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UNIVALENT FUNCTIONS OF ONE AND SEVERAL COMPLEX VARIABLES

Ph.D. Thesis Summary

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Contents

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Introduction

The geometric theory of complex variable functions was set as a separately branch of complex analysis in the XX-th century when the first important papers appeared in this domain, owed to P. Koebe [58], I.W. Alexander [2], L. Bieberbach [16].

The univalent function notion occupy a central role in geometric theory of analytic functions, first paper dating from 1907 owed to P. Koebe [58]. The study of univalent functions was continued by Plemelj [92], Gronwall [48] and Faber [31].

These days are many treaty and monographs dedicated to univalent functions study, of which we remember those of P. Montel [80], Z. Nehari [81], L.V. Ahlfors [1], Ch. Pommerenke [95], A.W. Goodman [39], P.L. Duren [30], D.J. Hallenbeck, T.H. MacGregor [50], S.S. Miller, P.T. Mocanu [75] and P.T. Mocanu, T. Bulboacă, Gr. Şt. Sălăgean [79].

The problem of results extension from the geometric theory of the functions of one to several complex variables was formulated first time by H. Cartan in the Appendix in the book of P. Montel published in 1933 [22].

The extension of the geometric properties of biholomorphic mappings was started in the 1960-1980 by the japanese mathematicians I. Ono [83], T. Higuchi [54], K. Kikuchi [57] and it was resumed in the last 20 years by J.A. Pfaltzgraff, T.J. Suffridge, C. FitzGerald, S. Gong, I. Graham, G. Kohr, H. Hamada, P. Liczberski, P. Curt.

In this thesis I introduced new classes of univalent functions respectively univalent mappings which I studied by using different methods.

In the following, in each chapter I selected the most relevant results, with the emphasis on my original contributions. The results from the first chapters are renumbered. Finally, full bibliography is included.

Chapter 1

Univalent functions of one complex variable

In this chapter are presented notions and elementary results from the geometric theory of univalent functions of one complex variable. Are treated some classes of univalent functions, the differential subordinations and superordinations methods, Loewner chains method and integral operators by Sălăgean type.

1.1 Elementary results in the theory of univalent functions

1.2 Analytic functions with positive real part

Analytic functions with positive real part have an important role in characterizing some special classes of univalent functions.

Definition 1.2.1 *By Charathèodory's functions class we understand:*

$$\mathcal{P} = \{p \in \mathcal{H}(U) : p(0) = 1,\ \operatorname{Re} p(z) > 0,\ z \in U\}.$$

1.3 Subordination. Subordination principle

Definition 1.3.1 *Let $f, g \in \mathcal{H}(U)$. We say that the function f is subordinate to g and we write $f \prec g$ or $f(z) \prec g(z)$ if there exists a function $w \in \mathcal{H}(U)$, with $w(0) = 0$ and $|w(z)| < 1,\ z \in U$ such that $f(z) = g(w(z)),\ z \in U$.*

1.4 Starlike functions. Convex functions

The notion of starlike function was introduced by J. Alexander [2] în 1915.

Definition 1.4.1 *Let $f : U \to \mathbb{C}$ be a holomorphic function with $f(0) = 0$. We say that f is starlike in U with respect to zero (or, in brief, starlike) if the function f is univalent in U and $f(U)$ is a starlike domain with respect to zero, meaning that for each $z \in U$ the segment between the origin and $f(z)$ lies in $f(U)$.*

Theorem 1.4.2 *[79] Let the function $f \in \mathcal{H}(U)$ with $f(0) = 0$. The function f is starlike if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0,\ \ z \in U. \tag{1.4.1}$$

Definition 1.4.3 *[79] We denote with S^* the class of functions $f \in \mathcal{A}$ which are starlike in the unit disk*

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re}\frac{zf'(z)}{f(z)} > 0,\ \ z \in U \right\}.$$

The notion of convex function was introduced by E. Study [110] în 1913.

Definition 1.4.4 *The function f is called convex in U (or, in brief convex) if the function f is univalent in U and $f(U)$ is a convex domain.*

Theorem 1.4.5 *[79] Let the function $f \in \mathcal{H}(U)$. Then f is convex if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re}\frac{zf''(z)}{f'(z)} + 1 > 0,\ \ z \in U. \tag{1.4.2}$$

Definition 1.4.6 *We denote with K the class of functions $f \in \mathcal{A}$ which are convex in the unit disk*

$$K = \left\{ f \in \mathcal{A} : \operatorname{Re}\frac{zf''(z)}{f'(z)} + 1 > 0,\ \ z \in U \right\}.$$

1.5 Functions whose derivative has a positive real part

1.6 Differential subordinations

The differential subordinations method (or admissible functions method) is one of the newest method used in the geometric theory of analytical functions. The bases of this theory where made by S.S. Miller şi P.T. Mocanu in the papers [73], [74].

Definition 1.6.1 *We denote by $\mathcal{Q}$ the set of functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where*

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\},$$

and are such $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

Lemma 1.6.2 *[56],[73] (I.S. Jack, S.S. Miller, P.T. Mocanu) Let $z_0 = r_0 e^{i\theta_0}$ with $0 < r_0 < 1$ and let $f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots$ a continuous function on $\overline{U}_{r_0}$ and analytic on $U_{r_0} \cup \{z_0\}$ with $f(z) \not\equiv 0$ and $n \geq 1$. If*

$$|f(z_0)| = \max\{|f(z)| : z \in \overline{U}_{r_0}\}$$

then there exists a real number m, $m \geq n$, such that

(i) $\dfrac{z_0 f'(z_0)}{f(z_0)} = m$

(ii) $\text{Re}\dfrac{z_0 f''(z_0)}{f'(z_0)} + 1 \geq m.$

Theorem 1.6.3 *[51] (D.J. Hallenbeck and St. Ruschweyh) Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$ such that $\text{Re}\,\gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt.$$

The function q is convex and it is the best (a, n) dominant.

1.7 Differential superordinations

The dual problem of differential subordinations, that of subordinations determination for differential superordinations was initiated in 2003 by S.S. Miller and P.T. Mocanu [76].

Theorem 1.7.1 *[76] Let h be a convex function in U, with $h(0) = a$, $\gamma \neq 0$, $\text{Re}\gamma \geq 0$ and $p \in \mathcal{H}[a, n] \cap \mathcal{Q}$. If $p(z) + \dfrac{zp'(z)}{\gamma}$ is univalent in U,*

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma}$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt.$$

The function q is convex and it is the best subordinant.

Theorem 1.7.2 *[76] Let q be a convex function in U and h is defined by*

$$h(z) = q(z) + \frac{zq'(z)}{\gamma}, \ z \in U$$

with $\gamma \neq 0$, $\text{Re}\gamma \geq 0$. If $p \in \mathcal{H}[a, n] \cap \mathcal{Q}$, and $p(z) + \frac{zp'(z)}{\gamma}$ is univalent function in U with

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma}, \ z \in U$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^n h(t) t^{\frac{\gamma}{n}-1} dt.$$

The function q is the best subordinant.

1.8 Subordination chains. The Loewner differential equation

The subordination chains method or Loewner chains was introduced in 1923 by Loewner [70] and developed by P.P. Kufarev [66] 1943, Ch. Pommerenke [96] 1965, G.S. Goodman [38] 1968.

Definition 1.8.1 *The function* $f : U \times [0, \infty) \to \mathbb{C}$, *with* $f(z, t)$ *of the form*

$$f(z,t) = e^t z + a_2(t)z^2 + \cdots, \quad |z| < 1$$

is called subordination chain (or Loewner chain) if $f(\cdot, t)$ *is holomorphic and univalent on* U, *for all* $t \in [0, \infty)$ *and*

$$f(z, s) \prec f(z, t) \tag{1.8.1}$$

for $0 \leq s \leq t < \infty$.

Theorem 1.8.2 *[96] A family of functions* $\{f(z, t)\}_{t \geq 0}$ *with* $f(0, t) = 0,\ f'(0, t) = e^t$ *is a Loewner chain if and only if the following conditions holds:*

(i) There exist $r \in (0, 1)$ *and a constant* $M \geq 0$ *such that* $f(\cdot, t)$ *is holomorphic on* U_r *for each* $t \geq 0$, *locally absolutely continuous in* $t \geq 0$, *locally uniformly with respect to* $z \in U_r$, *and*

$$|f(z, t)| \leq Me^t, \ |z| \leq r, \ t \geq 0.$$

(ii) There exist a function $p(z, t)$ *such that* $p(\cdot, t) \in \mathcal{P}$ *for each* $t \geq 0$, $p(z, \cdot)$ *is measurable on* $[0, \infty)$ *for each* $z \in U$, *for all* $z \in U_r$,

$$\frac{\partial f}{\partial t}(z, t) = zf'(z, t)p(z, t), \ \text{a.p.t. } t \geq 0. \tag{1.8.2}$$

(iii) For each $t \geq 0,\ f(\cdot, t)$ *is the analytic continuation of* $f(\cdot, t)\,|_{U_r}$ *to* U, *and furthermore this analytic continuation exists under the assumptions (i) and (ii).*

1.9 Integral operators

Gr. Şt. Sălăgean [104] defined, in 1983, two operators based on which were obtained over time remarkable results in the geometric univalent functions theory. We define the integral operator.

