A BIRKHOFF–LEWIS TYPE THEOREM FOR SOME HAMILTONIAN PDEs

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Abstract

In this paper we give an extension of the Birkhoff–Lewis theorem to some semilinear PDEs. Accordingly we prove existence of infinitely many periodic orbits with large period accumulating at the origin. Such periodic orbits bifurcate from resonant finite dimensional invariant tori of the fourth order normal form of the system. Besides standard nonresonance and nondegeneracy assumptions, our main result is obtained assuming a regularizing property of the nonlinearity. We apply our main theorem to a semilinear beam equation and to a nonlinear Schrödinger equation with smoothing nonlinearity.

1 Introduction

In 1934 Birkhoff and Lewis [BL34] (see also [Lew34, Mos77]) proved their celebrated theorem on existence of periodic orbits with large period close to elliptic equilibria of Hamiltonian systems[1]. Here we give a generalization of their result to some semilinear Hamiltonian PDEs.

Birkhoff–Lewis procedure consists in putting the system in fourth order (Birkhoff) normal form, namely in the form

$$H = H_0 + G_4 + R_5, \quad H_0 := \sum_{j=1}^{n} \omega_j \frac{p_j^2 + q_j^2}{2}, \tag{1}$$

where G_4 is a homogeneous polynomial of degree 2 in the actions $I_j := (p_j^2 + q_j^2)/2$ and R_5 is a remainder having a zero of fifth order at the origin. Then system (1) is a perturbation of the integrable system $H_0 + G_4$. Under a nondegeneracy condition (that also plays a

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[1] Actually [BL34] considers the neighborhood of an elliptic, non-constant, periodic orbit, but the scheme is essentially the same for elliptic equilibria.

fundamental role in KAM theory) the action to frequency map of this integrable system is one to one, and therefore there exist infinitely many resonant tori on which the motion is periodic. The question is: Do some of these periodic orbits persist under the perturbation due to the term R_5? Birkhoff–Lewis used the implicit function theorem and a topological argument to prove that there exists a sequence of resonant tori accumulating at the origin with the property that at least two periodic orbits bifurcate from each one of them.

In order to extend this result to infinite dimensional systems describing Hamiltonian PDEs one meets two difficulties: the first one is the generalization of Birkhoff normal form to PDEs and the second one is the appearing of a small denominator problem.

Here we decide to work in a way which is as straightforward as possible, so, instead of considering the standard Birkhoff normal form of the system, whose extension to PDEs is not completely understood at present[2], we consider its "seminormal form", namely the kind of normal form employed to construct lower dimensional tori. Precisely, having fixed a positive n, we split the phase variables in two groups, namely the variables with index smaller than n and the variables with index larger than n. We will denote by $\hat{z}$ the whole set of variables with index larger than n. We construct a canonical transformation putting the system in the form

$$H_0 + \overline{G} + \hat{G} + K \tag{2}$$

where $\overline{G}$ depends only on the actions, $\hat{G}$ is at least cubic in the variables $\hat{z}$ with index larger than n, and K has a zero of sixth order at the origin. The interest of such a seminormal form is that the normalized system $H_0+\overline{G}+\hat{G}$ has the invariant $2n$–dimensional manifold $\hat{z}=0$ which is filled by n–dimensional invariant tori.

Under a nondegeneracy condition, the frequencies of the flow in such tori cover an open subset of $\mathbf{R}^n$. We concentrate on the resonant tori filled by periodic orbits and we prove that at least n geometrically distinct periodic orbits of each torus survive the perturbation due to the term K. Since the orbits bifurcate from lower dimensional tori we have to impose a further nondegeneracy condition in order to avoid resonances between the frequency of the periodic orbit and the frequencies of the transversal oscillations.

The proof is based on a variational Lyapunov Schmidt reduction similar to that employed in [BBV03] and inspired to [ACE87]. It turns out that in the present case the range equation involves small denominators. To solve the corresponding problem we use an approach similar to that of [Bam00]. In particular we impose a strong condition on the small denominators and we show that, if the vector field of the nonlinearity is smoothing, then the range equation can be solved by the contraction mapping principle. Next, the kernel equation is solved by noting that it is the Euler-Lagrange equation of the action functional restricted to the solutions of the range equation. The restricted functional turns out to be defined on $\mathbf{T}^n$ and so existence and multiplicity of solutions (critical points) follows by the classical Lusternik-Schnirelmann theory.

Finally we apply the general theorem to the nonlinear beam equation

$$u_{tt} + u_{xxxx} + mu = f(u) \ . \tag{3}$$

[2]See however the recent works [Bam03,BG03].

with Dirichlet boundary conditions on a segment. We consider m as a parameter varying in the segment $[0, L]$ and we show that the assumptions of the abstract theorem are fulfilled provided one excludes from the interval a finite number of values of m. As a second application we will deal with a nonlinear Schrödinger equation with a smoothing nonlinearity of the type considered in [Pös02].

We recall that families of periodic solutions to Hamiltonian PDEs have been constructed by many authors (see e.g. [Kuk93m, Way90, Pös96b, CW93, Bou95, Bam00]). The main difference is that the periodic orbits of the above quoted papers are a continuation of the linear normal modes to the nonlinear system. In particular their period is close to one of the periods of the linearized system. Moreover (except in the resonant case, see [BP01,BB03]) each periodic solution involve only one of the linear oscillators[3].

On the contrary, the periodic orbits constructed in the present paper are the shadows of resonant tori; they are a purely nonlinear phenomenon, have long period, and moreover each periodic motion involves n linear oscillators that oscillate with amplitudes of the same order of magnitude.

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2 Main result

Consider a real Hamiltonian system with real[4] Hamiltonian function

$$H(z, \overline{z}) = \sum_{j \geq 1} \omega_j z_j \overline{z}_j + P(z, \overline{z}) \equiv H_0 + P \tag{4}$$

where P has a zero of third order at the origin and the symplectic structure is given by $\mathrm{i} \sum_j dz_j \wedge d\overline{z}_j$. Here z and $\bar{z}$ are considered as independent variables. Often we will write only the equation for z since the equation for $\bar{z}$ is obtained by complex conjugation. The formal Hamiltonian vector field of the system is $X_H(z, \overline{z}) := (\mathrm{i}\frac{\partial H}{\partial \overline{z}_j}, -\mathrm{i}\frac{\partial H}{\partial z_j})$, and therefore the equations of motion have the form

$$\dot{z}_j = \mathrm{i}\omega_j z_j + \mathrm{i}\frac{\partial P}{\partial \overline{z}_j} \ , \quad \dot{\overline{z}}_j = -\mathrm{i}\omega_j \overline{z}_j - \mathrm{i}\frac{\partial P}{\partial z_j}. \tag{5}$$

Define the complex Hilbert space

$$\mathcal{H}^{a,s}(\mathbf{C}) := \Big\{ w = (w_1, w_2, \ldots) \in \mathbf{C}^\infty \ \Big| \ ||w||^2_{a,s} := \sum_{j \geq 1} |w_j|^2 j^{2s} e^{2ja} < \infty \Big\}.$$

We fix $s \geq 0$ and $a \geq 0$ and we will study the system in the phase space

$$\mathcal{P}_{a,s} := \mathcal{H}^{a,s}(\mathbf{C}) \times \mathcal{H}^{a,s}(\mathbf{C}) \ni (z, \bar{z}) \ .$$

Fix any finite integer $n \geq 2$ and denote $\omega := (\omega_1, \ldots, \omega_n)$, $\Omega := (\omega_{n+1}, \omega_{n+2}, \ldots)$. We assume

[3] in the sense that all the other ones have a much smaller amplitude of oscillation.

