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The homogeneous shifts

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Abstract

A bounded linear operator T on a complex Hilbert space is called *homogeneous* if the spectrum of T is contained in the closed unit disc and all bi-holomorphic automorphisms of this disc lift to automorphisms of the operator modulo unitary equivalence. We prove that all the irreducible homogeneous operators are block shifts. Therefore, as a first step in classifying all of them, it is natural to begin with the homogeneous scalar shifts.

In this paper we determine all the homogeneous (scalar) weighted shifts. They consist of the unweighted bilateral shift, two one-parameter families of unilateral shifts (adjoints of each other), a one-parameter family of bilateral shifts and a two-parameter family of bilateral shifts. This classification is obtained by a careful analysis of the possibilities for the projective representation of the Möbius group associated with an irreducible homogeneous shift.

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1. Introduction

All Hilbert spaces in this paper are separable Hilbert spaces over the field of complex numbers. All operators discussed here are bounded linear operators on Hilbert spaces. The set of all unitary operators on a Hilbert space $\mathscr{H}$ will be denoted by $\mathscr{U}(\mathscr{H})$. When equipped with any of the usual operator topologies $\mathscr{U}(\mathscr{H})$ becomes a topological group. All these topologies induce the same Borel structure on $\mathscr{U}(\mathscr{H})$. We shall view $\mathscr{U}(\mathscr{H})$ as a Borel group with this structure.

$\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ will denote the set of all integers, non-negative integers and non-positive integers, respectively. $\mathbb{R}$ and $\mathbb{C}$ will denote the real and complex numbers. $\mathbb{D}$ and $\mathbb{T}$

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will denote the open unit disc and the unit circle in $\mathbb{C}$, and $\bar{\mathbb{D}}$ will denote the closure of $\mathbb{D}$ in $\mathbb{C}$. Möb will denote the Möbius group of all biholomorphic automorphisms of $\mathbb{D}$. Recall that Möb $= \{\varphi_{\alpha,\beta} : \alpha \in \mathbb{T}, \beta \in \mathbb{D}\}$, where

$$\varphi_{\alpha,\beta}(z) = \alpha \frac{z-\beta}{1-\bar{\beta}z}, \quad z \in \mathbb{D}. \tag{1.1}$$

Möb is topologised via its obvious identification with $\mathbb{T} \times \mathbb{D}$. With this topology, Möb becomes a topological group. Abstractly, it is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$ and to $\mathrm{PSU}(1,1)$.

Let us recall the following definition from [4]:

Definition 1.1. An operator T is called homogeneous if $\varphi(T)$ is unitarily equivalent to T for all φ in Möb which are analytic on the spectrum of T.

It was shown in [4, Lemma 2.2] that the spectrum of such an operator is either $\mathbb{T}$ or $\bar{\mathbb{D}}$, so that $\varphi(T)$ actually makes sense (and is unitarily equivalent to T) for all elements φ of Möb.

Many examples of homogeneous operators are already known. See, for instance, Refs. [4,5,10,20]. Also, there are interesting generalisations of the notion of homogeneity to commuting d-tuples of Hilbert space operators—modelled on bounded symmetric domains in $\mathbb{C}^d$. For preliminary works on these generalisations, see [2,3,11]. In [3] it was shown how usually difficult questions such as boundedness and subnormality become tractable in the presence of homogeneity. We have already remarked how the spectrum calculation of a single homogeneous operator becomes a triviality. This phenomenon becomes even more striking for tuples—where, as is well known, the explicit calculation of the Taylor spectrum is usually an impossibly difficult task. In [3] there are painless computations of the Taylor spectrum of various families of homogeneous operator tuples. Another possible direction for generalisation is to replace the group Möb by interesting subgroups, such as the Fuchsian groups. A beginning in this direction has been made in [21].

