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The relative uniform density of the continuous functions in the Baire functions, and of a divisible Archimedean ℓ-group in any epicompletion

Richard N. Ball [a,*], Anthony W. Hager [b,1]

[a] *Department of Mathematics and Computer Science, University of Denver, Denver, CO 80208, USA*

[b] *Department of Mathematics, Wesleyan University, Middletown, CT 06459, USA*

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Abstract

For a subset A of an ℓ-group B, $r(A, B)$ denotes the relative uniform closure of A in B. R^X denotes the ℓ-group of all real-valued functions on the set X, and when X is a topological space, $C^*(X)$ is the ℓ-group of all bounded continuous real-valued functions, and $B(X)$ is the ℓ-group of all Baire functions. We show that $B(X) = r(C^*(X), B(X)) = r(C^*(X), R^X)$. This would appear to be a purely order-theoretic construction of $B(X)$ from $C(X)$ within R^X. That result is then applied to the category **Arch** of Archimedean ℓ-groups, and its subcategory **W** of ℓ-groups with distinguished weak unit. In earlier work we have described the epimorphisms of these categories, characterized those objects with no epic extension (called epicomplete), and for **W**, constructed all epic embeddings into epicomplete objects (epicompletions) using Baire functions. Now this apparatus is combined with the equation above to make this contribution to the description of epimorphisms. In **Arch** or **W**, if a divisible ℓ-group A is epically embedded in an epicomplete ℓ-group B then $B = r(A, B)$. Examples are presented to show that, in each of **Arch** and **W**, the hypothesis that B be epicomplete cannot be dropped.

Keywords: Continuous real-valued function; Baire function; Relative uniform density; Archimedean lattice-ordered group; Epimorphism; Reflection

AMS classification: 54C30; 26A21; 06F20; 46A40; 18A40

We dedicate this paper to our friend Wis Comfort

The paper is organized as follows. Section 1 contains basic definitions and facts, the statement of the main technical Theorem 1, and the derivation therefrom of the equation in the abstract. Section 2 is devoted to the proof of Theorem 1, which involves some

* Corresponding author. Email: rball@du.edu.

1 Email: ahager@wesleyan.edu.

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intricacies of the Baire classification of functions. Section 3 is a brief recollection of some of our results on epimorphisms and epicompletions in Archimedean ℓ-groups. Section 4 is devoted to the issue of relative uniform density versus epimorphic embedding in the categories **W** and **Arch**, respectively. The analysis for **W** is needed for **Arch**.

1. Relative uniform convergence and Baire functions

Basic references on ℓ-groups and vector lattices (Riesz spaces) are [5] and [12]. For us, an ℓ-group will mean an Abelian ℓ-group. For B an ℓ-group or vector lattice, $A \leqslant B$ means that A is a sub-ℓ-group or sub-vector lattice of B. The ℓ-group B is *Archimedean* if $0 \leqslant ka \leqslant b$ for each $k \in N$ implies $a = 0$.

1.1. Relative uniform convergence

We recall some definitions and simple facts about relative uniform convergence from Sections 16 and 63 of [12], having made the easy modifications from vector lattices to ℓ-groups.

In an ℓ-group B, we say that a sequence $\{a_n\}$ *converges relatively uniformly to an element b with regulator u*, and write

$$a_n \to b\ (u),$$

provided that for each $k \in N$ there is some index $n(k) \in N$ such that $k|a_n - b| \leqslant u$ whenever $n \geqslant n(k)$. It follows quickly that for B Archimedean, relative uniform limits are unique, i.e., $a_n \to b\ (u)$ and $a_n \to c\ (v)$ imply $c = b$. *We henceforth assume all ℓ-groups are Archimedean.*

Suppose $A \subseteq B$. The *iterated relative uniform pseudo-closures* are defined as follows.

$$r_0(A, B) \equiv A,$$
$$r_1(A, B) \equiv \bigl\{b \in B : a_n \to b\ (u) \text{ for some } \{a_n\} \subseteq A \text{ and } u \in B\bigr\},$$
$$r_\alpha(A, B) \equiv \begin{cases} r_1(r_\gamma(A, B), B) & \text{for } \alpha = \gamma + 1, \\ \bigcup_{\gamma<\alpha} r_\gamma(A, B) & \text{for } \alpha \text{ a limit ordinal.} \end{cases}$$

When A is a sub-ℓ-group or sub-vector lattice of B, so is each $r_\alpha(A, B)$. In any event,

$$r_{\omega_1+1}(A, B) = r_{\omega_1}(A, B);$$

this set is denoted $r(A, B)$ and is called the *relative uniform closure of A in B*. When $r(A, B) = B$, A is said to be *relatively uniformly dense in B*. Note that when $A \leqslant B \leqslant C$ with A relatively uniformly dense in B and B relatively uniformly dense in C, then A is relatively uniformly dense in C. This is because $r(A, C) \supseteq r(A, B)$, so

$$r(A, C) \supseteq r\bigl(r(A, B), C\bigr).$$

1.2. Baire functions

We use [13] as a reference. R^X denotes all functions from X into R. In the pointwise operations and order, R^X is an Archimedean vector lattice and ring. In R^X, f_n *converges to* f *pointwise*, written

$$f_n \to f \quad \text{pointwise},$$

provided that $f_n(x) \to f(x)$ in R for each x in X.

Suppose that $L \subseteq R^X$. The *Baire classes derived from* L are defined as follows.

$$B_0 L \equiv L,$$
$$B_1 L \equiv \{b \in R^X : a_n \to b \text{ pointwise for some } \{a_n\} \subseteq L\},$$
$$B_\alpha L \equiv \begin{cases} B_1(B_\gamma L) & \text{for } \alpha = \gamma + 1, \\ \bigcup_{\gamma<\alpha} B_\gamma L & \text{for } \alpha \text{ a limit ordinal.} \end{cases}$$

It is easy to see that if L is a sub-vector lattice of R^X then so is each $B_\alpha L$. In any event $B_{\omega_1+1}L = B_{\omega_1}L$; this set is denoted BL and called *the Baire system derived from* L.

Suppose X has a topology. Then $C(X)$, the set of all continuous real-valued functions on X, is a sub-vector lattice and subring of R^X. $B_\alpha C(X)$ is denoted $B_\alpha(X)$ and $BC(X)$ is denoted $B(X)$.

For $L \subseteq R^X$, L^* denotes $\{f \in L\colon f \text{ is bounded}\}$. And L is said to be *uniformly complete* if $a_n \to b$ (1) with $\{a_n\} \subseteq L$ implies $b \in L$, where 1 denotes the function constantly 1. This just means L is closed under the usual uniform convergence over X of sequences. The following is the main technical result of the article.

Theorem 1. *Suppose that* L *is a uniformly complete sub-vector lattice of* R^X *which contains* 1. *Then*

$$B_1 L = r_3(L^*, B_1 L).$$

We defer the proof of Theorem 1 to Section 2. To proceed from Theorem 1 to our corollaries about $B(X)$ we need to recall that if L satisfies the hypotheses of Theorem 1 then so does each $B_\alpha L$ [13, 3.1], and we need the following lemma.