[4] i.e. if $\overline{z}$ is actually the complex conjugate to z then the Hamiltonian H takes real values.

(A) The frequencies grow at least linearly at infinity, namely there exists $a > 0$ and $d_1 \geq 1$ such that

$$\omega_j \sim a j^{d_1}.$$

(NR) For any $k \in \mathbf{Z}^n$, $l \in \mathbf{Z}^\infty$ with $|l| \leq 2$ and $|k| + |l| \leq 5$, one has

$$\omega \cdot k + \Omega \cdot l \neq 0. \tag{6}$$

(S) There exists a neighbourhood of the origin $\mathcal{U} \subset \mathcal{P}_{a,s}$ and $d \geq 0$ such that $X_P \in C^\infty(\mathcal{U}, \mathcal{P}_{a,s+d})$.

Remark 2.1 *In applications to PDEs, property (S) is usually a consequence of the smoothness of the Nemitsky operator defined by the nonlinear part of the equation. In order to ensure (S) one has usually to restrict to the case where the functions with Fourier coefficients in $\mathcal{P}_{a,s}$ is an algebra (with the product of convolution between sequences). This imposes some limitations to the choice of the indices a, s.*

Proposition 2.1 *Assume (A,NR,S). There exists a real analytic, symplectic change of variables $\mathcal{T}$ defined in some neighborhood $\mathcal{U}' \subset \mathcal{P}_{a,s}$ of the origin, transforming the Hamiltonian H in seminormal form up to order six, namely into*

$$H \circ \mathcal{T} \equiv \mathcal{H} = H_0 + \overline{G} + \hat{G} + K \tag{7}$$

with

$$\overline{G} = \frac{1}{2} \sum_{\min(i,j) \leq n} \overline{G}_{ij} |z_i|^2 |z_j|^2,$$

$\overline{G}_{ij} = \overline{G}_{ji}$, $\hat{G} = O(||\hat{z}||^3_{a,s})$ *where* $\hat{z} := (z_{n+1}, z_{n+2}, \ldots)$ *and* $K = O(||z||^6_{a,s})$*. Moreover*

$$X_{\overline{G}}, X_{\hat{G}}, X_K \in C^\infty(\mathcal{U}', \mathcal{P}_{a,s+d}) \ , \quad \|z - \mathcal{T}(z)\|_{a,s+d} \leq C \|z\|^2_{a,s} \ . \tag{8}$$

We defer the proof of this proposition to the Appendix.

The interest of such a seminormal form is that the system obtained by neglecting the reminder K has the invariant manifold $\hat{z} = 0$ on which the system is integrable.

As a variant with respect to the standard finite dimensional procedure we have left the third order term $\hat{G}$ but normalized the system up to order six (instead of five). This is needed in lemma 3.2.

We rewrite the Hamiltonian $\mathcal{H}$ in the form

$$\mathcal{H} := \omega \cdot I + \Omega \cdot Z + \frac{1}{2} A I \cdot I + (BI, Z) + \hat{G} + K \tag{9}$$

where $I := (|z_1|^2, \ldots, |z_n|^2)$, $Z := (|z_{n+1}|^2, |z_{n+2}|^2, \ldots)$ are the actions, A is the $n \times n$ matrix

$$A = (\overline{G}_{ij})_{1 \leq i,j \leq n} \tag{10}$$

and B is the $\infty \times n$ matrix

$$B = (\overline{G}_{ij})_{1 \leq j \leq n < i} \tag{11}$$

Remark 2.2 *Due to (8,9), one has* $|(BI)_j| \leq C|I|j^{-d}$ *for a suitable* C*. Indeed, since* $X_{\overline{G}}$ *maps* $\mathcal{P}_{a,s}$ *to* $\mathcal{P}_{a,s+d}$*, the operator* $z_j \mapsto (BI)_j z_j$ *maps* $\mathcal{P}_{a,s}$ *to* $\mathcal{P}_{a,s+d}$ *and therefore its eigenvalues* $(BI)_j$ *must fulfill the above property.*

Introduce action angle variables for the first n modes by $z_j = |z_j|e^{\mathrm{i}\phi_j} = \sqrt{I_j}e^{\mathrm{i}\phi_j}$ for $j = 1, \ldots, n$.

Perform the rescaling $I_j \to \eta^2 I_j$, $\phi_j \to \phi_j$ for $j = 1, \ldots, n$, $z_j \to \eta z_j$, $\overline{z}_j \to \eta \overline{z}_j$ for $j \geq n+1$ and divide the Hamiltonian by η^2. We get

$$\mathcal{H}(I, \phi, \hat{z}, \hat{\overline{z}}) = \omega \cdot I + \Omega \cdot Z + \eta \hat{G}_\eta + \eta^2 \Big(\frac{1}{2} AI \cdot I + (BI, Z)\Big) + \eta^4 K_\eta. \tag{12}$$

where $\hat{G}_\eta = O(||\hat{z}||^3_{a,s})$ and $K_\eta(z) = O(||z||^6_{a,s})$. We will still denote by $\mathcal{P}_{a,s} \equiv \mathbf{R}^n \times \mathbf{T}^n \times \mathcal{H}^{a,s} \times \mathcal{H}^{a,s}$ the phase space.

We will find periodic solutions of the Hamiltonian system (12) close to periodic solutions of the integrable Hamiltonian system

$$\dot{I} = 0, \quad \dot{\phi} = \omega + \eta^2(AI + B^T Z), \quad \dot{z}_j = \mathrm{i}\Big(\Omega_j + \eta^2 (BI)_j\Big) z_j, \quad j \geq n+1 \tag{13}$$

in which $\hat{G}_\eta$ and K_η are neglected. The manifold $\{\hat{z} = 0\}$ is invariant for the Hamiltonian system (13) and it is completely filled up by the invariant tori

$$\mathcal{T}(I_0) := \{I = I_0, \ \phi \in \mathbf{T}^n, \ \hat{z} = 0\}$$

on which the motion is linear with frequencies

$$\widetilde{\omega} \equiv \widetilde{\omega}(I_0) := \omega + \eta^2 A I_0.$$

Such a torus is linearly stable and the frequencies of small oscillation about the torus $\mathcal{T}(I_0)$ are the "shifted elliptic frequencies", namely

$$\widetilde{\Omega}_j(I_0) := (\Omega + \eta^2 B I_0)_j. \tag{14}$$

If all the $\widetilde{\omega}$'s are integer multiples of a single frequency, namely if

$$\widetilde{\omega} := \omega + \eta^2 A I_0 = \frac{1}{T} 2\pi k \in \frac{1}{T} 2\pi \mathbf{Z}^n, \tag{15}$$

then $\mathcal{T}(I_0)$ is a *completely resonant* torus, supporting the family of T-periodic motions

$$\mathcal{P} := \Big\{I(t) = I_0, \quad \phi(t) = \phi_0 + \widetilde{\omega} t, \quad \hat{z}(t) = 0\Big\}. \tag{16}$$

The whole family $\mathcal{P}$ will not persist in the dynamics of the complete Hamiltonian system (12). We will show that, under suitable assumptions, at least n geometrically distinct T-periodic solutions persist. More precisely, we will show that this happens for η small enough and for any choice of I_0 and T with

$$\|I_0\| \leq C \ , \qquad \frac{1}{\eta^2} \leq T \leq \frac{2}{\eta^2} \tag{17}$$

where C is independent of η, fulfilling

(H1) Equation (15) holds.