A further reason for our interest in homogeneity is its links with the theory of (projective unitary) representations. To any irreducible homogeneous operator is associated a representation of the Möbius group (see Definition 2.1 and Theorem 2.2). The proof of Theorem 2.2 presented here easily generalises to tuples. In this paper we exploit this relationship in order to classify homogeneous shifts. So far, this relationship is one way: using the well-understood representation theory of the Möbius group to answer operator theoretic questions. But it is expected that when this investigation moves over to tuples, the operator theory will begin to have significant impact on the representation theory. Another area where homogeneity may be of significance is the structure theory of invariant subspaces. If T is a homogeneous operator with associated representation π, then the group Möb acts on the lattice of invariant subspaces of T via π. This fact remains to be exploited. But it may be noted that both the unweighted unilateral shift as well as the Bergman shift are instances of homogeneous operators. While Beurling's theorem makes the structure of the lattice quite transparent in the first case, the results in [1,8] and indicate that the lattice for the Bergman shift also has many interesting properties in common with the unweighted shift. It may be expected that these results will generalise to the continuum $M^{(\lambda)},\ \lambda > 0$, of homogeneous shifts described in List 4.1 (the unweighted and Bergman shifts are the cases $\lambda = 1, 2$). Yet another instance of the interplay of homogeneity with other parts of Mathematics comes via the theory of characteristic functions. It can be shown that the Nagy–Foias characteristic function θ_T (cf. [13]) of any irreducible homogeneous contraction T is an (operator valued) analytic function on the unit disc which may be factorised in terms of two projective representations π_1, π_2 of Möb (yet another link with representations!): $\theta_T(z) = \pi_1(\varphi_z)^* C \pi_2(\varphi_z),\ z \in \mathbb{D}$. (Here $C := \theta_T(0)$ and φ_z is the unique involution in Möb which interchanges 0 and z.) In [5], this product formula is explicitly found for the homogeneous contractions $M^{(\lambda)},\ \lambda > 1$, thereby obtaining interesting explicit examples of operator valued inner functions.

Clearly, homogeneity is a unitary invariant. That is, if T_1 and T_2 are unitarily equivalent operators, then T_1 is homogeneous iff T_2 is. This raises the question of classifying homogeneous operators up to unitary equivalence. This paper marks the beginning of this classification programme. In the concluding section of this paper, it is shown (Theorem 5.1) that all irreducible homogeneous operators are block shifts (cf. Definition 2.2). So it is natural to begin the classification with the scalar weighted shifts (which are just the block shifts with one-dimensional blocks). This is what we do in this paper. For general information regarding scalar weighted shifts, the reader may consult the excellent survey article [17].

This paper is organised as follows. In Section 2, we prove that the only reducible homogeneous shift (with non-zero weights) is the unweighted bilateral shift (Theorem 2.1). So, to fulfil the objective of this paper, we need only classify the irreducible homogeneous shifts. To do so, we prove that with any irreducible homogeneous operator is associated an essentially unique projective unitary representation of Möb (Theorem 2.2). We also recall a general construction of homogeneous operators with given associated representations (Theorem 2.3). Finally, we present proofs (due to M. Ordower and V. Pati) of the uniqueness of the blocks of an irreducible block shift (Lemma 2.2).

Section 3 begins by recalling the definition of projective representations, their multipliers and the notion of equivalence for these objects. We point out in Theorem 3.1 that all the projective (unitary) representations of a connected semi-simple Lie group G are direct integrals of its irreducible projective representations. Also, the multipliers of G (modulo equivalence) are in a natural bijection with the characters of the fundamental group of G (Corollary 3.1). As a consequence of the proof of Theorem 3.1 (see Remark 3.1) the irreducible projective representations of such a group are obtainable as push-downs under the covering map of the ordinary irreducible representations of its universal covering group. Therefore, a complete list of the irreducible projective representations of Möb may easily be manufactured out of the known list (as obtainable from [16], for instance) of the irreducible representations of the universal cover of Möb. This is presented in List 3.1. We have parametrised the list in a convenient fashion in order to get a uniform description. This uniform description of the irreducible unitary projective representations of Möb will greatly simplify the proof of the main theorem presented in the final section. At this point List 3.1 is used to determine all the multipliers (modulo equivalence) of Möb (Theorem 3.2) and to determine (Corollary 3.2) when two representations in List 3.1 have identical multipliers. This is important for us since the direct integral of a family of irreducible projective representations defines a projective representation only if the members of this family have identical multipliers. Next, we define simple (projective unitary) representations (Definition 3.5). Theorem 3.1 and Corollary 3.2 allow us to classify the simple representations of Möb in Theorem 3.3. Experts in representation theory may find that most of Section 3 (with the exception of the material in Section 3.4) is "well known". However, the functional analyst reader (to whom this paper is addressed) may find it convenient to have this representation theoretic background available in a compact form. This is all the more important since we were unable to find a suitable reference for this material in the exact form in which we need it.