Definition 1.9.1 *Let* $f \in \mathcal{H}(U)$, $f(0) = 0$. *We define the integral operator* I^m, $m \in \mathbb{N}$, *by*

(i) $I^0 f(z) = f(z)$

(ii) $I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt$

(iii) $I^m f(z) = I(I^{m-1} f(z))$.

Chapter 2

Univalent functions of several complex variables

This chapter is dedicated to presenting results from the geometric theory of univalent functions of several complex variables and are treated some biholomorphic mappings classes, the automorphisms of the Euclidean unit ball, the Loewner chains method, generalizations of Roper-Suffridge extension operator, respectively Pfaltzgraff-Suffridge. Generally are presented just results obtained on the Euclidean unit ball from $\mathbb{C}^n$.

2.1 Holomorphic functions in $\mathbb{C}^n$. Biholomorphic mappings

2.2 The automorphisms of the Euclidean unit ball

2.3 Generalizations of functions with positive real part

Let $\mathbb{C}^n$ be the space of n-complex variables with respect to a given norm $\|\cdot\|$. For each $z \in \mathbb{C}^n \setminus \{0\}$, let

$$T(z) = \{l_z \in L(\mathbb{C}^n, \mathbb{C}) : \; l_z(z) = \|z\|, \; \|l_z\| = 1\}.$$

In the case of the Euclidean norm $\|\cdot\|$, if $z \in \mathbb{C}^n \setminus \{0\}$ and $l_z \in T(z)$, then

$$l_z(w) = \left\langle w, \frac{z}{\|z\|} \right\rangle, \quad w \in \mathbb{C}^n.$$

The following families play a key role in our discussion:

$$\mathcal{N}_0 = \{w : B^n \to \mathbb{C}^n, \; w \in \mathcal{H}(B^n), \; w(0) = 0, \; \mathrm{Re}\,\langle w(z), z\rangle \geq 0, \; z \in B^n\}$$
$$\mathcal{N} = \{w : B^n \to \mathbb{C}^n, \; w \in \mathcal{H}(B^n), \; w(0) = 0, \; \mathrm{Re}\,\langle w(z), z\rangle > 0, \; z \in B^n \setminus \{0\}\}$$
$$\mathcal{M} = \{w \in \mathcal{N} : \; Dw(0) = I_n\}.$$

2.4 Starlike mappings. Convex mappings

In this section are presented starlike mappings and convex mappings on the Euclidean unit ball, but the results are valid and in the case of arbitrary norm [36], [44].

Definition 2.4.1 *Let $f : B^n \to \mathbb{C}^n$ be a holomorphic mapping. We say that f is starlike if f is biholomorphic, $f(0) = 0$ and $f(B^n)$ is a starlike domain with respect to zero ($tf(z) \subseteq f(B^n)$ for all $z \in B^n$ and $t \in [0,1]$).*

Let $S^*(B^n)$ be the class of normalized starlike mappings on B^n.

The analytical characterization theorem of starlike mappings is given in 1955 by Matsuno [72]. The theorem was generalized to the unit ball of $\mathbb{C}^n$ with respect to an arbitrary norm and respectively to the unit ball of a complex Banach space by Suffridge [112], [113].

Theorem 2.4.2 *Let $f : B^n \to \mathbb{C}^n$ be a locally biholomorphic mapping such that $f(0) = 0$. Then f is starlike on B^n if and only if there is $h \in \mathcal{M}$ such that*

$$f(z) = Df(z)h(z), \quad z \in B^n$$

i.e.

$$\operatorname{Re}\langle [Df(z)]^{-1} f(z), z\rangle > 0, \quad z \in B^n \setminus \{0\}.$$

Definition 2.4.3 *We say that the mapping $f : B^n \to \mathbb{C}^n$ is convex if f is biholomorphic on B^n and $f(B^n)$ is a convex domain ($(1-t)f(z) + tf(w) \in f(B^n)$ for $z, w \in B^n$ and $t \in [0,1]$).*

Let $K(B^n)$ be the set of normalized convex mappings on the unit ball B^n.

The analytical characterization theorem of convexity in the case of locally biholomorphic mappings was obtained by Kikuchi in 1973 [57] and by Gong, Wang, Yu [37] in 1993.

Theorem 2.4.4 *Let $f : B^n \to \mathbb{C}^n$ be a locally biholomorphic mapping. Then f is convex if and only if*

$$1 - \operatorname{Re}\langle [Df(z)]^{-1} D^2 f(z)(v,v), z\rangle > 0,$$

for all $z \in B^n$ such that $\|v\| = 1$ and $\operatorname{Re}\langle z, v\rangle = 0$.

2.5 Loewner chains and the Loewner differential equation in $\mathbb{C}^n$

Pfaltzgraff [89] generalized Loewner chains to higher dimensions in 1974. Later contributions permitting generalizations to the unit ball of a complex Banach space by Poreda [99]. Some best-possible result concerning the Loewner chains in several complex variables were obtained by Graham, Hamada, Kohr [40], Graham, Kohr, Kohr [45], Graham, Kohr [44].

Definition 2.5.1 *A mapping $f : B^n \times [0,\infty) \to \mathbb{C}^n$ is called a subordination chain if it satisfies the following conditions (i) $f(\cdot,t) \in \mathcal{H}(B^n)$ and $Df(0,t) = \varphi(t)I_n$, $t \geq 0$, where $\varphi : [0,\infty) \to \mathbb{C}$ is a continuous function on $[0,\infty)$ such that $\varphi(t) \neq 0$, $t \geq 0$, $|\varphi(\cdot)|$ is strictly increasing on $[0,\infty)$, and $|\varphi(t)| \to \infty$ as $t \to \infty$;*

(ii) $f(\cdot,s) \prec f(\cdot,t)$, whenever $0 \leq s \leq t < \infty$, there exists a Schwarz mapping $v_{s,t}(\cdot) = v(\cdot,s,t)$, called the transition mapping associated to $f(z,t)$, such that $f(z,s) = f(v(z,s,t),t)$, $0 \leq s \leq t < \infty$, $z \in B^n$.

A subordination chain is called a univalent subordination chain (or a Loewner chain) if in addition $f(\cdot, t)$ is univalent on B^n for all $t \geq 0$. We say that $f(z,t)$ is a normalize Loewner chain if $Df(0,t) = e^t I_n$.

Theorem 2.5.2 *Let $f(z,t)$ be a Loewner chain. Then there is a mapping $h = h(z,t)$ such that $h(\cdot, t) \in \mathcal{M}$ for each $t \geq 0$, $h(z,t)$ is measurable in t for each $z \in B^n$, and for a.e. $t \geq 0$,*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \ \forall \ z \in B^n. \tag{2.5.1}$$

(there is a null set $E \subset (0,\infty)$ such that for all $t \in [0,\infty) \setminus E$, $\dfrac{\partial f}{\partial t}(\cdot, t)$ exist and is holomorphic on B^n). For $t \in [0,\infty) \setminus E$ and $z \in B^n$, (2.5.1) holds.

Moreover, if there exists a sequence $\{t_m\}_{m \in \mathbb{N}}$ such that $t_m > 0$, $t_m \to \infty$ and

$$\lim_{m \to \infty} e^{-t_m} f(z, t_m) = F(z) \tag{2.5.2}$$

locally uniformly on B^n, then

$$f(z,s) = \lim_{t \to \infty} e^t w(z,s,t)$$

locally uniformly on B^n for each $s \geq 0$, where $w(t) = w(z,s,t)$ is the solution of the initial value problem

$$\frac{\partial w}{\partial t} = -h(w,t), \ \text{a.p.t. } t \geq s, \ w(s) = z$$

for all $z \in B^n$.

As a consequence of the Theorem 2.5.2 we have the transition mappings which generates the Loewner chains.