Lemma 2. *The following hold in* R^X.

(1) *If* $f_n \to f$ (u) *then* $f_n \to f$ *pointwise.*

(2) *If* $f \geqslant 0$ *then* $f \wedge n \to f$ (f^2).

Proof. To prove (1) observe that for any x, $k|f_n(x) - f(x)| \leqslant u(x) \in R$ for $n \geqslant n(k)$. To prove (2) observe that for any x, $n|f(x) - f(x) \wedge n| \leqslant f^2(x)$, so we choose $n(k) = k$. □

Corollary 3. *For each* α, $C^*(X)$ *is relatively uniformly dense in* $B_\alpha(X)$. *In particular for* $\alpha = \omega_1$, $r(C^*(X), B(X)) = B(X)$.

Proof. The proof goes by induction on α; for $\alpha = 0$ we have $C^*(X)$ relatively uniformly dense in $B_0(X) = C(X)$ by part (2) of Lemma 2. Suppose $C^*(X)$ is relatively uniformly dense in $B_\alpha(X)$. Now $B_\alpha(X)$, indeed $B_\alpha(X)^*$, is relatively uniformly dense in $B_{\alpha+1}(X)$ by Theorem 1 applied to $B_\alpha(X)$. The result then follows by the transitivity of relative uniform density pointed out at the end of Subsection 1.1. □

Corollary 4. *The following hold in* R^X.

(1) $r_\alpha(C^*(X), R^X) \subseteq B_\alpha(X)$ *for each* α.
(2) $r(C^*(X), R^X) = B(X)$.

Proof. We prove part (1) by induction on α. Of course,

$$r_0\bigl(C^*(X), R^X\bigr) = C^*(X) \subseteq C(X) = B_0(X).$$

Suppose $r_\alpha(C^*(X), R^X) \subseteq B_\alpha(X)$. Then

$$\begin{aligned} r_{\alpha+1}\bigl(C^*(X), R^X\bigr) &= r_1\bigl(r_\alpha\bigl(C^*(X), R^X\bigr), R^X\bigr) \\ &\subseteq r_1\bigl(B_\alpha(X), R^X\bigr) \subseteq B_{\alpha+1}(X), \end{aligned}$$

where the first inclusion follows from the induction hypothesis and the second from part (1) of Lemma 2.

To prove part (2) first note that $r(C^*(X), R^X) \subseteq B(X)$ is the case $\alpha = \omega_1$ of part (1). Then $B(X)$ coincides with $r(C^*(X), B(X))$ by Corollary 3, while obviously

$$r\bigl(C^*(X), B(X)\bigr) \subseteq r\bigl(C^*(X), R^X\bigr). \qquad \square$$

2. Proof of Theorem 1

We require what is essentially the complete relationship between the Baire system as described in Subsection 1.2 and Baire measurability. We need some notation and terminology to articulate that.

For $S \subseteq X$, S' denotes $X \setminus S$ and $\chi(S)$ denotes the characteristic function of S. For $f \in R^X$, we denote the zero set and cozero set of f by

$$Zf \equiv \bigl\{x\colon\ f(x) = 0\bigr\} \quad \text{and} \quad \operatorname{coz} f \equiv (Zf)'.$$

For $L \subseteq R^X$,

$$ZL \equiv \{Zf\colon\ f \in L\} \quad \text{and} \quad \operatorname{coz} L \equiv \{\operatorname{coz} f\colon\ f \in L\}.$$

For $\Sigma \subseteq P(X)$, where $P(X)$ denotes the power set of X,

$$\Sigma_\sigma \equiv \Bigl\{\bigcup_n S_n\colon\ S_1, S_2, \ldots \in \Sigma\Bigr\}, \quad \text{and}$$

$$\Sigma_\delta \equiv \Bigl\{\bigcap_n S_n\colon\ S_1, S_2, \ldots \in \Sigma\Bigr\}.$$

$\ell^\infty(\Sigma)$ denotes the uniform completion of the linear span of $\{\chi(S)\colon\ S \in \Sigma\}$. (The uniform completion of $L \subseteq R^X$ consists of all uniform limits of sequences from L.)

2.1. A classical theorem

In the following theorem, due largely to Lebesgue and Hausdorff, the two parts say almost the same thing in that each implies the other with relative ease, but it is convenient for us to state both. Part (1) is quoted from [6, 3.2(4)], and part (2) is from [13, 3.5].

Theorem 5. *Suppose L is a uniformly complete sub-vector lattice of R^X containing* 1.
(1) $(B_1L)^* = \ell^\infty((ZL)_\sigma \cap (\operatorname{coz} L)_\delta)$.
(2) *For $f \in R^X$, $f \in B_1L$ if and only if $f^{-1}(G) \in (ZL)_\sigma$ for each open $G \subseteq R$.*

Remark 1. We record three observations about an ℓ-group L as in Theorem 5.
(1) $(\operatorname{coz} L)_\sigma = \operatorname{coz} L$ since

$$\operatorname{coz} \sum_n 2^{-n}\big(|f_n| \wedge 1\big) = \bigcup_n \operatorname{coz} f_n,$$

and dually, $(ZL)_\delta = ZL$.
(2) $E \in (ZL)_\sigma \cap (\operatorname{coz} L)_\delta$ means there are elements f_n' and g_n' in L with

$$E = \bigcup_n Zf_n = \bigcap_n \operatorname{coz} g_n,$$

and E satisfies this precisely when both E and E' lie in $(ZL)_\sigma$, or when both lie in $(\operatorname{coz} L)_\delta$.
(3) If $f, g \in L$ then $Zf \cap \operatorname{coz} g \in (ZL)_\sigma \cap (\operatorname{coz} L)_\delta$.

Lemma 6 [8, 3.3]. *Let L be as in Theorem* 5. *If $f \in L$ and $f \geqslant r > 0$ for some $r \in R$ then $1/f \in L$.*

2.2. The proof

We commence the proof proper of Theorem 1, whose idea is this. Part (1) of Theorem 5 says that

$$(B_1L)^* = \ell^\infty\big((ZL)_\sigma \cap (\operatorname{coz} L)_\delta\big),$$

which describes $(B_1L)^*$ as generated in certain successive steps from L; we shall simply describe each step in terms of relative uniform convergence. An additional step gives B_1L from $(B_1L)^*$. The running assumption is that L is a uniformly complete sub-vector lattice of R^X which contains 1. We proceed.

Proposition 7. *If $Z \in ZL$ then*
(1) *$\chi(Z) \in r_1(L^*, B_1L)$, and*
(2) *if $C \in \operatorname{coz} L$ then each of $\chi(Z)$, $1 - \chi(C') = \chi(C)$, and $\chi(Z) \wedge \chi(C) = \chi(Z \cap C)$ lies in $r_1(L^*, B_1L)$.*

Proof. From earlier observations, $r_1(L^*, B_1L)$ is a vector lattice containing 1. Thus part (2) follows from part (1). For proving part (1) note the following lemma.