In Section 4, we list the homogeneous scalar shifts with non-zero weights (List 4.1), along with their associated representations. Excepting for the operators in the Constant Characteristic family (cf. [4,6]), all these examples arise from the construction in Theorem 2.3. Though many of these examples were previously known, the two-parameter family of bilateral homogeneous shifts (dubbed the Complementary series examples) appears to be new.

In Section 5 we show that, as a consequence of Theorem 2.2, all irreducible homogeneous operators are block shifts (Theorem 5.1). Indeed, if T is an irreducible homogeneous operator with associated representation π (say) then the blocks of T are precisely the non-trivial $\mathbb{K}$-isotypic subspaces of the representation space of π. (Here $\mathbb{K}$ is the maximal compact subgroup of Möb.) This theorem acquires substance from the fact (Lemma 2.2) that the blocks of an irreducible block shift are uniquely determined by the operator. As a consequence of Theorems 5.1 and 3.3, it follows that (Lemma 5.1) the projective representation associated with an irreducible scalar weighted shift must be one of the representations from List 3.1. (With the exception of the Principal series representation $P_{1,0}$, all the representations in this list are irreducible.) Finally, we find out all the homogeneous operators associated with the representations in this list. In conjunction with Theorem 2.1 this proves (Theorem 5.2) that the homogeneous scalar shifts (with non-zero weights) are precisely the ones in List 4.1.

From the proof of Theorem 5.2, one sees that there is a natural correspondence ('associated') between the representations in List 3.1 and the operators in List 4.1. This correspondence fails to be a bijection (even a mapping) in several ways: each complementary series representation is associated with two operators (adjoint inverse of each other), all the principal series representations are associated with one and only one operator (namely the unweighted bilateral shift) and all the operators in the Constant characteristic series are associated with one and only one representation, namely $P_{1,0}$. However, in view of the uniqueness statement in Theorem 2.2, we can now turn around and define the Discrete (respectively

Complementary) series representations of Möb as the representations associated with the Discrete series (respectively Complementary series) operators in List 4.1.

2. Homogeneous operators and weighted shifts

2.1. Generalities

Let $*$ denote the involution (i.e., automorphism of order two) of Möb defined by

$$\varphi^*(z) = \overline{\varphi(\bar{z})}, \quad z \in \mathbb{D}, \quad \varphi \in \text{ Möb}. \tag{2.1}$$

Thus $\varphi^*_{\alpha,\beta} = \varphi_{\bar{\alpha},\bar{\beta}}$ for $(\alpha, \beta) \in \mathbb{T} \times \mathbb{D}$. It is known that essentially (i.e., up to multiplication by arbitrary inner automorphisms), $*$ is the only outer automorphism of Möb. It also satisfies $\varphi^*(z) = \varphi(z^{-1})^{-1}$ for $z \in \mathbb{T}$. It follows that for any operator T whose spectrum is contained in $\bar{\mathbb{D}}$, we have

$$\varphi(T^*) = \varphi^*(T)^*, \quad \varphi(T^{-1}) = \varphi^*(T)^{-1} \tag{2.2}$$

the latter in case T is invertible, of course. It follows immediately from (2.2) that the adjoint T^*—as well as the inverse T^{-1} in case T is invertible—of a homogeneous operator T is again homogeneous.

Clearly, a direct sum (more generally, direct integral) of homogeneous operators is again homogeneous.