(H2) There exist $\delta > 0$ and $\tau < d$ such that

$$|\widetilde{\Omega}_j T - 2\pi l| \geq \frac{\delta}{j^\tau}, \quad \forall l \in \mathbf{Z}, \ \forall j \geq n+1. \tag{18}$$

Proposition 2.2 *Fix $\tau > 1$. Assume (A),* $\det A \neq 0$ *and*

$$\hat{\Omega}_j := \left(\Omega - BA^{-1}\omega\right)_j \neq 0 \ , \quad \forall j \geq n+1, \tag{19}$$

then, for any $\eta > 0$, and almost any T fulfilling (17) there exists I_0 such that (H1,H2) hold.

Proof. Fix η, we define $I_0 := I_0(T)$ as a function of T so that that (15) is identically satisfied. Then we find T so that the non resonance property (18) holds. Fix η and define,

$$I_0 \ := \ I_0(T) := \frac{2\pi}{\eta^2 T} A^{-1}\Big(\Big[\frac{\omega T}{2\pi}\Big] - \frac{\omega T}{2\pi}\Big), \tag{20}$$

$$k \ := \ k(T) = \Big[\frac{\omega T}{2\pi}\Big], \tag{21}$$

where $[(x_1, \dots, x_n)] := ([x_1], \dots, [x_n])$ and $[x] \in \mathbf{Z}$ denotes the integer part of $x \in \mathbf{R}$. With the choice (20),(21), $\omega T + T\eta^2 A I_0 = 2\pi k$, and I_0 is of order 1 since $T\eta^2 \geq 1$.

We come to the nonresonance property (18). To study it remark that the function $T \to [\omega T/2\pi]$ is piecewise constant. Hence, for any $T_0 \in (\eta^{-2}, 2\eta^{-2})$ there exists an interval $\mathcal{I}_0 = (T_0 - a, T_0 + b) \subset [\eta^{-2}, 2\eta^{-2}]$ such that $[\omega T/2\pi] := k_0$ is constant for $T \in \mathcal{I}_0$. Moreover, the union of such intervals cover the whole set of values in which we are interested. We will construct a subset of full measure of $\mathcal{I}_0$, in which condition (H2) is fulfilled.

So, for fixed j, l consider the set

$$B_{jl}(\tau, \delta) := \left\{ T \in \mathcal{I}_0 \ : \ |\widetilde{\Omega}_j T - 2\pi l| < \delta / j^\tau \right\}. \tag{22}$$

Remark that

$$\widetilde{\Omega}_j T = \hat{\Omega}_j T + \left(2\pi B A^{-1}\left[\frac{\omega T}{2\pi}\right]\right)_j,$$

so that, in $\mathcal{I}_0$

$$\frac{d}{dT}\left(\widetilde{\Omega}_j T - 2\pi l\right) = \hat{\Omega}_j.$$

By (A) and remark 2.2 there exists C such that

$$\left|\widetilde{\Omega}_j\right| \geq C j^{d_1} \ , \quad \left|\hat{\Omega}_j\right| \geq C j^{d_1}. \tag{23}$$

Then B_{jl} is an interval with length $|B_{jl}|$ controlled by

$$|B_{jl}| < 2\frac{\delta}{C j^{\tau + d_1}}. \tag{24}$$

Fix j and estimate the number of l for which the set B_{jl} is (possibly) non empty. First remark that, due to (A), one has that, as T varies in $\mathcal{I}_0$, the quantity $\widetilde{\Omega}_j T$ varies in a segment of length smaller than Cj^{d_1}, with a suitable C. This means that there are at most Cj^{d_1} values of l which fall in such an interval (with redefined C). So, one has

$$\left|\bigcup_l B_{jl}\right| \leq \frac{C\delta}{j^\tau} \ . \tag{25}$$

Thus, provided $\tau > 1$ as we assumed, one has that

$$\left|\bigcup_{jl} B_{jl}\right| \leq C\delta. \tag{26}$$

By this estimate, the intersection over δ of such sets has zero measure, which is the thesis. □

Theorem 2.1 *Let T and I_0 fulfill (17) and (H1,H2), then, provided η is small enough, there exist n geometrically distinct periodic orbits of the Hamiltonian system $\mathcal{H}$ cfr. (12) with period T which are η^2 close in $\mathcal{P}_{a,s}$ to the torus $\mathcal{T}(I_0)$.*

Going back to the original system one has

Corollary 2.1 *Consider the Hamiltonian system (5) and fix a positive n. Assume that (A,NR,S) hold, that* $\det A \neq 0$ *(cf. (10)) and that $\hat{\Omega}_j \neq 0$ for all $j \geq n+1$ (cf. (19)). Finally assume $d > 1$.*
Then, for any positive $\eta \ll 1$ there exist at least n distinct periodic orbits $z^{(1)}(t), ..., z^{(n)}(t)$ with the following properties:

- $\|z^{(l)}(t)\|_{a,s} \leq C\eta$ *for $l = 1, ..., n$ and $t \in \mathbf{R}$;*
- $\|\Pi_{>n} z^{(l)}(t)\|_{a,s} \leq C\eta^2$ *for $l = 1, ..., n$ and $t \in \mathbf{R}$; here $\Pi_{>n}$ is the projector on the modes with index larger than n;*
- *The period T of $z^{(l)}$ does not depend on l and fulfills $\eta^{-2} \leq T \leq 2\eta^{-2}$.*

Remark 2.3 *If the integer numbers $(k_1, ..., k_n) = \widetilde{\omega} T/2\pi$ are relatively prime then T is the minimal period of the periodic solutions $z^{(l)}$. Indeed $z^{(l)}$ are T-periodic functions close to the functions defined in (16) which have minimal period T.*

3 Proof of theorem 2.1

Since the problem is Hamiltonian, any periodic solution of the system is a critical point of the action functional

$$S(I, \phi, \hat{z}, \hat{\bar{z}}) = \int_0^T \left(I \cdot \dot{\phi} + \mathrm{i} \sum_{j \geq n+1} z_j \dot{\bar{z}}_j - \mathcal{H}(I, \phi, \hat{z}, \hat{\bar{z}}) \right) dt \tag{27}$$

in the space of T-periodic, $\mathcal{P}_{a,s}$–valued functions. Here $\mathcal{H}$ is given by (12).