Lemma 8. *Suppose that A is a sub-vector lattice of R^X containing* 1, *that $f \in A$, and that $s < t$ in R. Then there is some $g \in A$ with $0 \leqslant g \leqslant 1$ and*

$$g(x) = \begin{cases} 0 & \text{if } f(x) \geqslant t, \\ 1 & \text{if } f(x) \leqslant s. \end{cases}$$

Proof. Set $g \equiv ((t - f) \vee 0)/(t - s) \wedge 1$. □

Now let $Z = Zf$ for $f \in L$. We can suppose $0 \leqslant f \leqslant \frac{1}{2}$. Then

$$\operatorname{coz} f = \bigcup_n \left\{x\colon\ f(x) \geqslant \frac{1}{n}\right\}.$$

By Lemma 8 choose for each n an $f_n \in L^*$ with $0 \leqslant f_n \leqslant 1$ and

$$f_n(x) = \begin{cases} 0 & \text{if } f(x) \geqslant \dfrac{1}{n}, \\ 1 & \text{if } f(x) \leqslant \dfrac{1}{n+1}. \end{cases}$$

Let

$$u(x) = \begin{cases} \dfrac{1}{f(x)} & \text{if } f(x) \neq 0, \\ 1 & \text{if } f(x) = 0. \end{cases}$$

Then $u \in B_\alpha L$ because $1/(f \vee f_n) \to u$ pointwise, and $1/(f \vee f_n) \in L$ by Lemma 6. Finally, one easily sees that $n \geqslant k$ implies $k|f_n - \chi(Z)| \leqslant u$, so $f_n \to \chi(Z)$ (u) in $B_1 L$. This completes the proof of Proposition 7. □

Proposition 9. *If $E \in (ZL)_\sigma \cap (\operatorname{coz} L)_\delta$ then $\chi(E) \in r_2(L^*, B_1 L)$.*

Proof. Since $E \in (ZL)_\sigma$ we can write E as $\bigcup_n Z_n$ with each Z_n in ZL, so $\chi(Z_n)$ lies in $r_1(L^*, B_1 L)$ by part (1) of Proposition 7. We can suppose $Z_n \subseteq Z_{n+1}$ for each n since ZL is closed under finite unions. Let E_1 be Z_1, and for $n > 1$ define E_n to be $Z_n - Z_{n-1} = Z_n \cap Z'_{n-1}$. Then the E_n's are disjoint, and each $\chi(E_n)$ lies in $r_1(L^*, B_1 L)$ by part (2) of Proposition 7.

Let v be the pointwise supremum $\bigvee_n n\chi(E_n)$. We show below that $v \in B_1 L$. It is easily seen that $n \geqslant k$ implies that $k|\chi(Z_n) - \chi(E)| \leqslant v$, whence $\chi(Z_n) \to \chi(E)$ (v), and therefore $\chi(E) \in r_2(L^*, B_1 L)$.

To see that $v \in B_1 L$ we use part (2) of Theorem 5. Let G be an open set in R. If $0 \notin G$ then $v^{-1}(G) = \bigcup\{E_k\colon\ k \in G\}$, while if $0 \in G$ then $v^{-1}(G) = \bigcup\{E_k\colon\ k \in G\} \cup E'$. Note that $E_k = Z_k \cap Z'_{k-1}$, and $Z'_{k-1} \in \operatorname{coz} L \subseteq (ZL)_\sigma$, so $E_k \in (ZL)_\sigma$. Also, $E' \in (ZL)_\sigma$ by part (2) of Remark 1. (This is the first use of the hypothesis that $E \in (\operatorname{coz} L)_\delta$.) In either case $v^{-1}(G) \in (ZL)_\sigma$. Thus Proposition 9 is proved. □

We conclude the proof of Theorem 1. By definition

$$\ell^\infty\big((ZL)_\sigma \cap (\operatorname{coz} L)_\delta\big) = uV,$$

V being the linear span of the $\chi(E)$'s and u denoting uniform completion, and part (1) of Theorem 5 says $(B_1L)^* = uV$. Proposition 9 says $V \subseteq r_2(L^*, B_1L)$, so $(B_1L)^* \subseteq ur_2(L^*, B_1L)$. By part (2) of Lemma 2 and the fact that B_1L is a ring [13, 3.1], we have $B_1L \subseteq r_1((B_1L)^*, B_1L)$, whence

$$B_1L \subseteq r_1\bigl(ur_2(L^*, B_1L), B_1L\bigr).$$

By Lemma 10 below, this last is just

$$r_1\bigl(r_2(L^*, B_1L), B_1L\bigr) \equiv r_3(L^*, B_1L),$$

and we are done. □

Lemma 10. *Let C be an ℓ-group, let $A \subseteq C$ and $0 \leqslant c_0 \in C$, and put*

$$u(c_0)A = \bigl\{c \in C : a_n \to c\ (c_0) \text{ for some } \{a_n\} \subseteq A\bigr\}.$$

(In this notation, the above uV is $u(1)V$, within B_1L.) Then

$$r_1\bigl(u(c_0)A, C\bigr) = r_1(A, C).$$

Proof. We need to show $\subseteq$, so consider $f \in r_1(u(c_0)A, C)$, i.e., $b_n \to f\ (v)$ for $v \in C$ and $\{b_n\} \subseteq u(c_0)A$. Given k there is $n(k)$ with $k|b_{n(k)} - f| \leqslant v$ and there is also $a_k \in A$ with $k|b_{n(k)} - a_k| \leqslant c_0$. Then

$$k|a_k - f| \leqslant k|a_k - b_{n(k)}| + k|b_{n(k)} - f| \leqslant v + c_0.$$

Thus $a_k \to f\ (v + c_0)$, so $f \in r_1(A, C)$. □

3. Some background on Archimedean ℓ-groups

While we would hope that our results 1, 3, and 4 above might be of interest in certain aspects of real analysis, our motivation for proving these was for further understanding of epimorphisms in Archimedean ℓ-groups. Now we sketch some preliminaries for that, and indicate the primitive connections between epics, Baire functions, and relative uniform density.

3.1. **Arch** *and* **W**

The category **Arch** has objects Archimedean ℓ-groups, and morphisms the ℓ-homomorphisms, i.e., the group and lattice homomorphisms. An object of **W** is a pair (A, e_A), where A is an Archimedean ℓ-group and e_A is a positive weak unit of A, meaning $e_A^\perp = (0)$, where

$$e_A^\perp \equiv \bigl\{a \in A\colon\ a \wedge e_A = 0\bigr\}.$$

A **W**-morphism $\varphi : (A, e_A) \to (B, e_B)$ is an ℓ-homomorphism $\varphi : A \to B$ with $\varphi(e_A) = e_B$.