Let I stand for either $\mathbb{Z}$, $\mathbb{Z}^+$ or $\mathbb{Z}^-$. Recall that an operator T on the Hilbert space $\mathscr{H}$ is called a weighted shift with weight sequence w_n, $n \in I$, if there is a distinguished orthonormal basis x_n, $n \in I$, such that $Tx_n = w_n x_{n+1}$ for all $n \in I$. T is called a bilateral shift, forward unilateral shift or backward unilateral shift according as $I = \mathbb{Z}, \mathbb{Z}^+$ or $\mathbb{Z}^-$. To avoid trivialities, we shall assume throughout that all the weights w_n are non-zero. Every weighted shift (with non-zero weights) is unitarily equivalent to a weighted shift whose weights are strictly positive. The unweighted unilateral (respectively bilateral) shift is the unilateral (respectively bilateral) weighted shift all whose weights are equal to 1.

2.2. The reducible case

As already stated, the object of this paper is to classify the homogeneous shifts up to unitary equivalence. We first dispose off the case of reducible homogeneous shifts. To do so, we need:

Lemma 2.1. *If T is a homogeneous operator such that T^k is unitary for some positive integer k then T is unitary.*

Proof. Let $\varphi \in$ Möb. Since $\varphi(T)$ is unitarily equivalent to T, it follows that $(\varphi(T))^k$ is unitarily equivalent to T^k and hence is unitary. In particular, taking $\varphi = \varphi_{1,\beta}$ (for a fixed but arbitrary $\beta \in \mathbb{D}$), we find that the inverse and the adjoint of $(T - \beta I)^k (I - \bar{\beta} T)^{-k}$ are equal. That is, $(T - \beta I)^{-k}(I - \bar{\beta}T)^k = (T^* - \bar{\beta}I)^k (I - \beta T^*)^{-k}$. Applying $\frac{\partial}{\partial \beta}$ to this equation and evaluating at $\beta = 0$, we get $T^{-(k+1)} = T^{*(k+1)}$. Cancelling the common factor $T^{-k} = T^{*k}$, we have $T^{-1} = T^*$. Thus T is unitary. $\square$

Theorem 2.1. *Up to unitary equivalence, the only reducible homogeneous weighted shift* (*with non-zero weights*) *is the unweighted bilateral shift* B.

Proof. We shall see in Section 4 that B is homogeneous. Being a non-trivial unitary, it is of course reducible. For the converse, let T be a reducible weighted shift with non-zero weights. Recall that by a theorem of R.L. Kelly and N.K. Nikolskii, any such operator T is a bilateral shift, and its weight sequence $w_n, n \in \mathbb{Z}$, is periodic, say with period $k \geqslant 1$. That is, $w_{n+k} = w_n$ for all n (see [14] as well as [7, Problem 129]). It follows that $T^k = cB^k$ where the scalar c is given by $c = w_1 \cdots w_k$. Without loss of generality (replacing T by a unitarily equivalent copy if necessary), we may assume $w_n > 0$ for all n in $\mathbb{Z}$. Thus $c > 0$. Since T is homogeneous, its spectral radius equals 1 as a consequence of Lemma 2.2 in [4]. Therefore, the spectral radius of T^k is $= 1$. Since B^k also has spectral radius $= 1$, it follows that $c = 1$ and therefore $T^k = B^k$. Thus T^k is unitary. Therefore, by Lemma 2.1, T is unitary. Hence $w_n = 1$ for all n. Thus $T = B$. $\square$

2.3. *Associated representations*

In this section we make use of the standard notions of projective unitary representations and their equivalence. However, for the sake of completeness we shall reproduce some of these definitions (along with some relevant results on these topics) in the following section:

Definition 2.1. If T is an operator on a Hilbert space $\mathscr{H}$ then a projective unitary representation π of Möb on $\mathscr{H}$ is said to be associated with T if the spectrum of T is contained in $\bar{\mathbb{D}}$ and

$$\varphi(T) = \pi(\varphi)^* T \pi(\varphi) \tag{2.3}$$

for all elements φ of Möb.

Clearly, if T has an associated representation then T is homogeneous. In the converse direction, we have the following theorem. It will be necessary in order to take care of the irreducible homogeneous shifts. The 'existence' part of this theorem is one of the main results in [11]. We include proofs of both parts for the sake of completeness and because the original existence proof in [11] uses a powerful selection theorem which is avoided here.