Corollary 2.5.3 *[28], [40], [44] Let $f : B^n \times [0,\infty) \to \mathbb{C}^n$ be a Loewner chain and $v = v(z,s,t)$ the transition mapping associated to $f(z,t)$. We suppose that $\{e^{-t} f(z,t)\}_{t \geq 0}$ is a normal family. Then for each $s \geq 0$, the limit*

$$f(z,s) = \lim_{t \to \infty} e^t v(z,s,t)$$

exist locally uniformly on B^n.

Corollary 2.5.4 *[28], [40], [46] Let $f(z,t)$ be a Loewner chain such that $\{e^{-t} f(z,t)\}_{t \geq 0}$ is a normal family on B^n. Then*

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|e^{-t} f(z,t)\| \leq \frac{\|z\|}{(1-\|z\|)^2}, \quad z \in B^n, \ t \geq 0. \tag{2.5.3}$$

In particular, if $f(z) = f(z,0)$ then

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2}, \ z \in B^n.$$

Theorem 2.5.5 *[46] Every sequence of Loewner chains $\{f_k(z,t)\}_{k \in \mathbb{N}}$, such that $\{e^{-t} f_k(z,t)\}_{t \geq 0}$ is a normal family on B^n for each $k \in \mathbb{N}$, contains a subsequence that converges locally uniformly on B^n for each fixed $t \geq 0$ to a Loewner chain $f(z,t)$ such that $\{e^{-t} f(z,t)\}_{t \geq 0}$ is normal family on B^n.*

2.6 Spirallike mappings

For a linear operator $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ we introduce the notation

$$m(A) = \min\{\mathrm{Re}\,\langle A(z), z\rangle : \ \|z\| = 1\}.$$

Definition 2.6.1 *Let $f : B^n \to \mathbb{C}^n$ be a normalized univalent mapping on B^n. Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ such that $m(A) > 0$. We say that f is spirallike relative to A if $e^{-tA} f(B^n) \subseteq f(B^n)$ for all $t \geq 0$, where*

$$e^{-tA} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k A^k. \tag{2.6.1}$$

Theorem 2.6.2 *Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ such that $m(A) > 0$ and let $f : B^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping. Then f is spirallike relative to A if and only if*

$$[Df(z)]^{-1} A f(z) \in \mathcal{N}.$$

In particular $A = e^{-i\alpha} I_n$ for $\alpha \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, Hamada and Kohr [52] have studied a class of spirallike mappings of type α.

In this case the condition (2.6.1) is

$$e^{-i\alpha} [Df(z)]^{-1} f(z) \in \mathcal{N}. \tag{2.6.2}$$

If $A = e^{-i\alpha}$, $\alpha \in (\pi/2, \pi/2)$ from Definition 2.6.1 we deduce the usual notion of spirallikeness of type α in the unit disk.The notion was introduced in 1932 by L. Špaček [107].

The next theorem given by Pommerenke [95] present a necessary and sufficient condition of spirallikeness of type α for holomorphic functions.

Theorem 2.6.3 *Assume f is a normalized holomorphic function on $U, \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $a = \tan\alpha$. Then f is a spirallike of type α if and only if*

$$F(z,t) = e^{(1-ia)t} f(e^{iat} z), \ z \in U, \ t \geq 0,$$

is a Loewner chain. In particular, f is starlike if and only if $F(z,t) = e^t f(z)$ is a Loewner chain.

Hamada şi Kohr [52] shows that spirallike mappings of type α can be embedded in Loewner chains.

Theorem 2.6.4 *Assume f is a normalized locally biholomorphic mapping on B^n, $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $a = \tan\alpha$. Then f is a spirallike mapping of type α if and only if*

$$F(z,t) = e^{(1-ia)t} f(e^{iat} z), \ z \in B^n, \ t \geq 0,$$

is a Loewner chain. In particular, f is a starlike mapping if and only if $F(z,t) = e^t f(z)$ is a Loewner chain.

2.7 Almost starlike mappings of order α

Another notion that will be used in the forthcoming sections is that of almost starlikeness of order α, where $\alpha \in [0, 1)$. The following definition was introduced by G. Kohr [59], [62] in the case of unit Euclidean ball for $\alpha = \frac{1}{2}$ and by Feng [32] in the case $\alpha \in [0, 1)$ and on the unit ball in a complex Banach space X. For our purpose, we present this notion only on the Euclidean setting.

Definition 2.7.1 *Suppose $0 \leq \alpha < 1$. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be almost starlike of order α if*

$$\mathrm{Re}\langle [Df(z)]^{-1} f(z), z\rangle > \alpha \|z\|^2, \; z \in B^n \setminus \{0\}. \tag{2.7.1}$$

In the case of one complex variable, the inequality (2.7.1) reduces to the following:

$$\mathrm{Re}\frac{f(z)}{zf'(z)} > \alpha, \; z \in U.$$

Q.H. Xu and T.S. Liu [68] proved the following characterization of almost starlikeness of order α in terms of Loewner chains.

Theorem 2.7.2 *Suppose f is a normalized holomorphic function in U and $0 \leq \alpha < 1$. Then f is an almost starlike function of order α if and only if*

$$F(z,t) = e^{\frac{t}{1-\alpha}} f(e^{\frac{\alpha}{\alpha-1}t} z), \; z \in U, \; t \geq 0,$$

is a Loewner chain. In particular, f is a starlike function (i.e., $\alpha = 0$) if and only if $F(z,t) = e^t f(z)$ is a Loewner chain.

The next result, again due to Q-H. Xu and T-S. Liu [68] is the generalization of Theorem 2.7.2 to the n-dimensional case. Note that this result was originally obtained on the unit ball of $\mathbb{C}^n$ with respect to an arbitrary norm.

Theorem 2.7.3 *Suppose f is a normalized locally biholomorphic mapping in B^n and $0 \leq \alpha < 1$. Then f is almost starlike of order α if and only if*

$$F(z,t) = e^{\frac{1}{1-\alpha}t} f(e^{\frac{\alpha}{\alpha-1}t} z), \; z \in B^n, \; t \geq 0,$$

is a Loewner chain. In particular, f is a starlike mapping (i.e., $\alpha = 0$) if and only if $F(z,t) = e^t f(z)$ is a Loewner chain.

2.8 Generalizations of the Roper-Suffridge extension operator. The Pfaltzgraff-Suffridge operator

In the euclidean n-dimensional space $\mathbb{C}^n$, let $\widetilde{z} = (z_2, \cdots, z_n)$ such that $z = (z_1, \widetilde{z})$.

The Roper-Suffridge extension operator is defined for normalized locally univalent functions on the unit disk U, by

$$\Phi_n(f)(z) = F(z) = \left(f(z_1), \sqrt{f'(z_1)}\widetilde{z}\right). \tag{2.8.1}$$

The branch of the square root is chosen such that $\sqrt{f'(0)} = 1$.

This operator was introduced in 1995 by Roper and Suffridge in purpose to extend an arbitrary convex function from the unit disk U to an convex mapping from the Euclidean unit ball of $\mathbb{C}^n$.

I. Graham, G. Kohr, M. Kohr [45] have generalized this operator to

$$\Phi_{n,\gamma}(f)(z) = \left(f(z_1), \left(f'(z_1)\right)^{\gamma} \widetilde{z}\right), \tag{2.8.2}$$

where $\gamma \in [0, \frac{1}{2})$ and the branch of the power function is chosen such that $(f'(z))^{\gamma} \mid_{z_1=0}= 1$.

Theorem 2.8.1 *[45] Suppose that $f \in S$ and $\gamma \in [0, \frac{1}{2}]$, then $\Phi_{n,\gamma}(f)$ can be embedded in a Loewner chain, where*

$$\Phi_{n,\gamma}(f)(z) = \left(f(z_1), \left(f'(z_1)\right)^{\gamma} \widetilde{z}\right), \ z = (z_1, \widetilde{z}) \in B^n$$

and $z_1 \in U$, $\widetilde{z} = (z_2, \cdots, z_n) \in \mathbb{C}^{n-1}$. The branch of the power function is chosen such that

$$\left(f'(z_1)\right)^{\gamma} \mid_{z_1=0}= 1.$$

In [42] I. Graham şi G. Kohr have introduced another Roper-Suffridge extension operator.

$$\Phi_{n,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\beta} \widetilde{z}\right), \tag{2.8.3}$$

where $\beta \in [0, 1]$. The power function is chosen such that $\left(\frac{f(z_1)}{z_1}\right)^{\beta} \mid_{z_1=0}= 1$.