Interest accrues to **W** as follows. **W** is a natural generalization of the category of $C(X)$'s, giving $C(X)$ the weak unit 1; **W** contains the category of Archimedean f-rings with identity, the identities being the weak units; **W** provides access to **Arch**—as in our present circumstances. Much of this has to do with a canonical representation available in **W**, which we now describe.

$R \cup \{\pm\infty\}$ is topologized and linearly ordered in the obvious way. For X a topological space, $D(X)$ consists of all continuous functions $f : X \to R \cup \{\pm\infty\}$ with $f^{-1}(R)$ dense in X. With the pointwise partial-ordering, $D(X)$ is a lattice, and has partial addition defined by declaring $f+g=h$ to mean that $f(x)+g(x)=h(x)$ when these three are real numbers. If a subset $A \subseteq D(X)$ is a sublattice, for which $f \in A$ implies $-f \in A$ and $f, g \in A$ with $f+g=h$ implies $h \in A$, and with $1 \in A$, then one sees that $(A, 1) \in |\mathbf{W}|$, and we call A a **W**-object in $D(X)$.

The exact condition that $D(X) \in |\mathbf{W}|$ is that each dense cozero-set of X is C^*-embedded [8]; then, one calls X a quasi-F space. It is important to note that if X is basically disconnected, i.e., if each cozero-set has open closure (see [7]), then X is quasi-F. The following is described in [2,4].

Theorem 11 (The Yosida Representation). *For each $(A, e_A) \in |\mathbf{W}|$ there is a compact Hausdorff space YA unique up to homeomorphism with these properties. There is a* **W**-*isomorphism $a \mapsto \widehat{a}$ of A onto a* **W**-*object $\widehat{A}$ in $D(YA)$, with $\widehat{A}$ separating the points of YA.*

In the sequel, unless misunderstanding is produced, we shall routinely identify $(A, e_A) \in$ **W** with its Yosida representation $(\widehat{A}, 1)$ and suppress explicit mention of the weak unit.

3.2. Epimorphisms and epicompletions

In a general category for the moment, a morphism $\varphi : A \to B$ is an *epimorphism*, or *epic*, if φ is right-cancellable, i.e., if $\psi_1\varphi = \psi_2\varphi$ implies $\psi_1 = \psi_2$; an object E is *epicomplete* if $\psi : E \to F$ epic and monic, i.e., left-cancellable, implies ψ is an isomorphism; an *epicompletion* of A is an epic and monic $\varphi : A \to E$ with E epicomplete.

In **W** and **Arch**, the monics are just the 1–1 morphisms, or embeddings, while the epics are characterized using Yosida representations in [2]. We need not recall that here (the details will be required only for Proposition 29), but the description yields Theorems 12 and 13 below.

Theorem 12 [2,4]. *In* **W** *or* **Arch**, *E is epicomplete if and only if E is divisible and both conditionally and laterally σ-complete.*

In **W**, E is epicomplete if and only if there is a compact basically disconnected X for which E is **W**-isomorphic to $D(X)$, and then $YE = X$.

The connection between **W**-epics and Baire functions is based on the following. Given $A \in |\mathbf{W}|$, A^* is the ℓ-ideal generated by the designated weak unit, i.e., with the view

$A \subseteq D(YA)$, $A^* = \{a \in A\colon\ a \text{ is bounded}\}$. Note that $A^* \subseteq C(YA) \subseteq B(YA)$, and the inclusions are W-embeddings, but $A \subseteq B(YA)$ only occurs if $A = A^*$, i.e., if the weak unit is what is called a strong unit.

Theorem 13 [4]. *Let $A \in |\mathbf{W}|$. There are ℓ-ideals I in $B(YA)$ for which the composite* **W**-*morphism $A^* \leqslant B(YA) \to B(YA)/I$ lifts to a* **W**-*embedding $\varphi_I : A \to B(YA)/I$, any such is a* **W**-*epicompletion of A, and each* **W**-*epicompletion of A is isomorphic over A to one of that form.*

Some comments on **Arch** versus **W** may be in order. According to [3], any **Arch**-object has **Arch**-epicompletions, but there would seem to be no concrete realization of these resembling Theorem 13. This stems from the situation regarding representations for **Arch**-objects. Representations in $D(X)$'s exist in abundance (see Chapter 7 of [12]), but there is none with a strong canonicity resembling Theorem 11. Finally we note the following, which has nothing to do with representations. See also [12, Section 63].

Proposition 14. *Let $\varphi, \psi : B \to C$ be ℓ-homomorphisms with C Archimedean.*

(1) *$b_n \to b\ (u)$ in B implies $\varphi(b_n) \to \varphi(b)\ (\varphi(u))$ in C.*
(2) *If $A \subseteq B$ and $\varphi|A = \psi|A$ then $\varphi|r(A, B) = \psi|r(A, B)$.*
(3) *A relatively uniformly dense embedding of Archimedean ℓ-groups is* **Arch**-*epic.*

Proof. To verify part (1) observe that

$$k|b_n - b| \leqslant u \Rightarrow k\left|\varphi(b_n) - \varphi(b)\right| \leqslant \varphi(u)$$

since φ is an ℓ-homomorphism. To prove part (2) we show by induction on α that $\varphi|r_\alpha(A, B) = \psi|r_\alpha(A, B)$ for each α. Part (2) is the case of $\alpha = \omega_1$, and the case $\alpha = 0$ is the hypothesis $\varphi|A = \psi|A$. Suppose $\varphi|r_\alpha(A, B) = \psi|r_\alpha(A, B)$. If $b \in r_{\alpha+1}(A, B)$ then $b_n \to b\ (u)$ for some $\{b_n\} \subseteq r_\alpha(A, B)$ and $u \in B$. By part (1) we have $\varphi(b_n) \to \varphi(b)\ (\varphi(u))$ and $\psi(b_n) \to \psi(b)\ (\psi(u))$, with $\varphi(b_n) = \psi(b_n)$ for each n. Since C is Archimedean, relative uniform limits in C are unique (see Subsection 1.1), so $\varphi(b) = \psi(b)$.

To verify part (3) suppose that $A \leqslant B$ is a relatively uniformly dense extension in **Arch**, i.e., $r_{\omega_1}(A, B) = B$, that $\varphi, \psi : B \to C$ are **Arch**-morphisms, i.e., $C \in |\mathbf{Arch}|$, and that $\varphi|A = \psi|A$. Then $\varphi|r_{\omega_1}(A, B) = \psi|r_{\omega_1}(A, B)$ by part (2). □

Note that the full converse of Proposition 14 certainly fails: $Z \leqslant R$ is not relatively uniformly dense, but is **Arch**-epic (see [2] if necessary). The following result represents new information which will find use elsewhere.

