranjan, D. Shakhmatov, The Markov–Zariski topology of an abelian group, J. Algebra 324 (6) (2010) 1125–1158.
[8] D. Dikranjan, I. Protasov, Counting maximal topologies on countable groups and rings, Topol. Appl. 156 (2008) 322–325.
[9] D. Dikranjan, D. Toller, Markov's problems through the looking glass of Zariski and Markov topologies, in: Ischia Group Theory 2010, Proceedings of the Conference, World Scientific Publ., Singapore, 2012, pp. 87–130.
[10] D. Dikranjan, D. Toller, The Markov and Zariski topologies of some linear groups, Topol. Appl. 159 (13) (2012) 2951–2972.
[11] G. Hesse, Zur Topologisierbarkeit von Gruppen, Dissertation, Univ. Hannover, Hannover, 1979.
[12] N. Hindman, I. Protasov, D. Strauss, Topologies on S determined by idempotents in βS, Topol. Proc. 23 (Summer) (1998) 155–190.
[13] T. Jech, Set Theory, Springer-Verlag, Berlin, 2003.
[14] M. Kilp, U. Knauer, A.V. Mikhalev, Monoids, Acts and Categories. With Applications to Wreath Products and Graphs, Expositions in Math., vol. 29, Walter de Gruyter, Berlin, 2000.

[15] A.A. Klyachko, A.Yu. Olshanskii, D.V. Osin, On topologizable and non-topologizable groups, Topol. Appl. 160 (16) (2013) 2104–2120.
[16] A.A. Klyachko, A.V. Trofimov, The number of non-solutions of an equation in a group, J. Group Theory 8 (6) (2005) 747–754.
[17] A.A. Markov, On unconditionally closed sets, Dokl. Akad. Nauk SSSR 44 (5) (1944) 196–197;
English translation: A. Markoff, On unconditionally closed sets, C.R. (Dokl.) Acad. Sci. URSS (N.S.) 44 (1944) 180–181.
[18] A.A. Markov, On free topological groups, Izv. Akad. Nauk SSSR, Ser. Mat. 9 (1) (1945) 3–64; English translation: Three papers on topological groups: I. On the existence of periodic connected topological groups. II. On free topological groups. III. On unconditionally closed sets, Am. Math. Soc. Transl. 30 (1950).
[19] A.A. Markov, On unconditionally closed sets, Mat. Sb. 18 (1) (1946) 3–28; English translation: Three papers on topological groups: I. On the existence of periodic connected topological groups. II. On free topological groups. III. On unconditionally closed sets, Am. Math. Soc. Transl. 30 (1950).
[20] A.Yu. Ol'shanskii, A remark on a countable nontopologized group, Vestn. Mosk. Univ., Ser. I., Mat. Meh. 3 (1980) 103.
[21] A.Yu. Ol'shanskii, Geometry of Defining Relations in Groups, Nauka, Moscow, 1989; English translation in Math. Appl., Sov. Ser., vol. 70, Kluwer Acad. Publishers, Dordrecht, 1991.
[22] K.-P. Podewski, Topologisierung algebraischer Strukturen, Rev. Roum. Math. Pures Appl. 22 (9) (1977) 1283–1290.
[23] B. Pospíšil, Remark on bicompact spaces, Ann. Math. (2) 38 (4) (1937) 845–846.
[24] O. Sipacheva, Unconditionally τ-closed and τ-algebraic sets in groups, Topol. Appl. 155 (4) (2008) 335–341.
[25] E. Zelenyuk, On topologies on groups with continuous shifts and inversion, Visn. Kyiv. Univ., Ser. Fiz.-Mat. Nauki 2 (2000) 252–256.
[26] Y. Zelenyuk, On topologizing groups, J. Group Theory 10 (2) (2007) 235–244.