Theorem 2.2. *If T is an* irreducible *homogeneous operator then T has a projective unitary representation of* Möb *associated with it. Further, this representation is uniquely determined* (*up to equivalence*) *by T.*

Proof. For any $\varphi \in$ Möb, let E_φ be the set of all unitary operators U such that $\varphi(T) = U^*TU$. Since T is homogeneous, E_φ is non-empty for each φ. Also, for $U_1, U_2 \in E_\varphi$, $U_1 U_2^{-1}$ commutes with T and hence (by the irreducibility of T) is a scalar unitary. Thus, each of the sets E_φ is a coset of the circle group $\mathbb{T}$ in $\mathcal{U}(\mathcal{H})$. Choose a Borel map $\pi :$ Möb$\to \mathcal{U}(\mathcal{H})$ such that $\pi(\varphi) \in E_\varphi$ for all φ. (For instance, this may be done as follows. Fix a countable dense subset $\{f_n\colon n = 1, 2, \ldots\}$ of the Hilbert space $\mathcal{H}$. Let E denote the subset of $\mathcal{U}(\mathcal{H})$ consisting of all unitaries U such that $\langle Uf_1, f_1 \rangle = \cdots \langle Uf_{n-1}, f_{n-1} \rangle = 0$ and $\langle Uf_n, f_n \rangle > 0$ for some $n = 1, 2, \ldots$. Clearly, E is a Borel subset of $\mathcal{U}(\mathcal{H})$ which meets every coset of $\mathbb{T}$ in a singleton. Therefore, we may choose $\pi(\varphi)$ to be the unique element of $E \cap E_\varphi$, for $\varphi \in$ Möb. By Theorem 4.5.2 in [18], if the graph of a map between standard Borel spaces is Borel in the product space, then the map is Borel. Since π defined here satisfies this requirement, it is a Borel map.) For $\varphi_1, \varphi_2 \in$ Möb, $\pi(\varphi_1)\pi(\varphi_2) \in E_{\varphi_1\varphi_2}$ and hence $\pi(\varphi_1\varphi_2)\pi(\varphi_2)^{-1}\pi(\varphi_1)^{-1}$ is a scalar. Thus, π is a projective unitary representation of Möb. By construction, it is associated with T.

Note that if π_1, π_2 are two projective unitary representations associated with the same operator T then for each φ in Möb, $\pi_1(\varphi)\pi_2(\varphi)^{-1}$ commutes with T. If T is irreducible then this implies that $\pi_1(\varphi)\pi_2(\varphi)^{-1}$ is a scalar (necessarily of modulus 1) for all φ. Thus, if T is an irreducible homogeneous operator, then the associated projective unitary representation is unique up to equivalence. □

For any projective unitary representation π of Möb, let $\pi^\#$ denote the projective representation of Möb obtained by composing π with the automorphism $*$ of Möb (cf. (2.1)). That is,

$$\pi^\#(\varphi) := \pi(\varphi^*), \quad \varphi \in \text{Möb}. \tag{2.4}$$

For future use, we note:

Proposition 2.1. *If the projective unitary representation π is associated with a homogeneous operator T then $\pi^\#$ is associated with the adjoint T^* of T. If, further, T is invertible, then $\pi^\#$ is associated with T^{-1} also. It follows that T and T^{*-1} have the same associated representation.*

Proof. This is more or less obvious from (2.2). □

2.4. A construction

Let us say that a projective unitary representation π of Möb is a *multiplier representation* if it is concretely realised as follows. π acts on a Hilbert space $\mathcal{H}$

containing a dense linear subspace $\mathcal{M}$ consisting of functions on $\mathbb{T}$. This subspace $\mathcal{M}$ is invariant under π and the action of π on $\mathcal{M}$ is given by $(\pi(\varphi)f)(z) = c(\varphi, z)f(\varphi^{-1}z)$ for $z \in \mathbb{T}$, $f \in \mathcal{M}$, $\varphi \in$ Möb. Here c is a suitable non-vanishing Borel function on Möb $\times\, \mathbb{T}$.

Theorem 2.3. *Suppose there is a multiplier representation π of* Möb *on $\mathcal{H}$. Let $\mathcal{M}$ be a dense π-invariant subspace as above. Suppose that the operator T given on $\mathcal{M}$ by*

$$(Tf)(x) = xf(x), \quad x \in \mathbb{T},\ f \in \mathcal{M},$$

leaves $\mathcal{M}$ invariant and has a bounded extension to $\mathcal{H}$. Then the extension T is homogeneous and π is associated with T.