Corollary 15. *For each α, $C^*(X) \leqslant B_\alpha(X)$ is* **Arch**-*epic.*

Proof. Corollary 3 and part (3) of Proposition 14. □

4. Epimorphisms and relative uniform density

4.1. **W**-*epics and relative uniform density*

Theorem 16. *Suppose that $A \leqslant E$ is a* **W**-*epicompletion, and that A is divisible. Then A is relatively uniformly dense in E.*

Proof. We invoke Theorem 13 to realize $A \leqslant E$ with a commutative diagram in **W**

$$\begin{array}{ccccc} A^* & \leqslant & C(YA) & \leqslant & B(YA) \\ \downarrow & & & & \downarrow{\scriptstyle q} \\ A & & \xrightarrow{\quad\varphi\quad} & & E \end{array}$$

in which q is a surjection and φ is an embedding. In these terms we are to show $\varphi(A)$ is relatively uniformly dense in E. For simplicity let $Y = YA$ and $B = B(YA)$.

First, A^* is divisible since A is, and A^* separates the points of Y since A does, by Theorem 11. By the Stone–Weierstrass Theorem, A^* is relatively uniformly dense in $C(Y)$ in the usual sense, regulated by 1, hence relatively uniformly dense. By Corollary 3, $C(Y)$ is relatively uniformly dense in B. Relative uniform density is transitive (see Subsection 1.1), so A^* is relatively uniformly dense in B, which is to say $B = r(A, B)$. (Alternatively, $C(Y) = ua^*$, and use Lemma 10.) Then

$$\begin{aligned} E &= q(B) && \text{since } q \text{ is surjective,} \\ &= q\big(r(A^*, B)\big) && \text{by the previous paragraph,} \\ &\subseteq r\big(q(A^*), E\big) && \text{by Proposition 14 (2),} \\ &= r\big(\varphi(A^*), E\big) && \text{since the diagram commutes,} \\ &\subseteq r\big(\varphi(A), E\big) && \text{since } A^* \subseteq A, \end{aligned}$$

showing that $\varphi(A)$ is relatively uniformly dense in E. □

Remark 2. We make some observations on Theorem 16.

(1) The **W**-embedding $Z \leqslant R$ noted at the end of Section 3 shows that the hypothesis that A be divisible cannot be dropped.

(2) That the hypothesis that E be epicomplete cannot be simply dropped is shown by any **W**-epic $A \leqslant E$ which is not **Arch**-epic by part (3) of Proposition 14. Several of these are exhibited in Section 8.7 of [2].

(3) Theorem 16 contains the assertion that $r(C^*(X), B(X)) = B(X)$ in Corollary 3 since $B(X)$ is an epicompletion of $C^*(X)$ [4].

4.2. **Arch**-*epics and relative uniform density*

Theorem 17. *Suppose that $A \leqslant E$ is an* **Arch**-*epicompletion, and that A is divisible. Then A is relatively uniformly dense in E.*

Proof. To say that $A \leqslant E$ is an epicompletion means that $A \leqslant E$ is epic and E is epicomplete. Whenever $A \leqslant E$ is epic the embedding is *coessential*, meaning that if an **Arch**-morphism $\varphi: E \to B$ has $\varphi|A = 0$ then $\varphi = 0$. This means that $\mathrm{ak}_E\, E = E$, where $\mathrm{ak}_E\, A$ denotes the least ideal in E containing A such that $E/\mathrm{ak}_E\, A$ is Archimedean. But $\mathrm{ak}_E\, A = r([A]_E, E)$, where

$$[A_E] = \{b \in E\colon\ |b| \leqslant a \text{ for some } a \in A\},$$

the ideal in E generated by A; see [12, pp. 85, 427]. This means that A will be relatively uniformly dense in E provided that $[A]_E \subseteq r(A, E)$. We set out to prove that.

Suppose that $A \leqslant E$ is an epicompletion and $b \in [A]^+$. Choose a in A with $0 \leqslant b \leqslant a$, and consider the induced embedding $A/a^\perp \leqslant E/a^\perp$. Upon designating the unit $a + a^\perp$, this is a **W**-epicompletion. It is epic by [2, 8.4.4], and **W**-epicomplete by [3, 3.9, 4.9]. By Theorem 16 here, $A/a^\perp \leqslant E/a^\perp$ is relatively uniformly dense. Since $0 \leqslant b \leqslant a$, this implies $b \in r(A, E)$ by Proposition 18. □

Proposition 18. *Suppose $A \leqslant E$, $a \in A^+$, and consider the induced embedding $A/a^\perp \leqslant E/a^\perp$. If $0 \leqslant x \leqslant a$ then for each α,*

$$x + a^\perp \in r_\alpha(A/a^\perp, E/a^\perp) \quad \textit{and} \quad 0 \leqslant x \leqslant a \Rightarrow x \in r_\alpha(A, E).$$

Proof. Denote cosets in $E/a^\perp$ by $\bar{x} \equiv x + a^\perp$, and $\overline{E} \equiv E/\bar{a}$. Note the following features of the quotient $E \to \overline{E}$.

(1) $0 \leqslant x, y \in a^{\perp\perp}$ and $\bar{x} \leqslant \bar{y}$ imply $x \leqslant y$.

(2) $x, y \in a^{\perp\perp}$ and $\bar{x} = \bar{y}$ imply $x = y$.

(3) For elements x_n and x in $a^{\perp\perp}$, if $\bar{x}_n \to \bar{x}\ (\overline{w})$ in $\overline{E}$ then $x_n \to x\ (w)$ in E.

To prove part (1) simply observe that

$$\bar{x} \leqslant \bar{y} \Leftrightarrow x + a^\perp \leqslant y + a^\perp \Leftrightarrow (x - y) \vee 0 \in a^\perp, \quad \text{and}$$
$$0 \leqslant (x - y) \vee 0 \leqslant x \in a^{\perp\perp} \Rightarrow (x - y) \vee 0 \in a^{\perp\perp}.$$

Since $a^\perp \cap a^{\perp\perp} = \{0\}$ we get $(x - y) \vee 0 = 0$, i.e., $x \leqslant y$. (Alternatively, embed E into $E/a^\perp \times \prod_{\mathcal{U}} E/u^\perp$, where $\mathcal{U}$ is maximal among the pairwise disjoint subsets of E^+ which are disjoint from $\{a\}$.) Part (2) is an immediate consequence of part (1), since $|x| \in a^{\perp\perp}$ if and only if $x \in a^\perp$. For part (3) we note that, by part (1),

$$k|\bar{x}_n - \bar{x}| \leqslant \overline{w} \Rightarrow k|x_n - x| \leqslant w. \tag{$*$}$$

We now prove by induction on α that

$$\bar{x} \in r_\alpha(\overline{A}, \overline{E}) \quad \text{and} \quad 0 \leqslant x \leqslant a \Rightarrow x \in r_\alpha(A, E).$$

To start the induction consider $\bar{x} \in r_0(\overline{A}, \overline{E})$, $0 \leqslant x \leqslant a$, and let $y \in A$ satisfy $\bar{y} = \bar{x}$. By replacing y by $(y \vee 0) \wedge a$ if necessary, we may assume that $0 \leqslant y \leqslant a$. Then $x = y$ by part (2), so $x \in A = r_0(A, E)$.