Theorem 2.8.2 *[43] Suppose that $f \in S$ and $\beta \in [0, 1]$, then $\Phi_{n,\alpha}(f)$ can be embedded in a Loewner chain, where*

$$\Phi_{n,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\beta} \widetilde{z}\right), \ z \in (z_1, \widetilde{z}) \in B^n,$$

and $z_1 \in U$, $\widetilde{z} = (z_2, \cdots, z_n) \in \mathbb{C}^{n-1}$. The branch of the power function is chosen such that $\left(\frac{f(z_1)}{z_1}\right)^{\beta} \mid_{z_1=0}= 1$.

In 2002, I. Graham, H. Hamada, G. Kohr and T. Suffridge [41] have given another generalization of Roper-Suffridge extension operator by type

$$\Phi_{n,\beta,\gamma}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\beta} (f'(z_1))^{\gamma}\widetilde{z}\right), \tag{2.8.4}$$

where $\beta \in [0, 1]$ and $\gamma \in [0, \frac{1}{2}]$ such that $\beta + \gamma \leq 1$.The branches of the power function are chosen such that $\left(\frac{f(z_1)}{z_1}\right)^{\beta} \mid_{z_1=0}= 1$ şi $(f'(z_1))^{\gamma} \mid_{z_1=0}= 1$.

Theorem 2.8.3 *[41] Assume that $f \in S$ and $\beta \in [0, 1]$, $\gamma \in [0, \frac{1}{2}]$ with $\gamma + \beta \leq 1$, then*

$$\Phi_{n,\beta,\gamma}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\beta} (f'(z_1))^{\gamma}\widetilde{z}\right), \ z = (z_1, \widetilde{z}) \in B^n,$$

and $z_1 \in U$, $\widetilde{z} = (z_2, \cdots, z_n) \in \mathbb{C}^{n-1}$. The branches of the power functions are chosen such that $\left(\frac{f(z_1)}{z_1}\right)^{\beta} \mid_{z_1=0}= 1$ and $(f'(z_1))^{\gamma} \mid_{z_1=0}= 1$.

For $n \geq 1$, the set $z' = (z_1, \cdots, z_n) \in \mathbb{C}^n$ and $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$.

Another operator which extend univalent (locally) mappings from B^n to univalent (locally) mappings from B^{n+1} is the operator introduced by Pfaltzgraf and Suffridge [91].

Definition 2.8.4 *The Pfaltzgraff-Suffridge extension operator $\Psi_n : \mathcal{L}S_n \to \mathcal{L}S_{n+1}$ is defined by*

$$\Psi_n(f)(z) = \left(f(z'), z_{n+1}[J(z')]^{\frac{1}{n+1}}\right), \ z = (z', z_{n+1}) \in B^{n+1}. \tag{2.8.5}$$

The branches of the power functions are chosen such that

$$[J_f(z')]^{\frac{1}{n+1}} \mid_{z'=0} = 1.$$

Chapter 3

Applications of differential subordinations and superordinations methods

In this chapter are defined classes of univalent functions on the unit disk U of the complex plane. For defining some classes we use the integral operator $I^m f$ presented in the paragraph 1.9. Using the differential subordinations and superordinations method are highlighted the properties of this classes and also are presented concrete examples of differential subordinants.

The results of this chapter are original and are included in the papers [4], [5], [6], [7], [8], [9], [10], [13].

3.1 Differential subordinations defined by integral operator

In this paragraph is defined a new class of univalent functions, using the integral operator $I^m f$. Using the differential subordinations method are highlighted the properties of this class.

Definition 3.1.1 *If $0 \leq \alpha < 1$ and $m \in \mathbb{N}$, let $\mathcal{I}_n^m(\alpha)$ denote the class of functions $f \in \mathcal{A}_n$ which satisfy the inequality:*

$$\mathrm{Re}[I^m f(z)]' > \alpha.$$

Remark 3.1.2 *For $m = 0$, we obtain*

$$\mathrm{Re} f'(z) > \alpha, \ \ z \in U.$$

When $f \in \mathcal{A}$ we obtain the next definition.

Definition 3.1.3 *If $0 \leq \alpha < 1$ and $m \in \mathbb{N}$, let $\mathcal{I}^m(\alpha)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the inequality:*

$$\mathrm{Re}[I^m f(z)]' > \alpha.$$

If $m = 0$ then $\mathcal{I}^0(\alpha)$ is the class of functions with boundary rotation.

Theorem 3.1.4 *[10] If $0 \le \alpha < 1$ and $m, n \in \mathbb{N}$, then we have*

$$\mathcal{I}_n^m(\alpha) \subset \mathcal{I}_n^{m+1}(\delta),$$

where

$$\delta(\alpha, n) = 2\alpha - 1 + 2(1-\alpha)\frac{1}{n}\beta(\frac{1}{n})$$

and

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}dt.$$

The result is sharp.

For the class $\mathcal{I}^m(\alpha)$we obtain the next result.

Corollary 3.1.5 *[4] If $0 \le \alpha < 1$ and $m \in \mathbb{N}$, then we have*

$$\mathcal{I}^m(\alpha) \subset \mathcal{I}^{m+1}(\delta),$$

where

$$\delta = \delta(\alpha) = 2\alpha - 1 + 2(1-\alpha)\ln 2$$

and this result is sharp.

Theorem 3.1.6 *[10] Let q be a convex function, $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + nzq'(z), \ z \in U.$$

If $f \in \mathcal{A}_n$ and verifies the differential subordination

$$[I^m f(z)]' \prec h(z) \tag{3.1.1}$$

then

$$\left[I^{m+1} f(z)\right]' \prec q(z), \ z \in U$$

and this result is sharp.

Theorem 3.1.7 *[10] Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, $h'(0) \neq 0$, which verifies the inequality*

$$\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2n}, \ z \in U, \ n \in \mathbb{N}^*.$$

If $f \in A_n$ and verifies the differential subordination

$$[I^m f(z)]' \prec h(z), \tag{3.1.2}$$

then

$$\left[I^{m+1} f(z)\right]' \prec q(z), \ z \in U$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt, \ z \in U.$$

The function q is convex and is the best dominant.

Theorem 3.1.8 *[10] Let q be a convex function, $q(0) = 1$, and*

$$h(z) = q(z) + nzq(z).$$

If $f \in \mathcal{A}_n$ and verifies the differential subordination

$$[I^m f(z)]' \prec h(z) \tag{3.1.3}$$

then

$$\frac{I^m f(z)}{z} \prec q(z), \ z \in U, z \neq 0.$$

The result is sharp.

Theorem 3.1.9 *[10] Let $h \in \mathcal{H}(U)$, $h(0) = 0$, $h'(0) \neq 0$ which satisfy the inequality*

$$\operatorname{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2n}, \ z \in U.$$

If $f \in A_n$ and verifies the differential subordination

$$[I^m f(z)]' \prec h(z) \tag{3.1.4}$$

then

$$\frac{I^m f(z)}{z} \prec q(z), \ z \in U, \ z \neq 0,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt, \ z \in U.$$

The function q is convex and is the best dominant.

Remark 3.1.10 *In the particular case $n = 1$ this theorems were studied in [4] and are included in the thesis as corollaries.*

Remark 3.1.11 *For the differential operator similar results were obtained in the papers [85], [86], [88].*

3.2 A class of univalent functions obtained using the integral operator applied to meromorphic functions

For $k \geq 0$, we denote by Σ_k the class of meromorphic functions defined on $\dot{U}$ by form

$$f(z) = \frac{1}{z} + \sum_{n=k}^{\infty} a_n z^n.$$

Definition 3.2.1 *If $0 \leq \alpha < 1$, $k \in \mathbb{Z}_+$ and $m \in \mathbb{N}$, let $\Sigma_k(\alpha, m)$ denote the class of function $f \in \Sigma_k$ which satisfy the inequality*

$$\operatorname{Re}\left[I^m(z^2 f(z))\right]' > \alpha, \ z \in \dot{U}. \tag{3.2.1}$$

Theorem 3.2.2 *[6] If $0 \leq \alpha < 1$, $k \in \mathbb{Z}_+$ and $m \in \mathbb{N}$ then*

$$\Sigma_k(\alpha, m) \subset \Sigma_k(\delta, m+1), \tag{3.2.2}$$

where

$$\delta = \delta(\alpha, m) = 2\alpha - 1 + 2(1-\alpha)\frac{1}{k+1}\beta\left(\frac{1}{k+1}\right)$$

and

$$\beta(x) = \int_0^z \frac{t^{x-1}}{1+t} dt.$$

Theorem 3.2.3 *[6] Let q be a convex function, $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + z(k+1)q'(z), \;\; z \in U.$$

If $f \in \Sigma_k(\alpha, m)$ and verifies the differential subordination

$$\left[I^m(z^2 f(z))\right]' \prec h(z), \;\; z \in \dot{U} \tag{3.2.3}$$

then

$$\left[I^{m+1}(z^2 f(z))\right]' \prec q(z), \;\; z \in \dot{U}$$

and this result is sharp.