We look for a periodic solution $\zeta := (\phi, I, \hat{z}, \hat{\bar{z}})$ of the form

$$\phi(t) = \phi_0 + \widetilde{\omega} t + \psi(t), \quad I(t) = I_0 + J(t), \tag{28}$$

where $(\psi, J, \hat{z}, \hat{\bar{z}})$ are periodic functions of period T taking values in the covering space $\mathbf{R}^n \times \mathbf{R}^n \times \mathcal{H}^{a,s} \times \mathcal{H}^{a,s}$ of $\mathcal{P}_{a,s}$ (that for simplicity will still be denoted by $\mathcal{P}_{a,s}$). Hence $(\psi,\ J,\ \hat{z})$ must satisfy (in the sequel for simplicity of notation we will only consider the equation for $\hat{z}$)

$$\nabla_\phi S(\zeta) = 0 \quad \Longleftrightarrow \quad \dot{J} = R_\phi(\zeta) \tag{29}$$

$$\nabla_I S(\zeta) = 0 \quad \Longleftrightarrow \quad \dot{\psi} - \eta^2 A\, J = R_I(\zeta) \tag{30}$$

$$\nabla_{\bar{z}_j} S(\zeta) = 0 \quad \Longleftrightarrow \quad \dot{z}_j - \mathrm{i}\widetilde{\Omega}_j z_j = (R_{\bar{z}})_j(\zeta) \tag{31}$$

where

$$\begin{cases} R_\phi(\zeta) & := & -\eta^4 \partial_\phi K_\eta(I_0 + J, \phi_0 + \widetilde{\omega} t + \psi, \hat{z}) - \eta \partial_\phi \hat{G}_\eta \\ R_I(\zeta) & := & \eta^2 B^T Z + \eta^4 \partial_I K_\eta(I_0 + J, \phi_0 + \widetilde{\omega} t + \psi, \hat{z}) + \eta \partial_I \hat{G}_\eta \\ (R_{\bar{z}})_j(\zeta) & = & \mathrm{i}\eta^2 (BJ)_j z_j + \mathrm{i}\eta \partial_{\bar{z}_j} \hat{G}_\eta + \mathrm{i}\eta^4 \partial_{\bar{z}_j} K_\eta(I_0 + J, \phi_0 + \widetilde{\omega} t + \psi, \hat{z}). \end{cases} \tag{32}$$

Remark that, since one expects J, ψ and $\hat{z}$ to be small (they will turn out to be of order η^2), and by proposition 2.1 one has $\partial_{\bar{z}} \hat{G}_\eta(\zeta) = O(\|\hat{z}\|^2_{a,s})$, $\partial_\phi \hat{G}_\eta(\zeta) = O(\|\hat{z}\|^3_{a,s})$, $\partial_I \hat{G}_\eta(\zeta) = O(\|\hat{z}\|^3_{a,s})$ the r.h.s. of (29,30,31) is actually small with respect to the period $T \leq 2\eta^{-2}$, see lemma 3.2.

Define the Hilbert space $H^1_P\left((0,T); \mathcal{P}_{a,s}\right)$ of the T-periodic $\mathcal{P}_{a,s}$ valued periodic function of class H^1. In order to simplify notations we will denote this space by $H^1_{P,s}$. Denote

$$|J|^2_{L^2,T} := \frac{1}{T}\int_0^T |J|^2\, dt\ , \quad |\psi|^2_{L^2,T} := \frac{1}{T}\int_0^T |\psi|^2\, dt, \tag{33}$$

$$||w||^2_{L^2,T,a,s} := \frac{1}{T}\int_0^T ||w(t)||^2_{a,s}\, dt\ , \tag{34}$$

$$\|\zeta\|_{L^2,T,a,s} := |J|_{L^2,T} + |\psi|_{L^2,T} + ||w||_{L^2,T,a,s}\ . \tag{35}$$

We will endow $H^1_P\left((0,T); \mathcal{P}_{a,s}\right) \equiv H^1_{P,s}$ with the norm

$$\|\zeta\|_{T,a,s} := \|\zeta\|_{L^2,T,a,s} + T\|\dot{\zeta}\|_{L^2,T,a,s}. \tag{36}$$

Remark 3.1 *With this choice one has*

$$\|\zeta(t)\|_{\mathcal{P}_{a,s}} \leq C\|\zeta\|_{T,a,s}\ , \quad \forall t \in \mathbf{R}$$

with a constant independent of T. Therefore, with this choice of the norm, the space $H^1_{P,s}$ is a 'Banach algebra', and the T, a, s norm of the product of any component of a vector ζ with any component of a vector ζ' is bounded by $C\|\zeta\|_{T,a,s}\|\zeta'\|_{T,a,s}$ with a constant C independent of T.

We will consider the system (29,30,31) as a functional equation in $H^1_{P,s}$.

Remark 3.2 *As a consequence of (8) and remark 3.1, the map* $\zeta \mapsto R(\zeta) := (R_\phi(\zeta), R_I(\zeta), R_{\overline{z}}(\zeta))$ *is a* C^∞ *map from* $H^1_{P,s}$ *to* $H^1_{P,s+d}$.

We are going to use the method of Lyapunov–Schmidt decomposition in order to solve (29,30,31). To this end remark that the kernel of the linear operator $\mathcal{L}$ at l.h.s. of (29,30,31) is given by $(\phi, 0, 0)$ with constant $\phi \in \mathbf{T}^n$. The range of $\mathcal{L}$ is the space of the functions $\zeta = (\psi, J, \hat{z})$ with $\psi(t)$ having zero mean value. So, there is a natural decomposition of $H^1_{P,s}$ in Range+Kernel. Explicitly we write

$$\zeta = (\phi + \psi, J, \hat{z}) = (\psi, J, \hat{z}) + (\phi, 0, 0) \equiv \zeta_R + \phi$$

with ψ having zero mean value and ϕ being constant. Then we fix ϕ, take the projection of the system (29,30,31) on the range and solve it. The solution is a function $\zeta_R(\eta, \phi)$. Finally we insert this function in the variational principle in order to find critical points of S.

3.1 The Range equation

The range equation has the form

$$\begin{cases} \dot{J} & = & R_\phi(\zeta) - \langle R_\phi(\zeta) \rangle \\ \dot{\psi} - \eta^2 A\, J & = & R_I(\zeta) \\ \dot{z}_j - \mathrm{i}\widetilde{\Omega}_j z_j & = & (R_{\overline{z}})_j(\zeta), \end{cases} \tag{37}$$

where $\langle R_\phi(\zeta) \rangle := (1/T) \int_0^T R_\phi(\zeta) dt$. We look for its solution in the range, namely in the space

$$\overline{H}^1_{P,s} \subset H^1_{P,s}$$

of the functions $\zeta_R \equiv (\psi, J, \hat{z})$ with ψ having zero average.

First of all we analyze the linear problem defined by the left hand side of (37). A "small denominator problem" appears since inverting this linear system the denominators $\widetilde{\Omega}_j T - 2\pi l$, $j \geq n+1$, $l \in \mathbf{Z}$ are present. So, define the linear operator

$$\mathcal{L}(\psi, J, \hat{z}) \equiv \mathcal{L}\zeta_R := (\dot{J}, \dot{\psi} - \eta^2 A J, \dot{w}_j - \mathrm{i}\widetilde{\Omega}_j w_j)$$

and study the equation

$$\mathcal{L}\zeta_R = (\widetilde{\psi}, \widetilde{J}, \widetilde{w}) \tag{38}$$

with $(\widetilde{\psi}, \widetilde{J}, \widetilde{w}) \in \overline{H}^1_{P,s+\tau}$ given.