Now assume that $(*)$ holds for α, and consider $\bar{x} \in r_{\alpha+1}(\overline{A}, \overline{E})$, $0 \leqslant x \leqslant a$. So there are elements $\bar{x}_n \in r_\alpha(\overline{A}, \overline{E})$ with $\bar{x}_n \to \bar{x}\ (\bar{u})$. Let $z_n = (x_n \wedge 0) \wedge a$. Then for all $n \in N$ we have $0 \leqslant z_n \leqslant a$ and $\bar{z}_n = \bar{x}_n$, and by [12, 16.2] we have $\bar{z}_n \to \bar{x}\ (\bar{v})$ for some regulator

$\bar{v}$. By the induction hypothesis $z_n \in r_\alpha(A, E)$ for all n, and $z_n \to x$ (v) by part (3). This completes the proof of Proposition 18, and hence also of Theorem 17. □

That Theorem 17 includes Theorem 16 is true, but less obvious than one might expect. Any **W**-epicompletion is an **Arch**-epicompletion for the following reasons. A **W**-epic $A \leqslant B$, with B a ring with identity e_B, is **Arch**-epic [2, 8.5.2]. And **W**-epicomplete objects are $D(X)$'s (Theorem 12) and hence rings.

4.3. An example

We give here an example of an epimorphic extension $A \leqslant B$ in **Arch** which is not relatively uniformly dense. Let X denote $[0, 1]$, let $\{q_n\colon n \in N\}$ be an enumeration of the rational points of X, and let P denote the irrational points of X. For each $n \in N$ define r_n by

$$r_n(x) \equiv \begin{cases} \dfrac{1}{|x - q_n|}, & x \neq q_n, \\ \infty & x = q_n. \end{cases}$$

Let R denote $\{\sum_{i=1}^n k_i r_i\colon k_i \in Z, n \in N\}$, the subgroup of $D(X)$ generated by the r_n's, let C denote $C(X)$, and let

$$A \equiv C + R = \{c + r\colon c \in C, r \in R\}.$$

We show that A is an ℓ-group in $D(X)$ in Proposition 21, for which two lemmas are required.

Lemma 19. *A is a subset of $D(X)$.*

Proof. Clearly each r_n lies in $D(X)$, and, for $c \in C$ and $r \equiv \sum_{i=1}^n k_i r_i$, the extension

$$(c + r)(x) \equiv \begin{cases} \infty & \text{if } x = q_i \text{ for some } i \text{ such that } k_i > 0, \\ -\infty & \text{if } x = q_i \text{ for some } i \text{ such that } k_i < 0, \\ c(x) + r(x) & \text{otherwise,} \end{cases}$$

can readily be seen to lie in $D(X)$ as well. □

Lemma 20. *If N_0 and N_1 are disjoint finite subsets of N, if $0 > k_i \in Z$ for all $i \in N_0 \cup N_1$, and if $c \in C$, then*

$$\left(c + \sum_{i \in N_0} k_i r_i\right) \vee \sum_{i \in N_1} k_i r_i$$

is bounded below.

Proof. The disjoint finite sets $\{q_i\colon i \in N_0\}$ and $\{q_i\colon i \in N_1\}$ are contained in disjoint open sets U_0 and U_1. Now $c + \sum_{N_0} k_i r_i$ is bounded below on $U_0' \equiv X \setminus U_0$ because it is continuous and finite there, and $\sum_{N_1} k_i r_i$ is likewise bounded below on U_1'. Because $U_0' \cup U_1' = X$, the lemma follows. □

Proposition 21. *A is an ℓ-group in $D(X)$.*

Proof. A is a group by construction, and is contained in $D(X)$ by Lemma 19. What we must show is that A is a sublattice of $D(X)$, i.e., that $a \vee 0 \in A$ for all $a \in A$. Given $a \in A$, find $c \in C$ and $r \in R$ such that $a = c + r$. Writing r as $\sum_{i=1}^{n} k_i r_i$ for $k_i \in Z$, set

$$r_+ \equiv \sum_{k_i > 0} k_i r_i, \quad \text{and} \quad r_- \equiv \sum_{k_i < 0} k_i r_i,$$

with the understanding that $r_+ = 0$ in case $k_i \leqslant 0$ for $1 \leqslant i \leqslant n$, and likewise for r_-. Set $d \equiv (c + r_-) \vee (-r_+)$. Then d is bounded below by Lemma 20, and is bounded above because it is formed from functions which are bounded above. Therefore $d \in C$, so

$$\begin{aligned} a \vee 0 &= (c + r) \vee 0 = (c + r_- + r_+) \vee (-r_+ + r_+) \\ &= (c + r_-) \vee (-r_+) + r_+ = d + r_+ \in A. \qquad \square \end{aligned}$$

We now change our point of view to $C(P)$, and view the elements of A to be present in $C(P)$ by restriction. Let

$$b_0 \equiv \sum_{i=1}^{\infty} 2^{-i} \sin\left(\frac{1}{x - q_i}\right).$$

This function b_0 is clearly continuous and bounded on P, but it cannot be continuously extended to any rational point of X. In fact, no element of the form $c + nb_0$, $c \in C$ and $n \neq 0$, can be so extended. Furthermore, although an element of the form $a + nb_0$, $a \in A$ and $n \neq 0$, can be continuously extended to the poles of a, it cannot be continuously extended to more than finitely many rational points.

Let B denote the ℓ-subgroup of $C(P)$ generated by $A \cup \{b_0\}$. It is in this B that A is embedded epimorphically but not relatively uniformly densely.

Lemma 22. *Suppose $b = \bigvee_I \bigwedge_J b_{ij}$ for some finite subset $\{b_{ij}\} \subseteq B$. Then there is a family $\{U_{ij}\}$ of pairwise disjoint open sets whose union is dense in P such that $b(x) = b_{ij}(x)$ for all $x \in U_{ij}$, $i \in I$, and $j \in J$.*

Proof. In the simple case in which $b = b_1 \wedge b_2$, set

$$U_1 \equiv \operatorname{coz}(b_2 - b_1)^+ \quad \text{and} \quad U_2 \equiv P \setminus \operatorname{cl} U_1.$$

A simple induction yields the slightly more complicated case in which $b = \bigwedge_J b_j$ for finite sets $\{b_j\} \subseteq B$. In the general case $b = \bigvee_I \bigwedge_J b_{ij}$, first write $b = \bigvee_I b_i$ for $b_i \equiv \bigwedge_J b_{ij}$, then use the dual of the aforementioned case to find pairwise disjoint open sets $\{U_i\}$ whose union is dense in P such that $b(x) = b_i(x)$ for all $x \in U_i$ and all $i \in I$. Then for each $i \in I$ use this case again to find pairwise disjoint open sets $\{V_{ij}\colon j \in J\}$ such that $\bigcup_J V_{ij}$ is dense in P and $b_i(x) = b_{ij}(x)$ for all $x \in V_{ij}$ and all $j \in J$. Finally, the sets we seek are

$$U_{ij} \equiv U_i \cap V_{ij}, \quad i \in I,\ j \in J. \qquad \square$$

Lemma 23. *Suppose $a_n \to b\,(r)$ for some sequence $\{a_n\}$ in A, some $b \in B$, and some $r \in B^+$. Then there is a positive integer n such that b can be continuously extended to $X \setminus \{q_i\colon 1 \leqslant i \leqslant n\}$.*

Proof. Since every element of B has an upper bound in R, we may assume $r \in R$, say $r = \sum_{i=1}^{n} k_i r_i$. The fact that $r > 0$ implies that each k_i is nonnegative, and, when viewed as a function on X, r is finite at each point of $X_n \equiv X \setminus \{q_i\colon 1 \leqslant i \leqslant n\}$. It follows that b can be extended continuously to this set.