Theorem 3.2.4 *[6] Let q be a convex function with $q(0) = 1$ and*

$$h(z) = q(z) + z(k+1)q'(z), \;\; z \in U.$$

If $f \in \Sigma_k(\alpha, n)$ and verifies the differential subordination

$$\left[I^m(z^2 f(z))\right]' \prec h(z), \;\; z \in \dot{U} \tag{3.2.4}$$

then

$$\frac{I^m(z^2 f(z))}{z} \prec q(z), \;\; z \in \dot{U}$$

and this result is sharp.

Theorem 3.2.5 *[6] Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, and $h'(0) \neq 0$ which verifies the inequality*

$$\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \;\; z \in U.$$

If $f \in \Sigma_k(\alpha, m)$ and verifies the differential subordination

$$\left[I^m(z^2 f(z))\right]' \prec h(z), \;\; z \in \dot{U} \tag{3.2.5}$$

then

$$\left[I^{m+1}(z^2 f(z)\right]' \prec q(z), \;\; z \in \dot{U}$$

where

$$q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t) t^{\frac{1}{k+1}-1} dt, \;\; z \in U. \tag{3.2.6}$$

The function q is convex and it is the best $(1, k+1)$ dominant.

Theorem 3.2.6 *[6] Let $h \in \mathcal{H}(U)$ with $h(0) = 1$, $h'(0) \neq 0$, which verifies the inequality*

$$\operatorname{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \ z \in U.$$

If $f \in \Sigma_k(\alpha, m)$ and verifies the differential subordination

$$\left[I^m(z^2 f(z))\right]' \prec h(z), \ z \in \dot{U} \tag{3.2.7}$$

then

$$\frac{I^m(z^2 f(z))}{z} \prec q(z), \ z \in \dot{U}$$

where

$$q(z) = \frac{1}{(k+1)z^{\frac{1}{k+1}}} \int_0^z h(t)t^{\frac{1}{k+1}-1}dt, \ z \in U.$$

The function q is convex and is the best $(1, k+1)$ dominant.

3.3 A class of starlike functions of order α

Definition 3.3.1 *Let $0 \leq \alpha < 1$ and $f \in \mathcal{A}_n$ such that*

$$\frac{f(z)f'(z)}{z} \neq 0, \ 1 + \frac{zf'(z)}{f(z)} \neq 0, \ z \in U.$$

We say that the function f is in the class M_β^n, $\beta \in \mathbb{R}$, if the function $F : U \to \mathbb{C}$ given by

$$g(z) = f(z)\left[\frac{1}{2}\frac{zf'(z)}{f(z)}\left(1 + \frac{zf'(z)}{f(z)}\right)\right]^\beta$$

is starlike of order α.

Remark 3.3.2 *If $\beta = \dfrac{1}{2}$ in Definition 3.3.1 we obtain the class $M^n(\alpha)$(see [84]).*

Remark 3.3.3 *If $\beta = 0$ then $g(z) = f(z)$ and $M_\beta^n(\alpha) = S^*$.*

Theorem 3.3.4 *[13] For each $\alpha, \beta \in \mathbb{R}$ with $\alpha \in [0, 1)$ and $\beta > 0$ we have the following inclusion*

$$M_\beta^n(\alpha) \subset S^*(\alpha).$$

For the particular case of $\beta = \dfrac{1}{2}$ we obtain the next corollary.

Corollary 3.3.5 *[84] For each real number $0 \leq \alpha < 1$ and we have the following inclusion*

$$M^n(\alpha) \subset S^*(\alpha).$$

3.4 Differential superordinations defined by integral operator

The results of this paragraph are obtained with the differential superordinations method.

Theorem 3.4.1 *[9] Let $h \in \mathcal{H}(U)$ be a convex function in U, with $h(0) = 1$ and $f \in \mathcal{A}_n$, $n \in \mathbb{N}^*$.*

Assume that $[I^m f(z)]'$ is univalent with $[I^{m+1} f(z)] \in \mathcal{H}[1, n] \cap \mathcal{Q}$.

If

$$h(z) \prec [I^m f(z)]', \ z \in U \tag{3.4.1}$$

then

$$q(z) \prec [I^{m+1} f(z)]', \ z \in U \tag{3.4.2}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is convex and is the best subordinant.

Theorem 3.4.2 *[9] Let $h \in \mathcal{H}(U)$ be a convex function in U, with $h(0) = 1$ and $f \in \mathcal{A}_n$.*

Assume that $[I^m f(z)]'$ is univalent with $\frac{I^m f(z)}{z} \in \mathcal{H}[1, n] \cap \mathcal{Q}$.

If

$$h(z) \prec [I^m f(z)]', \ z \in U \tag{3.4.3}$$

then

$$q(z) \prec \frac{I^m f(z)}{z}, \ z \in U \tag{3.4.4}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is convex and is the best subordinant.

Theorem 3.4.3 *[9] Let q be a convex function in U and h defined by*

$$h(z) = q(z) + zq'(z), \ z \in U.$$

If $f \in \mathcal{A}_n$, $[I^{m+1}]'$ is univalent in U, $[I^{m+1} f(z)]' \in \mathcal{H}[1, n] \cap \mathcal{Q}$ and

$$h(z) \prec [I^{m+1} f(z)]' \tag{3.4.5}$$

then

$$q(z) \prec [I^{m+1} f(z)]' \tag{3.4.6}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is the best subordinant.

Theorem 3.4.4 *[9] Let q a convex function in U and h defined by*

$$h(z) = q(z) + zq'(z).$$

If $f \in \mathcal{A}_n$, $[I^m f(z)]'$ is univalent in U, $\frac{I^m f(z)}{z} \in \mathcal{H}[1, n] \cap \mathcal{Q}$ and

$$h(z) \prec [I^m f(z)]' \tag{3.4.7}$$

then

$$q(z) \prec \frac{I^m f(z)}{z} \tag{3.4.8}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is convex and is the best subordinant.

Remark 3.4.5 *In the particular case $n = 1$ this theorems are included in the thesis as corollaries.*

3.5 Subordinants of some differential superordinations

In this section are set some differential superordinations using the integral operator by type Sălăgean and concrete examples of convex functions.

Theorem 3.5.1 *[8] Let $R \in (0, 1]$ and let h be convex in U, defined by*

$$h(z) = 1 + Rz + \frac{Rz}{2 + Rz} \tag{3.5.1}$$

with $h(0) = 1$.

Let $f \in \mathcal{A}_n$ and suppose that $[I^m f(z)]'$ is univalent and $[I^{m+1} f(z)]' \in \mathcal{H}[1, n] \cap \mathcal{Q}$.
If

$$h(z) \prec [I^m f(z)]', \ z \in U \tag{3.5.2}$$

then

$$q(z) \prec [I^{m+1} f(z)]', \ z \in U, \tag{3.5.3}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \left(1 + Rt + \frac{Rt}{2 + Rt}\right) t^{\frac{1}{n}-1} dt, \tag{3.5.4}$$

$$q(z) = 1 + \frac{Rz}{n + 1} + R\frac{1}{n} M(z) \frac{1}{z^{\frac{1}{n}}}$$

and

$$M(z) = \int_0^z \frac{t^{\frac{1}{n}}}{2 + Rt} dt.$$

The function q is convex and is the best subordinant.

If $n = 1$, from Theorem 3.5.1 we obtain the next corollary.

Corollary 3.5.2 *[8] Let $R \in (0, 1]$ and let h be convex in U, defined by*

$$h(z) = 1 + Rz + \frac{Rz}{2 + Rz}$$

with $h(0) = 1$.

Let $f \in \mathcal{A}$ and suppose that $[I^m f(z)]'$ is univalent and $[I^{m+1} f(z)]' \in \mathcal{H}[1, 1] \cap \mathcal{Q}$. If

$$h(z) \prec [I^m f(z)]', \ z \in U$$

then

$$q(z) \prec [I^{m+1} f(z)]', \ z \in U,$$

where

$$q(z) = \frac{1}{z} \int_0^z \left(1 + Rt + \frac{Rt}{2 + Rt}\right) dt,$$

$$q(z) = 1 + \frac{Rz}{2} + RM(z)\frac{1}{z}$$

and

$$M(z) = \frac{z}{R} - \frac{2}{R^2} \ln(2 + Rz) + \frac{2}{R} \ln 2, \ z \in U.$$

The function q is convex and is the best subordinant.