Lemma 3.1 *Assume (H2). If*

$$\widetilde{\zeta}_R \equiv (\widetilde{\psi}, \widetilde{J}, \widetilde{w}) \in \overline{H}^1_{P,s+\tau} \quad \text{i.e. in } H^1_{P,s+\tau} \text{ with } \int_0^T \widetilde{\psi}(t)\ dt = 0,$$

then the equation (38) has a unique solution

$$\zeta_R \equiv (\psi, J, w) \in \overline{H}^1_{P,s}.$$

Moreover, for $T \in (\eta^{-2}, 2\eta^{-2})$ and a constant $C := C(\delta)$

$$\|\zeta_R\|_{T,a,s} \le \frac{C}{\eta^2}\|\widetilde{\zeta}_R\|_{T,a,s+\tau}.$$

Proof. Since A is symmetric and invertible it has an orthonormal basis of eigenvectors $e_1, \ldots, e_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. In these coordinates $J(t) = \sum_{k=1}^n J_k(t)e_k$, $\psi(t) = \sum_{k=1}^n \psi_k(t)e_k$ and the solution ζ_R of (38) with $\psi_0 = 0$ has Fourier coefficients

$$J_{kl} = \frac{T\widetilde{\psi}_{kl}}{\mathrm{i}2\pi l} \quad \text{for} \quad l \ne 0, \qquad J_{k0} = -\frac{\widetilde{J}_{k0}}{\eta^2\lambda_k},$$

$$\psi_{kl} = T\frac{\widetilde{J}_{kl} + \eta^2 J_{kl}\lambda_k}{\mathrm{i}2\pi l} \quad \text{for} \quad l \ne 0,$$

and, for $j \ge n+1$

$$w_{jl} := \frac{T\widetilde{w}_{jl}}{\mathrm{i}(2\pi l - \widetilde{\Omega}_j T)}.$$

We find

$$|J|^2_{L^2,T} = \sum_{kl}|J|^2_{kl} = \sum_k\Big(\frac{\widetilde{J}_{k0}}{\eta^2\lambda_k}\Big)^2 + \sum_{k,l\ne 0}\Big(\frac{T\widetilde{\psi}_{kl}}{\mathrm{i}2\pi l}\Big)^2 \le \frac{C}{\eta^4}|\widetilde{J}|^2_{L^2,T} + CT^2|\widetilde{\psi}|^2_{L^2,T}. \tag{39}$$

A similar estimate for $|\psi|_{L^2,T}$ holds. Moreover

$$|\dot{\psi}|_{L^2,T} \le \eta^2|J|_{L^2,T} + |\widetilde{\psi}|_{L^2,T} \le C(|\widetilde{J}|_{L^2,T} + |\widetilde{\psi}|_{L^2,T}), \tag{40}$$

using (39). Finally, the solution $w = (w_j)_{j\ge n+1}$ of (38) is

$$w_j(t) = \sum_{l\in\mathbf{Z}}\frac{T\widetilde{w}_{jl}}{\mathrm{i}(2\pi l - \widetilde{\Omega}_j T)}e^{\mathrm{i}(2\pi/T)lt}$$

where $\widetilde{w}_j(t) = \sum_{l\in\mathbf{Z}}\widetilde{w}_{jl}e^{\mathrm{i}(2\pi/T)lt}$. From (H2) we get

$$||w||_{L^2,T,a,s} \le C\frac{T}{\delta}||\widetilde{w}||_{L^2,T,a,s+\tau}, \qquad ||\dot{w}||_{L^2,T,a,s} \le C\frac{T}{\delta}||\dot{\widetilde{w}}||_{L^2,T,a,s+\tau}. \tag{41}$$

By (39), (40) and (41) the last estimate of the lemma follows. □

Thus $\mathcal{L}^{-1}$ defines a linear bounded operator $L : \overline{H}^1_{P,s+\tau} \to \overline{H}^1_{P,s}$.

In order to find a solution $\zeta_R = (\psi, J, \hat{z})$ of the range equation it is sufficient to find a fixed point of

$$\zeta_R = \Phi(\zeta_R) := L(N(\zeta_R;\phi)) \tag{42}$$

in the space $\overline{H}^1_{P,s}$, where $N := N(\zeta_R;\phi)$ denotes the right hand side of (37).

Lemma 3.2 *Assume $d > \tau$. Then there exists a constant C sufficiently large such that $\forall \eta \ll 1$ the map Φ is a contraction of a ball of radius $C\eta^2$.*

Proof. Consider a $\zeta_R \in \overline{H}^1_{P,s}$ with $\|\zeta_R\|_{T,a,s} \leq \rho$ with some positive (small) ρ. Since $H^1_{P,s}$ is an algebra with constants independent of T (c.f. remark 3.1), one has, by (32),

$$\|N(\zeta_R)\|_{T,a,s+d} \leq C(\eta^4 + \eta\rho^2)$$

with a suitable C. Therefore, by lemma 3.1 one has

$$\|\Phi(\zeta_R)\|_{T,a,s} \leq \|\Phi(\zeta_R)\|_{T,a,s+d-\tau} \leq C\left(\eta^2 + \frac{\rho^2}{\eta}\right)$$

which is smaller than ρ provided $C(\eta^2 + \rho^2/\eta) < \rho$, which is implied e.g. by $\rho = 2C\eta^2$ and η small enough.

Similarly one estimates the Lipschitz constant of Φ by the norm of its differential. Such a differential is bounded in a ball of radius ρ by $C(\eta^2 + \rho/\eta)$, from which the thesis follows. □

Corollary 3.1 *There exists a unique smooth function $\mathbf{T}^n \ni \phi \mapsto \zeta_R(\phi, \eta) \in \overline{H}^1_{P,s}$ solving (37) and fulfilling*

$$\|\zeta_R(\phi, \eta)\|_{T,a,s} \leq C\eta^2 \ .$$

3.2 The kernel equation

The geometric interpretation of the construction of the previous subsection is that we have found a submanifold $\mathcal{T}^n \equiv (\phi, \zeta_R(\phi, \eta)) \subset H^1_{P,s}$, diffeomorphic to an n dimensional torus, on which the partial derivative of the action functional S with respect to the variables ζ_R vanishes. Consider now the restriction S_n of S to $\mathcal{T}^n$. At a critical point of S_n all the partial derivatives of the complete functional S vanish, and therefore we can conclude that such point is critical also for the non restricted functional.

By standard Lusternik-Schnirelmann theory there exist at least n geometrically distinct T-periodic solutions, i.e. solutions not obtained one from each other simply by time-translations. Indeed, restrict S_n to the plane $E := [\widetilde{\omega}]^{\perp}$ orthogonal to the periodic flow $\widetilde{\omega} = (1/T)2\pi k$ with $k \in \mathbf{Z}^n$. The set $\mathbf{Z}^n \cap E$ is a lattice of E, and hence S_n defines a functional $S_{n|\Gamma}$ on the quotient space $\Gamma := E/(\mathbf{Z}^n \cap E) \sim \mathbf{T}^{n-1}$.