Proof. Let U be a sufficiently small neighbourhood of the identity in Möb so that $\varphi(T)$ makes sense for all $\varphi \in U$. According to Bagchi and Misra [4, Lemma 2.2], it suffices to verify that $T\pi(\varphi) = \pi(\varphi)\varphi(T)$ for all $\varphi \in U$. Notice that for $\varphi \in U$, the action of $\varphi(T)$ on $\mathcal{M}$ is just multiplication by φ: $(\varphi(T)f)(x) = \varphi(x)f(x)$. Since $\mathcal{M}$ is a dense subspace, it suffices to verify that for any $x \in \mathbb{T}$, $f \in \mathcal{M}$,

$$xc(\varphi, x)f(\varphi^{-1}(x)) = c(\varphi, x)(y \mapsto \varphi(y)f(y))(\varphi^{-1}(x)).$$

But this is trivial. $\square$

This is easy but basic construction is from Proposition 2.3 of [4]. To apply this theorem, we only need a good supply of what we have called multiplier representations of Möb. Notice that most of the irreducible projective representations of Möb (as concretely presented in List 3.1) are multiplier representations.

2.5. Block shifts

Although this paper is essentially about ordinary weighted shifts, along the way we shall need the following more general notion:

Definition 2.2. Let T be a bounded operator on a Hilbert space $\mathcal{H}$. Then T is called a block shift if there is an orthogonal decomposition $\mathcal{H} = \oplus_{n \in I} W_n$ of $\mathcal{H}$ into non-trivial subspaces W_n, $n \in I$, such that $T(W_n) \subseteq W_{n+1}$ for all n in I. Here $I = \mathbb{Z}, \mathbb{Z}^+$ or $\mathbb{Z}^-$. We say that T is a bilateral, forward unilateral or backward unilateral block shift according as $I = \mathbb{Z}, \mathbb{Z}^+$ or $\mathbb{Z}^-$. The subspaces $W_n, n \in I$, are called the blocks of T. In the case of a backward block shift T, it is understood that $T(W_0) = \{0\}$. Notice that the adjoint of a forward block shift is a backward block shift and vice versa.

Note that the weighted shifts are simply the block shifts all whose blocks are one dimensional. To distinguish them from more general block shifts, they are sometimes called the scalar shifts.

One might imagine that the block shifts are too general a class to be of much significance. Indeed, one might think that most (if not all) operators can be realised as block shifts. Therefore, the following result, showing that block shifts (at least the irreducible ones) have a very rigid structure, comes as a surprise. In the concluding section we shall see that all irreducible homogeneous operators are block shifts.

Lemma 2.2. *If T is an irreducible block shift then the blocks of T are uniquely determined by T.*

Proof (Due to Marc Ordower). Fix an element $\alpha \in \mathbb{T}$ of infinite order (i.e., α is not a root of unity). Let $V_n,\ n \in I$, be blocks of T. Define a unitary operator S by $Sx = \alpha^n x$ for $x \in V_n,\ n \in I$. Notice that by our assumption on α the eigenvalues $\alpha^n,\ n \in I$, of S are distinct and the blocks V_n of T are precisely the eigenspaces of S. If $W_n,\ n \in J$, are also blocks of T then define another unitary S_1 by replacing the blocks V_n by the blocks W_n in the definition of S. A simple computation shows that we have $STS^* = \alpha T = S_1 T S_1^*$ and hence $S_1^* S$ commutes with T. Since $S_1^* S$ is unitary and T is irreducible, it follows from Schur's Lemma that $S_1^* S$ is a scalar. That is, $S_1 = \beta S$ for some $\beta \in \mathbb{T}$. Therefore, S_1 has the same eigenspaces as S. Thus, the blocks of T are uniquely determined as the eigenspaces of S. □

Remark 2.1. After we conjectured (and were unable to prove) the validity of Lemma 2.2, our colleague V. Pati found a proof in the case of unilateral shifts. Finally, Marc Ordower found the beautiful proof (presented above) which works in all cases. In fact, this proof works equally well for commuting tuples of operators.