If $R = 1$, the Theorem 3.5.1 becomes:

Corollary 3.5.3 *[5] Let h be convex in U, defined by*

$$h(z) = 1 + z + \frac{z}{2 + z} \tag{3.5.5}$$

with $h(0) = 1$.

Let $f \in \mathcal{A}_n$ and suppose that $[I^m f(z)]'$ is univalent and $[I^{m+1} f(z)]' \in \mathcal{H}[1, n] \cap \mathcal{Q}$. If

$$h(z) \prec [I^m f(z)]', \ z \in U \tag{3.5.6}$$

then

$$q(z) \prec [I^{m+1} f(z)]', \ z \in U, \tag{3.5.7}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \left(1 + t + \frac{t}{2 + t}\right) t^{\frac{1}{n} - 1} dt, \tag{3.5.8}$$

$$q(z) = 1 + \frac{z}{n + 1} + \frac{1}{n} M(z) \frac{1}{z^{\frac{1}{n}}}$$

and

$$M(z) = \int_0^z \frac{t^{\frac{1}{n}}}{2 + t} dt.$$

The function q is convex and it is the best subordinant.

For $n = 1$ and $R = 1$ we have the next corollary.

Corollary 3.5.4 *[5] Let h be convex in U, defined by*

$$h(z) = 1 + z + \frac{z}{2+z}$$

with $h(0) = 1$.

Let $f \in \mathcal{A}$ and suppose that $[I^m f(z)]'$ is univalent and $[I^{m+1} f(z)]' \in \mathcal{H}[1,1] \cap \mathcal{Q}$.

If

$$h(z) \prec [I^m f(z)]', \ z \in U$$

then

$$q(z) \prec [I^{m+1} f(z)]', \ z \in U,$$

where

$$q(z) = \frac{1}{z}\int_0^z \left(1 + t + \frac{t}{2+t}\right) dt,$$

$$q(z) = 1 + \frac{z}{2} + M(z)\frac{1}{z}$$

and

$$M(z) = z - 2\ln(2+z) + \ln 2, \ z \in U.$$

The function q is convex and it is the best subordinant.

Theorem 3.5.5 *[8] Let $R \in (0,1]$ and let h be convex in U, defined by*

$$h(z) = 1 + Rz + \frac{Rz}{2+Rz}$$

with $h(0) = 1$. Let $f \in \mathcal{A}_n$ and suppose that $[I^m f(z)]'$ is univalent and $\frac{I^m f(z)}{z} \in \mathcal{H}[1,n] \cap \mathcal{Q}$.

If

$$h(z) \prec [I^m f(z)]', \ z \in U, \tag{3.5.9}$$

then

$$q(z) \prec \frac{I^m f(z)}{z}, \ z \in U, \tag{3.5.10}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z \left(1 + Rt + \frac{Rt}{2+Rt}\right) t^{\frac{1}{n}-1} dt =$$

$$= 1 + \frac{Rz}{n+1} + R\frac{1}{n}M(z)\frac{1}{z^{\frac{1}{n}}}$$

and

$$M(z) = \int_0^z \frac{t^{\frac{1}{n}}}{2+Rt} dt, \ z \in U.$$

The function q is convex and it is the best subordinant.

By customizing for $n = 1$ and $R = 1$ we obtain the next corollaries.

Corollary 3.5.6 *[8] Let $R \in (0, 1]$ and let h be convex in U, defined by*

$$h(z) = 1 + Rz + \frac{Rz}{2 + Rz}$$

with $h(0) = 1$. Let $f \in \mathcal{A}$ and suppose that $[I^m f(z)]'$ is univalent and $\frac{I^m f(z)}{z} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$. If

$$h(z) \prec [I^m f(z)]', \ z \in U,$$

then

$$q(z) \prec \frac{I^m f(z)}{z}, \ z \in U,$$

where

$$q(z) = \frac{1}{z} \int_0^z \left(1 + Rt + \frac{Rt}{2 + Rt}\right) dt =$$
$$= 1 + \frac{Rz}{2} + RM(z)\frac{1}{z}$$

and

$$M(z) = \frac{z}{R} - \frac{2}{R^2} \ln(2 + Rz) + \frac{2}{R} \ln 2, \ z \in U.$$

The function q is convex and it is the best subordinant.

Corollary 3.5.7 *[5] Let h be convex in U, defined by*

$$h(z) = 1 + z + \frac{z}{2 + z}$$

with $h(0) = 1$. Let $f \in \mathcal{A}_n$ and suppose that $[I^m f(z)]'$ is univalent and $\frac{I^m f(z)}{z} \in \mathcal{H}[1, n] \cap \mathcal{Q}$. If

$$h(z) \prec [I^m f(z)]', \ z \in U, \tag{3.5.11}$$

then

$$q(z) \prec \frac{I^m f(z)}{z}, \ z \in U, \tag{3.5.12}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \left(1 + t + \frac{t}{2 + t}\right) t^{\frac{1}{n} - 1} dt =$$
$$= 1 + \frac{z}{n + 1} + \frac{1}{n} M(z) \frac{1}{z^{\frac{1}{n}}}$$

and

$$M(z) = \int_0^z \frac{t^{\frac{1}{n}}}{2 + t} dt, \ z \in U.$$

The function q is convex and it is the best subordinant.

Corollary 3.5.8 *[5] Let h be convex in U, defined by*

$$h(z) = 1 + z + \frac{z}{2+z}$$

with $h(0) = 1$. Let $f \in \mathcal{A}$ and suppose that $[I^m f(z)]'$ is univalent and $\frac{I^m f(z)}{z} \in \mathcal{H}[1,1] \cap \mathcal{Q}$.

If

$$h(z) \prec [I^m f(z)]', \ z \in U,$$

then

$$q(z) \prec \frac{I^m f(z)}{z}, \ z \in U,$$

where

$$q(z) = \frac{1}{z}\int_0^z \Big(1 + t + \frac{t}{2+t}\Big) dt = $$
$$= 1 + \frac{z}{2} + M(z)\frac{1}{z}$$

and

$$M(z) = z - 2\ln(2+z) + 2\ln 2, \ z \in U.$$

The function q is convex and it is the best subordinant.

Remark 3.5.9 *In the case of Sălăgean differential operator, similar results for the function*

$$h(z) = 1 + Rz + \frac{Rz}{2+Rz}$$

were obtained by A. Cătaş in [24].

Theorem 3.5.10 *[7] Let*

$$h(z) = \frac{1 + (2\alpha - 1)z}{1+z}, \ z \in U.$$

be a convex function in U, with $h(0) = 1$.

Let $f \in \mathcal{I}^m(\alpha)$, and suppose that $[I^m f(z)]'$ is univalent and

$$\left[I^{m+1} f(z)\right]' \in \mathcal{H}[1,1] \cap \mathcal{Q}.$$

If

$$h(z) \prec [I^m f(z)]', \ z \in U, \tag{3.5.13}$$

then

$$q(z) \prec [I^{m+1} f(z)]', \ z \in U,$$

where

$$q(z) = 2\alpha - 1 + 2(1-\alpha)\frac{\log(1+z)}{z}. \tag{3.5.14}$$

The function q is convex and it is the best subordinant.

Theorem 3.5.11 *[7] Let*

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}.$$

be convex. Let $f \in \mathcal{I}^m(\alpha)$, *and suppose that* $[I^m f(z)]'$ *is univalent and*

$$\frac{I^m f(z)}{z} \in \mathcal{H}[1, 1] \cap \mathcal{Q}.$$

If

$$h(z) \prec [I^m f(z)]', \ z \in U \tag{3.5.15}$$

then

$$q(z) \prec \frac{I^m f(z)}{z}, \ z \in \dot{U}$$

where

$$q(z) = 2\alpha - 1 + 2(1 - \alpha)\frac{\log(1 + z)}{z}.$$

The function q *is convex and it is the best subordinant.*

Chapter 4

Almost starlikeness of complex order λ

In this chapter is approached the notion by almost starlikeness of complex order λ in $\mathbb{C}^n$ and in $\mathbb{C}$. Are presented characterization theorems for almost starlikeness of complex order λ with the help of Loewner chains, sufficient conditions for almost starlikeness of complex order λ, results of compactness, concrete example and results regarding the preservations of almost starlikeness of complex order λ by generalizing the Roper-Suffridge extension operator, respectively Pfaltzgraff-Suffridge.