Due to the invariance of S_n with respect to the time shift, a critical point of $S_{n|\Gamma}$ is also a critical point of $S_n : \mathbf{T}^n \to \mathbf{R}$. By the Lusternik-Schnirelman category theory since $\mathrm{cat}\Gamma = \mathrm{cat}\mathbf{T}^{n-1} = n$, we can define the n min-max critical values $c_1 \leq c_2 \leq \ldots \leq c_n$ for $S_{n|\Gamma}$. If the critical levels c_i are all distinct, the corresponding T-periodic solutions are geometrically distinct, since their actions c_i are all different. On the other hand, if some min-max critical level c_i coincide, then, by the Lusternik-Schnirelmann theory, $S_{n|\Gamma}$ possesses infinitely many critical points. However *not* all the corresponding T-periodic solutions are necessarily geometrically distinct, since two different critical points could belong to the same orbit. Nevertheless, since a periodic solution can cross Γ at most a finite number of times, the existence of infinitely many geometrically distinct orbits follows. For further details see [BBV03].

This concludes the proof of the theorem 2.1.

4 Applications

4.1 The nonlinear beam equation

Consider the beam equation

$$u_{tt} + u_{xxxx} + mu = f(u) \tag{43}$$

subject to hinged boundary conditions

$$u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0, \tag{44}$$

where the nonlinearity $f(u)$ is a real analytic, odd function of the form

$$f(u) = au^3 + \sum_{k\geq 5} f_k u^k, \qquad a \neq 0.$$

The beam equation (43) is a Hamiltonian PDE with associated Hamiltonian

$$H = \int_0^\pi \frac{u_t^2}{2} + \frac{u_{xx}^2}{2} + \frac{mu^2}{2} + g(u)\ dx$$

where $g(u) := \int_0^u f(s)\ ds$ is a primitive of f.

Write the system in first order form

$$\begin{cases} \dot{u} &= v \\ \dot{v} &= -u_{xxxx} - mu + f(u) \end{cases} \tag{45}$$

The standard phase space[5] for (45) is $\mathcal{F}_s := H^s_C \times H^{s-2}_C \ni (u,v)$, where H^s_C is the space of the functions which extend to skew symmetric H^s periodic functions over $[-\pi,\pi]$. Note that $H^s_C = \{u(x) = \sum_{j\geq 1} u_j \sin(jx) \mid \sum_{j\geq 1} |u_j|^2 j^{2s} < +\infty\}$. It is then immediate to realize that, due to the regularity and the skewsymmetry of the vector field of the nonlinear part, f defines a smoothing operator, namely a smooth map from $\mathcal{F}_s$ to $\mathcal{F}_{s+2}$, provided $s \geq 1$.

Here we are also interested in spaces of analytic functions, namely functions whose Fourier coefficients belong to $\mathcal{H}^{a,s}$ with some positive a. It is easy to see that the smoothing property of the nonlinearity holds also for these spaces.

[5] An equivalent definition makes use of the so called compatibility conditions required for the smoothness of solutions of second order equations with Dirichlet boundary conditions, see e.g. [Bre93], theorem X.8.

Introduce coordinates $q = (q_1, q_2, \ldots,)$, $p = (p_1, p_2, \ldots,)$ through the relations

$$u(x) = \sum_{j\geq 1} \frac{q_j}{\sqrt{\omega_j}} \phi_j(x), \qquad v(x) = \sum_{j\geq 1} p_j \sqrt{\omega_j} \phi_j(x),$$

where $\phi_j(x) = \sqrt{2/\pi} \sin(jx)$ and

$$\omega_j^2 = j^4 + m. \tag{46}$$

Remark also that

$$\omega_j \sim j^2 \ .$$

Passing to complex coordinates

$$z_j := \frac{q_j + \mathrm{i} p_j}{\sqrt{2}}, \qquad \overline{z}_j := \frac{q_j - \mathrm{i} p_j}{\sqrt{2}},$$

the Hamiltonian takes the form (4) and the nonlinearity fulfills (S) with $s \geq 1$ a suitable a, depending on the anayticity strip of f, and $d = 2$ (for more details see [Pös96b]-[GY03]).

In order to verify the nonresonance property we use m as a parameter belonging to the set $[0, L]$ with an arbitrary L.

Lemma 4.1 *There exists a finite set $\Delta \subset [0, L]$ such that, if $m \in [0, L] \backslash \Delta$ then condition (NR) holds.*

Proof. First remark that, due to the growth property of the frequencies there is at most a finite number of vectors $l \in \mathbf{Z}^2$ at which $\omega \cdot k + \Omega \cdot l$ is small. It follows that, having fixed an arbitrary constant C, there is at most a finite set of k's and l's over which $|\omega \cdot k + \Omega \cdot l| < C$. Denote by $\mathcal{S}$ such a set.

For $(k, l) \in \mathcal{S}$ consider

$$f_{kl}(m) = \omega(m) \cdot k + \Omega(m) \cdot l$$

since f_{kl} is an analytic function it has only isolated zeros. So at most finitely many of them fall in $[0, L]$. The set Δ is the union over $k, l \in \mathcal{S}$ of such points. Fix $m \in [0, L] \backslash \Delta$. $\square$

Then one can put the system in seminormal form. The explicit computation was essentially done in [KP96] (see also [Pös96b,GY03]), obtaining that the matrixes A and B are given by

$$A = \frac{6}{\pi} \begin{pmatrix} \frac{3}{\omega_1^2} & \frac{4}{\omega_1\omega_2} & \cdots & \frac{4}{\omega_1\omega_n} \\ \frac{4}{\omega_2\omega_1} & \frac{3}{\omega_2^2} & \cdots & \frac{4}{\omega_2\omega_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{4}{\omega_1\omega_n} & \frac{4}{\omega_n\omega_2} & \cdots & \frac{3}{\omega_n^2} \end{pmatrix}, \ B = \frac{6}{\pi} \begin{pmatrix} \frac{4}{\omega_{n+1}\omega_1} & \cdots & \frac{4}{\omega_{n+1}\omega_n} \\ \frac{4}{\omega_{n+2}\omega_1} & \cdots & \frac{4}{\omega_{n+2}\omega_n} \\ \vdots & \vdots & \vdots \end{pmatrix}. \tag{47}$$

Remark that, defining the matrixes $S_1 := \mathrm{diag}(\omega_1, ..., \omega_n)$ and $S_2 := \mathrm{diag}(\omega_{n+1}, \omega_{n+2}, ...)$ one can write $A = \frac{6}{\pi} S_1^{-1} \widetilde{A} S_1^{-1}$, $\quad B = \frac{6}{\pi} S_2^{-1} \widetilde{B} S_1^{-1}$ with

$$\widetilde{A} = \begin{pmatrix} 3 & 4 & \dots & 4 \\ 4 & 3 & \dots & 4 \\ \dots & \dots & \dots & \dots \\ 4 & 4 & \dots & 3 \end{pmatrix}, \ \widetilde{B} = \begin{pmatrix} 4 & \dots & 4 \\ 4 & \dots & 4 \\ \vdots & \vdots & \vdots \end{pmatrix}. \tag{48}$$

With these expressions at hand it is immediate to verify that $\det A \neq 0$. For what pertains $\hat{\Omega}_j$ by exactly the same argument in the proof of lemma 4.1 one has that they are different from zero except for at most finitely many values of $m \in [0, L]$.

Thus, provided m does not belong to a finite subset of $[0, L]$, theorem 2.1 and its corollary 2.1 apply.