We elaborate a bit on the last assertion. Because $a_n \to b\ (r)$ on P it follows easily that

$$\frac{a_n}{r} \to \frac{b}{r}\ (1)$$

on P. But because the approximations a_n/r extend continuously to X_n and the convergence is uniform, the limit b/r must also extend to X_n. But then $b = r(b/r)$ must extend to X_n as well. □

Lemma 24. *Suppose $a_n \to b\,(r)$ for some sequence $\{a_n\}$ in A, some $b \in B$, and some $r \in B^+$. Then there is a finite subset $\{\bar{a}_k\colon k \in K\} \subseteq A$ and a family $\{U_k\}$ of pairwise disjoint open sets whose union is dense in P such that $b(x) = a_k(x)$ for all $x \in U_k$ and all $k \in K$.*

Proof. b can be expressed as $\bigvee_I \bigwedge_J (a_{ij} + n_{ij} b_0)$ for finite sets $\{a_{ij}\} \subseteq A$ and $\{n_{ij}\} \subseteq N$. By Lemma 22 there is a collection $\{U_{ij}\}$ of pairwise disjoint open subsets whose union is dense in P and which satisfies

$$b(x) = a_{ij}(x) + n_{ij} b_0(x), \quad x \in U_{ij},\ i \in I,\ j \in J.$$

Now as we remarked above, no function of the form $a + nb_0$, $a \in A$ and $n \neq 0$, can be continuously extended to more than finitely many rational points. Thus if $U_{ij} \neq \emptyset$ then it follows from Lemma 23 that $n_{ij} = 0$. Therefore the desired collection is $\{a_{ij}\colon U_{ij} \neq \emptyset\}$. □

Proposition 25. *A is relatively uniformly closed in B.*

Proof. Suppose $a_n \to b\,(r)$ for a sequence $\{a_n \subseteq A\}$, an element $b \in B$, and some $0 \leqslant r \in B$. We first show that if b is bounded then it must lie in C. Let n be the positive integer given by Lemma 23 such that b can be continuously extended to $X \setminus \{q_i\colon 1 \leqslant i \leqslant n\}$. We show that b can be continuously extended to any q_i, $1 \leqslant i \leqslant n$. Let $\{\bar{a}_k\} \subseteq A$ and $\{U_k\}$ be the finite sets given by Lemma 24, and let K_0 denote $\{k_0 \in K\colon q_i \in \operatorname{cl} U_{k_0}\}$.

It is sufficient to demonstrate that $\{\bar{a}_{k_0}(q_i)\colon k_0 \in K_0\}$ contains a single real number v, for in that case we obtain the desired extension by setting $b(q_i) = v$. So assume for contradiction that this set contains more than one point. Choose open intervals V_{k_0} containing $\bar{a}_{k_0}(q_i)$ such that the intervals of distinct points are disjoint. Then use the continuity of each $\bar{a}_{k_0}$ to find a single interval (u, v) containing q_i but not q_j for $j \neq i$, $1 \leqslant j \leqslant n$, and satisfying $\bar{a}_{k_0}(p) \in V_{k_0}$ for all $p \in (u, v)$ and all $k_0 \in K_0$. Now

$$\left\{\bar{a}_{k_0}(q_i)\colon U_{k_0} \cap (u, q_i) \neq \emptyset\right\}$$

must be a singleton, since otherwise the extension of b to the interval $(u, q_i)_X$ violates the Intermediate Value Theorem. It follows that b has a left limit at q_i, and similarly that it has a right limit at q_i, but that the two limits are different. That is, b must have a jump discontinuity at q_i. But this cannot happen, for the sequences $\{q_i + 1/(n\pi)\}$ and $\{q_i - 1/(n\pi)\}$ both lie in P and converge to q_i from opposite directions, and

$$\lim_{n\to\infty} b_0\left(q_i + \frac{1}{n\pi}\right) = \sum_{j\neq i} 2^{-j} \sin\left(\frac{1}{q_i - q_j}\right) = \lim_{n\to\infty} b_0\left(q_i - \frac{1}{n\pi}\right).$$

From this fact it follows that each $b_1 \in B$ satisfies

$$\lim_{n\to\infty} b_1\left(q_i + \frac{1}{n\pi}\right) = \lim_{n\to\infty} b_1\left(q_i - \frac{1}{n\pi}\right),$$

a contradiction which completes the proof that if b is bounded then it lies in C.

Now suppose that b is unbounded, and write it in the form

$$b = \bigvee_I \bigwedge_J (a_{ij} + n_{ij} b_0)$$

for subsets $\{a_{ij}\} \subseteq A$ and $\{n_{ij}\} \subseteq Z$. By Proposition 21 there are functions $r \in R$ and $c \in C$ for which $\bigvee_I \bigwedge_J a_{ij} = c + r$. We claim that $b - r$ is bounded. In fact, if m and l are positive integers such that $|c| \leqslant m$ and $|n_{ij}| \leqslant l$ for all $i \in I$ and $j \in J$ then $|b - r| \leqslant m + l$. For if $i \in I$ and $p \in P$ then

$$\begin{aligned}\bigwedge_J \big(a_{ij}(p) - r(p) + n_{ij} b_0(p)\big) &\leqslant \bigwedge_J \big(a_{ij}(p) - r(p)\big) + n_{ij} b_0(p)\\ &\leqslant \bigwedge_J (a_{ij} - r)(p) + l,\end{aligned}$$

with the result that

$$\begin{aligned}(b - r)(p) &= \bigvee_I \bigwedge_J (a_{ij} - r + n_{ij})(p) \leqslant \bigvee_I \bigwedge_J (a_{ij} - r)(p) + l\\ &= c(p) + l,\end{aligned}$$

and likewise $(b - r)(p) \geqslant c(p) - l$. From this the claim follows, and, since $(a_n - r) \to (b - r)$ (r), it follows from the bounded case that $b - r \in C$, i.e., that $b \in C + R = A$. □

It remains to show in Proposition 30 that the embedding of A in B is an epimorphism in **Arch**. For that purpose we require several preliminary results and some notation.