The results are original and are contained in the papers [11], [12], [14], [15].

4.1 Almost starlike functions and mappings of complex order λ. Characterization by Loewner chains

Definition 4.1.1 *Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \le 0$. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be almost starlike of complex order λ if*

$$\mathrm{Re}\Big\{(1-\lambda)\langle [Df(z)]^{-1} f(z), z\rangle\Big\} > -\mathrm{Re}\lambda \|z\|^2,\ z \in B^n \setminus \{0\}. \tag{4.1.1}$$

It is easy to see that in the case of one variable, the above relation becomes

$$\mathrm{Re}\left[(1-\lambda)\frac{f(z)}{zf'(z)}\right] > -\mathrm{Re}\lambda,\ z \in U. \tag{4.1.2}$$

We denote by $S^*_\lambda(B^n)$ the set of almost starlike mappings of complex order λ.

Example 4.1.2 *Let $\lambda \in \mathbb{C}$ be such that $\mathrm{Re}\lambda \le 0$, and let $f : U \to \mathbb{C}$ given by $f(z) = z(1 + \frac{1+\lambda}{1-\lambda} z)^{-\frac{2}{1+\lambda}}$, $z \in U$, the branch of the power function is chosen such that*

$$\left(1 + \frac{1+\lambda}{1-\lambda} z\right)^{-\frac{2}{1+\lambda}} |_{z=0} = 1.$$

Then f is almost starlike of complex order λ on U.

Example 4.1.3 *(i) Let f_j be an almost starlike function of complex order λ on the unit disc U for $j = 1, \dots, n$. Then $f : B^n \to \mathbb{C}^n$ given by $f(z) = (f_1(z_1), \dots, f_n(z_n))$ is almost starlike of complex order λ on B^n.*

(ii) In particular, if $f_j(z_j)$ is given by Example 4.1.2 for $j = 1, \dots, n$, then $f(z) = (f_1(z_1), \dots, f_n(z_n))$ is almost starlike of complex order λ on B^n.

The following result provides a necessary and sufficient condition for almost starlikeness of complex order λ on U in terms of Loewner chains.

Theorem 4.1.4 *[14] Let $f : U \to \mathbb{C}$ be a normalized holomorphic function and let $\lambda \in \mathbb{C}$ be such that $\mathrm{Re}\lambda \leq 0$. Then f is almost starlike of complex order λ if and only if*

$$F(z,t) = e^{(1-\lambda)t} f(e^{\lambda t} z),\ z \in U,\ t \geq 0,$$

is a Loewner chain. In particular, f is a starlike function (i. e. $\lambda = 0$) if and only if $F(z,t) = e^t f(z)$ is a Loewner chain.

Remark 4.1.5 *In view of Theorem 4.1.1 we obtain Theorem 2.6.3 in the case of $\lambda = i \tan \alpha$, $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Also, if $\lambda = \alpha/(\alpha - 1)$, where $\alpha \in [0, 1)$, Theorem 4.1.1 reduces to Theorem 2.7.2.*

Corollary 4.1.6 *[14] Let $f(z)$ be an almost starlike function of complex order λ. Then*

$$\frac{|z|}{(1+|z|)^2} \leq |e^{-\lambda t} f(e^{\lambda t} z)| \leq \frac{|z|}{(1-|z|)^2},\ z \in U,\ t \geq 0.$$

In particular, if $t = 0$ then

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2},\ z \in U.$$

Theorem 4.1.7 *[14] Let $f : B^n \to \mathbb{C}^n$ be a normalized holomorphic mapping and let $\lambda \in \mathbb{C}$ be such that $\mathrm{Re}\lambda \leq 0$. Then f is almost starlike mapping of complex order λ if and only if*

$$F(z,t) = e^{(1-\lambda)t} f(e^{\lambda t} z),\ z \in B^n,\ t \geq 0,$$

is a Loewner chain. In particular, f is a starlike mapping (i. e. $\lambda = 0$) if and only if $F(z,t) = e^t f(z)$ is a Loewner chain.

Remark 4.1.8 *In view of Theorem 4.1.7, we obtain Theorem 2.6.4 for $\lambda = i \tan \alpha$ and $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Also, if $\lambda = \alpha/(\alpha - 1)$, where $\alpha \in [0, 1)$, Theorem 4.1.7 reduces to Theorem 2.7.3. Of course, if $\lambda = 0$ in Theorem 4.1.7, we obtain the usual characterization of starlikeness in terms of Loewner chains.*

From Theorem 4.1.7 and the growth result for the class of all Loewner chains $F(z,t)$ such that $\{e^{-t} F(\cdot, t)\}_{t \geq 0}$ is a normal family on B^n (see [45]), we obtain the following corollary.

Corollary 4.1.9 *[14] Let $f : B^n \to \mathbb{C}^n$ be an almost starlike mapping of complex order λ. Then*

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2},\ z \in B^n.$$

Theorem 4.1.10 *[11] The set $S^*_\lambda(B^n)$ is compact.*

4.2 Applications with generalizations of the Roper-Suffridge extension operator

In this section we obtain various results related to the preservation of the notion of almost starlikeness of complex order λ by the generalized Roper-Suffridge extension operators. These results also provide concrete examples of almost starlike mappings of complex order λ on the unit ball in $\mathbb{C}^n$.

Theorem 4.2.1 *[14] Assume that f is an almost starlike function of complex order λ on U and $F_\gamma(z) = \Phi_{n,\gamma}(f)(z)$ is defined as in Theorem 2.8.1. Then F_γ is an almost starlike mapping of complex order λ on B^n.*

Theorem 4.2.2 *[14] Let f be an almost starlike function of complex order λ on U, and let $F_\beta(z) = \Phi_{n,\beta}(f)(z)$ be given as in Theorem 2.8.2. Then F_β is an almost starlike mapping of complex order λ on B^n.*

Theorem 4.2.3 *[14] Assume that f is an almost starlike function of complex order λ on U, and $F_{\beta,\gamma}(z) = \Phi_{n,\beta,\gamma}(f)(z)$ is defined as in Theorem 2.8.3. Then $F_{\beta,\gamma}$ is an almost starlike mapping of complex order λ on B^n.*

We present a different method for preservation of almost starlikeness of complex order λ by generalizations of Roper-Suffridge extension operator. For this we present two classical results.

Lemma 4.2.4 *[30] Let f be a holomorphic function on the unit disc U. Then $\mathrm{Re} f(z) \geq 0$, $\forall z \in U$, if and only if there exists an increasing function, μ, on $[0, 2\pi]$, which satisfies $\mu(2\pi) - \mu(0) = \mathrm{Re} f(0)$, such that*

$$f(z) = \int_0^2 \pi \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} d\mu(\theta) + i\mathrm{Im} f(0), \ z \in U.$$

Lemma 4.2.5 *[68],[106] Suppose $w \in \mathbb{C}$, then we have*

1. $\mathrm{Re}(1 - w^2)(1 - \overline{w})^2 = (1 - |w|^2)|1 - w|^2$;
2. $\mathrm{Re}(1 + 2w - w^2)(1 - \overline{w})^2 = (1 - |w|^2)^2 - 2|w|^2|1 - w|^2$.

Theorem 4.2.6 *[15] If f is an almost starlike function of complex order λ on the unit disc U, then*

$$F(z) = \Phi_{n,\beta,\gamma}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1} \right)^\beta (f'(z_1))^\gamma z' \right)$$

is an almost starlike mapping of complex order λ on B^n, where $z = (z_1, z') \in B^n$, $z_1 \in U$, $z' = (z_2, \cdots, z_n)' \in C^{n-1}$, $\beta \in [0, 1]$, and $\gamma \in \left[0, \frac{1}{2}\right]$ such that $\beta + \gamma \leq 1$, $f(z_1) \neq 0$ when $z_1 \in U \setminus \{0\}$ and $\left(\frac{f(z_1)}{z_1}\right)^\beta$, $(f'(z_1))^\gamma$ satisfy $\left(\frac{f(z_1)}{z_1}\right)^\beta |_{z_1=0} = 1$, respectively $(f'(z_1))^\gamma |_{z_1=0} = 1$.

If $\lambda = 0$, then the Theorem 4.2.6 is the result of starlike mappings.