4.2 A nonlinear Schrödinger equation

Consider the space H^s_C as in the previous section. Following Pöschel [Pös02] we define a smoothing operator as follows. Fix a sequence $\{\rho_j\}_{j\geq 1}$ with the property

$$\forall j \geq 1 \quad \rho_j \neq 0 \ , \ \text{and} \quad |\rho_j| \leq C j^{-d/2} \ , \quad d > 1. \tag{49}$$

Consider the even, 2π periodic function $\rho(x) := \sum_j \rho_j \cos(jx)$ and define

$$\Gamma : H^s_C \to H^{s+d/2}_C \ , \quad \Gamma u := \rho * u \tag{50}$$

where the star denotes convolution (it is defined first extending the function u to an odd 2π periodic function).

Remark 4.1 *It is easy to see that, expanding u in Fourier series*

$$u(x) = \sum_{j \geq 1} z_j \sin(jx),$$

the j-th Fourier coefficient of Γu is proportional to $\rho_j z_j$.

Consider the Hamiltonian system with Hamiltonian function

$$H(u, \overline{u}) = \int_0^\pi |u_x|^2 + P\left(|\Gamma u|^2\right) \tag{51}$$

with P an analytic function having a zero of order two at the origin. So the equations of motion are

$$-\mathrm{i}\dot{u} = u_{xx} + \Gamma\left(P'\left(|\Gamma u|^2\right)\Gamma u\right). \tag{52}$$

Inserting the Fourier expansion of u the Hamiltonian takes the form (4) with $\omega_j = j^2$ and the vector field fulfills (S) with d given by (49).

Then one has that (NR) is here violated. So in principle one cannot expect proposition 2.1 to hold for this system. However it is clear that one can use a similar procedure to obtain a weaker result in which the function $\overline{G}$ is in resonant normal form, i.e. it contains only monomyals Poisson commuting with H_0. Now, an explicit computation, identical to that done in [KP96], shows that the so obtained normal form *depends on the actions only*. So one can proceed in the construction and try to look for resonant tori and the corresponding periodic orbits.

The explicit computation shows that also in this case the matrices A and B have the structure $A = \alpha S_1 \widetilde{A} S_1$, $B = 2\alpha S_2 \widetilde{B} S_1$ with the matrixes $\widetilde{A}$ and $\widetilde{B}$ still given by (48), matrixes $S_1 := \mathrm{diag}(\rho_1, ..., \rho_n)$ and $S_2 := \mathrm{diag}(\rho_{n+1}, \rho_{n+2}, ...)$ and a suitable constant α. So the determinant of A is still different from zero. The frequencies $\hat{\Omega}_j$ have now the structure

$$\hat{\Omega}_j(\rho) = j^2 - \rho_j a(\rho) \ , \quad \forall j \geq n+1$$

where a is a function of $\rho_1, ..., \rho_n$. So, except for exceptional choices of $\{\rho_j\}_{j \geq n+1}$ the nondegeneracy conditions are fulfilled and the conclusions of theorem 2.1 and its corollary apply.

5 Appendix: Proof of proposition 2.1

The idea is to proceed as in the proof of the standard Birkhoff normal form theorem, i.e. by successive elimination of the nonresonant monomials. As a variant with respect to the standard procedure one does not eliminate terms which are at least cubic in the variables $\hat{z}$. Remark that the estimates involved in the proofs are much more complicated than in the finite dimensional case.

To start with expand P in Taylor series up to order five: $P = P_3 + P_4 + P_5$+higher order terms. Then we begin by looking for the transformation simplifying P_3. So write

$$P_3 \quad = \quad P_3^1 + O(\|\hat{z}\|^3) \ ,$$

where P_3^1 is composed by the first three terms of the Taylor expansion of P_3 in the variables $\hat{z}$ only (so it contains only terms of degree 0,1 and 2 in such variables). We use the Lie transform to eliminate from P_3^1 all the nonresonant terms, i.e. we make a canonical transformation which is the time 1 flow Φ^1 of an auxiliary Hamiltonian system with a Hamiltonian function χ of degree 3. By considering the Taylor expansion of Φ^1 at zero one has

$$H \circ \Phi^1 = H_0 + P_3^1 + \{\chi, H_0\} + O(\|z\|^4) + O(\|\hat{z}\|^3) \ . \tag{53}$$

One wants to determine χ so that $P_3^1 + \{\chi, H_0\}$ depends on the actions $|z_j|^2$ only. To this end we proceed as usual in the theory of Birkhoff normal form.

Denote by $x = (x_1, ..., x_n) \equiv (z_1, ..., z_n)$ the first n variables and take χ to be a homogeneous polynomial of degree three. Write

$$\chi = \sum_{|j_1|+|j_2|+|j_3|+|j_4|=3} \chi_{j_1 j_2 j_3 j_4} x^{j_1} \overline{x}^{j_2} \hat{z}^{j_3} \overline{\hat{z}}^{j_4} \tag{54}$$

with multindexes j_1, j_2, j_3, j_4. For a multi index $j_l \equiv (j_{l,1}, ..., j_{l_n})$ we used the notation $|j_l| := |j_{l,1}| + \ldots |j_{l,n}|$ and $x^{j_l} := x_1^{j_{l,1}} \ldots x_n^{j_{l,n}}$, and similarly for a multi index with infinitely many components. So, one has

$$\{\chi, H_0\} = \sum_{|j_1|+|j_2|+|j_3|+|j_4|=3} \mathrm{i}\left(\omega \cdot (j_1 - j_2) + \Omega \cdot (j_3 - j_4)\right) \chi_{j_1 j_2 j_3 j_4} x^{j_1} \overline{x}^{j_2} \hat{z}^{j_3} \overline{\hat{z}}^{j_4} .$$

Write now

$$P_3^1 = \sum_{|j_1|+|j_2|+|j_3|+|j_4|=3} P_{j_1 j_2 j_3 j_4} x^{j_1} \overline{x}^{j_2} \hat{z}^{j_3} \overline{\hat{z}}^{j_4} \tag{55}$$

and remark that the indexes are here subjected to the further limitation $|j_3| + |j_4| \leq 2$. So one is led to the choice

$$\chi_{j_1 j_2 j_3 j_4} := \frac{-P_{j_1 j_2 j_3 j_4}}{\mathrm{i}\left(\omega \cdot (j_1 - j_2) + \Omega \cdot (j_3 - j_4)\right)} \ , \quad j_1 - j_2 + j_3 - j_4 \neq 0 \tag{56}$$

and zero otherwise. Remark that, due to the assumption (NR) the denominators appearing in the above expression are all different from zero. Moreover, since due to the growth of the frequencies they are actually bounded away from zero. Then in order to conclude the proof (at least for what concerns the elimination of the third order part) one has to ensure that the function χ is well defined and that it has a smooth Hamiltonian vector field. The terms of χ of different degree in $\hat{z}$ have to be treated in a different way, so we will denote by χ_0, χ_1, χ_2 the homogeneous parts of degree 0,1 and 2 respectively with respect to the variables $\hat{z}$.

We need a few lemmas.

Lemma 5.1 *Let $\mathbf{R}^n \ni x \mapsto f(x) \in \ell^2$ be a homogeneous bounded polynomial of degree r. Write*

$$f(x) = \sum_{j \in \mathbf{N}^n, |j|=r} \sum_{k \geq 1} f_{jk} x^j e_k$$

where e_k is the standard basis of ℓ^2. Let $\{\rho_{j,k}\}_{j \in \mathbf{N}^n}^{k \geq 1}$ be a sequence with the property $|\rho_{jk}| \geq C$ and define a function g by

$$g(x) = \sum_{j \in \mathbf{N}^n, |j|=r} \sum_{k \geq 1} \frac{f_{jk}}{\rho_{jk}} x^j e_k. \tag{57}$$

Then there exists C such that $\|g(x)\| \leq C \|x\|^r$.