We have described A in such a way that it coincides with its Yosida representation (Theorem 11), so that YA can be taken to be X. Let Y denote YB, the Yosida space of B, which we regard as another compactification of P. Then the embedding of A in B is realized by a unique continuous surjection $\tau : Y \to X$ [2, 8.2.4 (b)]. What this means is that for all $a \in A$ and $y \in Y$,

$$a_X\big(\tau(y)\big) = a_Y(y),$$

where a_X refers to the image of a in the Yosida representation of A as an ℓ-group in $D(X)$, and a_Y refers to the image of a in the Yosida representation of B as an ℓ-group in $D(Y)$.

(The symbol a with no subscript refers to a_Y.) The restriction of τ to P is the inclusion of P in X, and the task immediately at hand is to verify that τ takes the growth of Y to the growth of X, i.e., that $\tau^{-1}(p) = \{p\}$ for each $p \in P$.

Lemma 26. *For each $b \in B$ and $\varepsilon > 0$ there is an element $d \in C(P)$ such that $|b - d| \leqslant \varepsilon$, and such that d can be continuously extended to all but a finite number of rational points of X.*

Proof. Given $b = \bigvee_I \bigwedge_J (a_{ij} + n_{ij} b_0)$ and $\varepsilon > 0$, let n be a positive integer large enough that $n \geqslant |n_{ij}|$ for all $i \in I$ and $j \in J$. Then find another positive integer m big enough that $1/2^m \leqslant \varepsilon/n$. Define

$$b_m \equiv \sum_{i=1}^{m} 2^{-i} \sin\left(\frac{1}{x - q_i}\right) \in C(P),$$

a function which can be continuously extended to $X \setminus \{q_i\colon 1 \leqslant i \leqslant m\}$. Then

$$|b_0 - b_m| = \left| \sum_{i=m+1}^{\infty} 2^{-i} \sin\left(\frac{1}{x - q_i}\right) \right| \leqslant \sum_{i=m+1}^{\infty} 2^{-i} = \frac{1}{2^m} \leqslant \frac{\varepsilon}{n}.$$

Set $d \equiv \bigvee_I \bigwedge_J (a_{ij} + n_{ij} b_m)$. Then

$$d \leqslant \bigvee_I \bigwedge_J \left(a_{ij} + n_{ij}\left(b_0 + \frac{\varepsilon}{n}\right)\right) \leqslant \bigvee_I \bigwedge_J (a_{ij} + n_{ij} b_0 + \varepsilon) =$$
$$= \bigvee_I \bigwedge_J (a_{ij} + n_{ij} b_0) + \varepsilon = b_0 + \varepsilon,$$

and likewise $d \geqslant b_0 - \varepsilon$. □

A consequence of the following lemma is that $C \equiv C(X)$ is *order dense* in B, i.e., for all $0 < b \in B$ there is some $0 < c \in C$ such that $c \leqslant b$. It follows that A is order dense in B also.

Lemma 27. *For each $0 < b \in B$ and $p \in P$ such that $b(p) > 0$ there is some $c \in C \equiv C(X)$ such that $0 < c \leqslant b$ and $c(p) > 0$.*

Proof. Suppose $b(p) = \varepsilon$, and let $d \in C(P)$ satisfy Lemma 26 for $\varepsilon/4$. Now $d(p) \geqslant 3\varepsilon/4$, and since d can be continuously extended to some open interval of X containing p, there must be another open interval S of X containing p such that $d(s) \geqslant \varepsilon/2$ for all $s \in S$. Then for such s we have

$$b(s) \geqslant d(s) - \frac{\varepsilon}{4} \geqslant \frac{\varepsilon}{2} - \frac{\varepsilon}{4} = \frac{\varepsilon}{4}.$$

Choose a continuous function $c \in C$ such that $0 < c \leqslant \varepsilon/4$, $c(p) > 0$, and $c(x) = 0$ for all $x \in X \setminus S$. Such a c must satisfy the lemma. □

Lemma 28. *τ takes the growth of Y to the growth of X.*

Proof. Suppose for contradiction that $\tau(y) = p \in P$ for some $y \in Y \setminus P$. Since B separates the points of Y, there is some $0 < b \in B$ such that $b(p) > 0$ but $b(y) = 0$. Let c be an element of C given by Lemma 27 such that $0 < c \leqslant b$ and $c(p) > 0$. But since $c \in A$ and τ realizes the embedding of A in B, we arrive at the contradiction

$$0 < c_X(p) = c_X\big(\tau(y)\big) = c_Y(y) \leqslant b(y) = 0. \qquad \square$$

Proposition 29. *The embedding of A in B is an epimorphism in* **W**.

Proof. We appeal to the fundamental Theorem 8.3.2 of [2]. We claim that R serves as a countable set of epi-indicators for each $b \in B$. For if points $y_1 \neq y_2$ of Y satisfy $\tau(y_1) = \tau(y_2)$ then, since τ takes growth to growth, it must be true that $\tau(y_1) = \tau(y_2) = q_n$ for some rational point $q_n \in X$. But then $r_n(y_1) = r_n(y_2) = \infty$. $\square$

Proposition 30. *The embedding A in B is an epimorphism in* **Arch**.

Proof. Suppose $\alpha_i : B \to D$ are morphisms in **Arch** such that $\alpha_1|A = \alpha_2|A$. To show that $\alpha_1 = \alpha_2$ it is clearly sufficient to show that $\alpha_1(b_0) = \alpha_2(b_0)$. So suppose for the sake of argument that $\alpha_1(b_0) \neq \alpha_2(b_0)$. Since $-1 \leqslant b_0 \leqslant 1$, it follows that

$$d^\perp + \alpha_1(b_0) \neq d^\perp + \alpha_2(b_0),$$

in $D/d^\perp$, where d denotes $\alpha_1(1) = \alpha_2(1)$. But if $\alpha : D \to D/d^\perp$ is the projection defined by $\alpha(d') \equiv d^\perp + d'$, and if we regard $D/d^\perp$ as a **W** object by equipping it with unit $d^\perp + d$, then the maps $\alpha\alpha_1$ and $\alpha\alpha_2$ violate Proposition 29. $\square$

We conclude with a question. Let us say that E is absolutely relatively uniformly closed in **Arch** if $E \leqslant A$ in **Arch** implies $r(E, A) = E$. Then Theorem 17 implies that an absolutely relatively uniformly closed **Arch** object is epicomplete. For if $E \leqslant A$ is epic, and we choose any epicompletion $A \leqslant B$ of A (see Section 3), then $E \leqslant B$ is an epicompletion, hence relatively uniformly dense by Theorem 17, so $r(E, B) = B$, meaning $E = B$, so $E = A$.

The question is whether the converse holds, i.e., whether every epicomplete **Arch** object is absolutely relatively uniformly closed. Part (2) of Proposition 14 would seem to be trying to say that, but does not quite. The issues are closely related to those of "saturations versus absolute closures" mentioned in [11, 1.7, 1.8].

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