Corollary 4.2.7 *[41] If f is a starlike function on the unit disc U, then*

$$F(z) = \Phi_{n,\beta,\gamma}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1} \right)^{\beta} (f'(z_1))^{\gamma} z' \right)$$

is a starlike mapping on B^n, where $z = (z_1, z') \in B^n$, $z_1 \in U$, $z' = (z_2, \cdots, z_n)' \in C^{n-1}$, $\beta \in [0,1]$, and $\gamma \in \left[0, \frac{1}{2}\right]$ such that $\beta + \gamma \le 1$, $f(z_1) \neq 0$ when $z_1 \in U \setminus \{0\}$ and $\left(\frac{f(z_1)}{z_1}\right)^{\beta}$, $(f'(z_1))^{\gamma}$ satisfy $\left(\frac{f(z_1)}{z_1}\right)^{\beta} |_{z_1=0}= 1$, respectively $(f'(z_1))^{\gamma} |_{z_1=0}= 1$.

For $\beta = 0$, we obtain the next corollary.

Corollary 4.2.8 *[15] If f is an almost starlike function of complex order λ, $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda \le 0$ on the unit disc U, then*

$$F(z) = \Phi_{n,\gamma}(f)(z) = (f(z_1), (f'(z_1))^{\gamma} z')$$

is an almost starlike mapping of complex order λ on B^n, where $\gamma \in \left[0, \frac{1}{2}\right]$, and $(f'(z_1))^{\gamma}$ satisfies $(f'(z_1))^{\gamma} |_{z_1=0}= 1$.

If $\gamma = 0$, from Theorem 4.2.6 we obtain the next corollary.

Corollary 4.2.9 *[15] If f is an almost starlike function of complex order λ, $\mathrm{Re}\lambda \le 0$, $\lambda \in \mathbb{C}$ on the unit disc U, then*

$$F(z) = \Phi_{n,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1} \right)^{\beta} z' \right)$$

is an almost starlike mapping of complex order λ on B^n, $\beta \in [0,1]$, $\left(\frac{f(z_1)}{z_1}\right)^{\beta} |_{z_1=0}= 1$.

4.3 Almost starlikeness of complex order λ and the Pfaltzgraff-Suffridge extension operator

In the next theorem is proved that the Pfaltzgraff-Suffridge extension operator preserve the almost starlikeness of complex order λ.

Theorem 4.3.1 *[11] Assume f is an almost starlike mapping of complex order λ on B^n. Then $F = \Psi_n(f)$ is also an almost starlike mapping of complex order λ on B^{n+1}.*

In particular, if $\lambda = i \tan \alpha$ in Theorem 4.3.1, we obtain the following result.

Corollary 4.3.2 *[11] Assume f is a spirallike mapping of type α on B^n. Then $F = \Psi_n(f)$ is a spirallike mapping of type α on B^{n+1}.*

Next, if we consider $\lambda = \dfrac{\alpha}{\alpha - 1}$ in the proof of Theorem 4.3.1, we obtain the following particular case.

Corollary 4.3.3 *[11] Assume f is an almost starlike mapping of order α on B^n. Then $F = \Psi_n(f)$ is an almost starlike mapping of order α on B^{n+1}.*

Another consequence of Theorem 4.3.1 is given in the following result due to Graham, Kohr and Pfaltzgraff [47]. This result provides a positive answer to the question of Pfaltzgraff and Suffridge regarding the preservation of starlikeness under the operator Φ_n.

Corollary 4.3.4 *Assume that $f \in S^*(B^n)$. Then $F = \Psi_n(f) \in S^*(B^{n+1})$.*

Example 4.3.5 *Let $f_j,\ j = 1, \ldots, n$ be almost starlike functions of complex order λ. It is not difficult to deduce that the mapping $f : B^n \to \mathbb{C}^n$ given by $f(z') = (f_1(z_1), \ldots, f_n(z_n))$ is almost starlike mapping of complex order λ on B^n. By Theorem 4.3.1, $F : B^{n+1} \to \mathbb{C}^{n+1}$ given by*

$$F(z) = \Big(f_1(z_1), \ldots, f_n(z_n), z_{n+1} \prod_{j=1}^{n} [f_j'(z_j)]^{\frac{1}{n+1}}\Big),\ z = (z', z_{n+1}) \in B^n$$

is almost starlike of complex order λ on B^{n+1}. For example, the mapping

$$F(z) = \Big(z_1\Big(1 + \frac{1+\lambda}{1-\lambda} z_1\Big)^{-\frac{2}{1+\lambda}}, \ldots, z_n\Big(1 + \frac{1+\lambda}{1-\lambda} z_n\Big)^{-\frac{2}{1+\lambda}},$$
$$z_{n+1} \prod_{j=1}^{n} \Big[(1 - z_j)\Big(1 + \frac{1+\lambda}{1-\lambda} z_j\Big)\Big]^{-\frac{3+\lambda}{(1+n)(1+\lambda)}}\Big)$$

is almost starlike of complex order λ on B^{n+1}.

Theorem 4.3.6 *[11] The set $\Psi_n[S^*_\lambda(B^n)]$ is compact.*

4.4 Sufficient conditions for almost starlikeness of complex order λ

In this paragraph are presented sufficient conditions of almost stalikeness of complex order λ in the unit disk U and in the Euclidean unit ball B^n, conditions obtained with Lemma 1.6.2 (Jack-Miller and Mocanu) also by the similar result for n-dimensional case give by P. Liczberski in the case of Euclidean unit ball [62], [67].

Lemma 4.4.1 *Let $f \in H(B^n)$ with $f(0) = 0$. If*

$$\|f(z_0)\| = \max\{\|f(z)\| : \|z\| \le \|z_0\|\},\ z_0 \in B^n,$$

then there are some real numbers m, s, $s \ge m \ge 1$, such that the following relations holds

$$\langle Df(z_0)(z_0), z_0\rangle = m\|f(z_0)\|^2, \tag{4.4.1}$$

$$\|Df(z_0)(z_0)\| = s\|f(z_0)\| \tag{4.4.2}$$

and

$$\mathrm{Re}\langle D^2 f(z_0)(z_0, z_0), z_0\rangle \ge m(m-1)\|f(z_0)\|^2. \tag{4.4.3}$$

Moreover, for $n > 1$, $m = s$ if and only if

$$Df(z_0)(z_0) = mf(z_0).$$

Theorem 4.4.2 *[12] Let $f : U \to \mathbb{C}$ normalized holomorphic function and $\lambda \in \mathbb{C}$ with* $\mathrm{Re}\lambda \leq 0$*, such that*

$$\left|\frac{f''(z)}{f'(z)}\right| < \frac{1}{1+|1-\lambda|},\ z \in U.$$

Then f is an almost starlike function of complex order λ.

In the case $\lambda = 0$, we obtain the next sufficient condition of starlikeness in the unit disk U.

Corollary 4.4.3 *Let $f : U \to \mathbb{C}$ a holomorphic function, with $f(0) = f'(0) - 1 = 0$, such that*

$$\left|\frac{f''(z)}{f'(z)}\right| < \frac{1}{2},\ z \in U.$$

Then f is a starlike function.

The n-dimensional version of Theorem 4.4.2 is given by the next result.

Theorem 4.4.4 *[12] Let $f : B^n \to \mathbb{C}^n$ normalized locally biholomorphic mapping and $\lambda \in \mathbb{C}$ with* $\mathrm{Re}\lambda \leq 0$*, such that*

$$\|[Df(z)(z)]^{-1}D^2f(z)(z,\cdot)\| < \frac{1}{1+|1-\lambda|},\ z \in B^n.$$

Then f is an almost starlike mapping of complex order λ.

For particular values of λ from Theorem 4.4.4 are obtained some interesting results. If $\lambda = 0$ from Theorem 3.5.5 we obtain the next corollary.

Corollary 4.4.5 *Let $f : B^n \to \mathbb{C}^n$ normalized locally biholomorphic mapping such that*

$$\|[Df(z)(z)]^{-1}D^2f(z)(z,\cdot)\| < \frac{1}{2},\ z \in B^n.$$

Then f is a starlike mapping.

In the particular case $\lambda = -1$ we obtain the following result for almost starlikeness of order $1/2$.

Corollary 4.4.6 *Let $f : B^n \to \mathbb{C}^n$ normalized locally biholomorphic mapping such that*

$$\|[Df(z)(z)]^{-1}D^2f(z)(z,\cdot)\| < \frac{1}{3},\ z \in B^n.$$

Then f is a starlike mapping of order $1/2$.

For $\lambda = i\tan\alpha$ we obtain a sufficient conditions for spirallikeness of type α, $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Corollary 4.4.7 *Let $f : B^n \to \mathbb{C}^n$ normalized locally biholomorphic mapping such that*

$$\|[Df(z)(z)]^{-1}D^2f(z)(z,\cdot)\| < \frac{1}{2},\ z \in B^n.$$

Then f is a spirallike mapping of type α, $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

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