Proof. Write $g(x) = \sum_j g_j x^j$ and remark that the cardinality of the set over which the sum is carried out is finite. We estimate each of the vectors g_j's. One has

$$\|g_j\|^2 = \sum_k \left(\frac{f_{kj}}{\rho_{kj}}\right)^2 \leq \frac{1}{C^2} \sum_k f_{kj}^2 = \frac{1}{C^2} \|f_j\|^2.$$

Now the norms of the vectors f_j are bounded and therefore the thesis follows. □

Remark 5.1 *By the same proof, the same result holds if the space ℓ^2 is substituted by the spaces $\mathcal{H}^{a,s}$.*

Lemma 5.2 *Let $\mathbf{R}^n \times \ell^2 \ni (x,z) \mapsto f(x,z) \in \mathbf{R}$ be a homogeneous bounded polynomial of degree r in x, linear and bounded in z. Write*

$$f(x,z) = \sum_{\substack{k\geq 1 \\ j\in\mathbf{N}^n, |j|=r}} f_{jk} x^j z_k .$$

Let $\{\rho_{j,k}\}_{j\in\mathbf{N}^n}^{k\geq 1}$ be as above and define a function g by

$$g(x,z) = \sum_{\substack{k\geq 1 \\ j\in\mathbf{N}^n, |j|=r}} \frac{f_{jk}}{\rho_{jk}} x^j z_k . \tag{58}$$

Then there exists C such that $|g(x,z)| \leq C\|x\|^r \|z\|$.

Proof. Just write $g(x,z) = \sum_j g_j(z) x^j$. Fix j and study the linear functional $g_j(z)$, one has

$$|g_j(z)| = \left| \sum_{k\geq 1} f_{jk} \frac{z_k}{\rho_{jk}} \right| \leq \|f_j\| \|z/\rho\|$$

where f_j is defines in analogy to g_j, its norm is the norm as a linear functional, and z/ρ is the vector of ℓ^2 with k-th component equal to z_k/ρ_{jk}. From this inequality, summing over j, the thesis follows. □

In order to estimate the vector field of χ_2 we will need the following lemma:

Lemma 5.3 *[Lemma A.1 of [Pös96a]]. If $A = (A_{kl})$ is a bounded linear operator on ℓ^2, then also $B = (B_{kl})$ with*

$$B_{kl} := \frac{|A_{kl}|}{1 + |k-l|} \ , \tag{59}$$

is a bounded linear operator on ℓ^2.

For the proof we refer to [Pös96a].

Lemma 5.4 *Let $\mathbf{R}^n \times \ell^2 \ni (x,z) \mapsto f(x,z) \in \ell^2$ be a homogeneous bounded polynomial of degree r in x linear and bounded in z. Write*

$$f(x,z) = \sum_{\substack{k,l\geq 1 \\ j\in\mathbf{N}^n, |j|=r}} f_{jkl} x^j z_k e_l .$$

Let $\{\rho_{j,k,l}\}_{j\in\mathbf{N}^n}^{k,l\geq 1}$ be a sequence fulfilling

$$|\rho_{jkl}| \geq C_1 (1 + |k-l|) \tag{60}$$

and define a function g by

$$g(x,z) = \sum_{\substack{k,l\geq 1\\ j\in \mathbf{N}^n, |j|=r}} \frac{f_{jkl}}{\rho_{jkl}} x^j z_k. \tag{61}$$

Then there exists C such that $\|g(x,z)\| \leq C\|x\|^r \|z\|$.

Proof. Write $g(x,z) = \sum_j g_j(z) x^j$. Fix j and apply lemma 5.3 to such operators obtaining the result. □

Remark 5.2 *An identical statement holds for functions from $\mathbf{R}\times\mathcal{H}_{a,s}$ to $\mathcal{H}_{a,s+d}$. To obtain the proof just remark that the boundedness of a linear operator $B = (B_{kl})$ (g_j in the proof) as an operator from $\mathcal{H}^{a,s}$ to $\mathcal{H}^{a,s+d}$ is equivalent to the boundedness of $\tilde{B} := (v_k B_{kl} s_l)$ as an operator from ℓ^2 to itself, where v_k, s_l are suitable weights.*

With the above lemmas at hand it easy to estimate the vector field of χ. We treat explictly only χ_1.

Lemma 5.5 *Let χ_1 be the component linear in $\hat{z}$ and $\overline{\hat{z}}$ of the function χ defined by (56). Then there exists a constant C such that its vector field is bounded by*

$$\|X_{\chi_1}(z,\overline{z})\|_{a,s+d} \leq C\|z\|^2_{a,s}.$$

Proof. Write χ_1 as follows

$$\chi_1(x,\overline{x},\hat{z},\overline{\hat{z}}) = \langle \chi_{01}(x,\overline{x}); \hat{z}\rangle_{\ell^2} + \left\langle \chi_{10}(x,\overline{x}); \overline{\hat{z}}\right\rangle_{\ell^2}.$$

Consider the first term. Separating the $x, \overline{x}$ and $\overline{\hat{z}}$ components, its vector field is given by

$$\left(\mathrm{i}\left\langle \frac{\partial \chi_{01}}{\partial \overline{x}}; \hat{z}\right\rangle_{\ell^2}, -\mathrm{i}\left\langle \frac{\partial \chi_{01}}{\partial \overline{x}}; \hat{z}\right\rangle_{\ell^2}, -\mathrm{i}\chi_{01}(x,\overline{x})\right).$$

Explicitly χ_{01} is given by

$$\sum_{\substack{|j_1|+|j_2|=2\\ l\geq n+1}} \frac{-P_{j_1 j_2 e_l}}{\mathrm{i}(\omega\cdot(j_1-j_2)+\Omega_l)} x^{j_1}\overline{x}^{j_2} e_l.$$

It follows that each of the x (and $\overline{x}$) components of the vector field has the structure considered in lemma 5.2, which therefore gives the estimate of such part of the vector field. Concerning the $\overline{\hat{z}}$ component lemma 5.1 applies and gives the result. The remaining components can be treated exactly in the same way. □

The estimate of the vector fields of χ_0 and χ_2 are obtained in a similar way. In order to apply lemma 5.4 to the estimate of the vector field of χ_2 one has just to remark that from (A) and (NR) one has the estimate

$$|\omega\cdot k + \Omega_j - \Omega_l| \geq C(1+|j-l|) .$$

Thus we have the following

Proposition 5.1 *The vector field of the function χ defined by (56) fulfills the inequality*

$$\|X_\chi(z,\bar{z})\|_{a,s+d} \leq C\|z\|^2_{a,s} \ .$$

Then by standard existence and uniqueness theory one has that such vector fields defines a unique smooth time 1 flow in a neighbourhood of the origin both in $\mathcal{P}_{a,s}$ and in $\mathcal{P}_{a,s+d}$. It follows that the transformation is well defined. Transforming the vector field of H one gets a vector field having the same smoothness properties of the original one. Thus one can iterate the construction and eliminate also the unwanted terms of degree four and five. This concludes the proof of proposition 2.1.

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