Though limited in scope, Pati's proof has the advantage of being 'constructive': it gives an explicit description of the blocks of an irreducible unilateral block shift T in terms of the operator (in comparison, Ordower's proof is non-constructive). We present a brief sketch of this proof.

Let T be an irreducible forward block shift. (To get the proof for the backward case, just apply the following to the adjoint.) Let $\mathcal{S}$ be the multiplicative semi-group generated by T and T^*. Any element S of $\mathcal{S}$ can be written as a word in the letters T and T^*. Define the weight $w(S)$ of S to be the number of T^*'s minus the number of T's in such a word (although the expression of S as a word need not be unique, looking at the action of S on the blocks, it is clear that its weight is well defined). Then it can be shown that the initial block V_0 of T is the intersection of the kernels of the elements of $\mathcal{S}$ of weight 1. Also, for $n > 0$, the nth block V_n is the closed span of the images of V_0 under the elements of $\mathcal{S}$ of weight $-n$.

3. Projective representations and multipliers

3.1. Generalities

Throughout, G is a locally compact second countable topological group. (Later in this section, we specialise to connected Lie groups. However, in this paper, our interest is in the case of the Möbius group and its universal cover.)

Definition 3.1. A measurable function $\pi : G \to \mathcal{U}(\mathcal{H})$ is called a projective representation of G on the Hilbert space $\mathcal{H}$ if there is a function (necessarily Borel) $m : G \times G \to \mathbb{T}$ such that

$$\pi(1) = I, \quad \pi(g_1 g_2) = m(g_1, g_2)\pi(g_1)\pi(g_2) \tag{3.1}$$

for all g_1, g_2 in G. (More precisely, such a function π is called a projective unitary representation of G; however, from now on, we shall drop the adjective 'unitary' since all representations considered in this paper are unitary.)

Definition 3.2. Two projective representations π_1, π_2 of G on the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ (respectively) will be called equivalent if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ and a function (necessarily Borel) $\gamma : G \to \mathbb{T}$ such that $\pi_2(g)U = \gamma(g)U\pi_1(g)$ for all $g \in G$.

We shall identify two projective representations if they are equivalent. Recall that a projective representation π of G is called *irreducible* if the unitary operators $\pi(g)$, $g \in G$, have no common non-trivial reducing subspace. Clearly, equivalence respects this property.

The function m associated with the projective representation π via (3.1) is called the multiplier of π. Clearly, $m : G \times G \to \mathbb{T}$ is a Borel map. In view of Eq. (3.1), m satisfies

$$m(g, 1) = 1 = m(1, g), \quad m(g_1, g_2)m(g_1 g_2, g_3) = m(g_1, g_2 g_3)m(g_2, g_3). \tag{3.2}$$

for all group elements g, g_1, g_2, g_3. Any Borel function m from $G \times G$ into $\mathbb{T}$ satisfying Eq. (3.2) is called a multiplier on the group G. The multipliers form an abelian group under pointwise multiplication. This is called the multiplier group of G.

Recall that π is called an ordinary representation (and we drop the adjective "projective") if its multiplier is the constant function 1. The ordinary representation π which sends every group element to the identity operator on a one-dimensional Hilbert space is called the identity (or trivial) representation. The following definition of equivalence of multipliers is standard (see for instance [15,19]):

Definition 3.3. Two multipliers m and $\tilde{m}$ on the group G are called equivalent if there is a Borel function $\gamma : G \to \mathbb{T}$ such that $\gamma(g_1 g_2)\tilde{m}(g_1, g_2) = \gamma(g_1)\gamma(g_2)m(g_1, g_2)$ for all g_1, g_2 in G.

Clearly, equivalent projective representations have equivalent multipliers. The multipliers equivalent to the trivial multiplier (viz. the constant function 1) are called exact. The exact multipliers form a subgroup of the multiplier group. The quotient is called the second cohomology group $H^2(G, \mathbb{T})$ (with respect to the trivial action of G on $\mathbb{T}$). See [12] for the relevant cohomology theory. We shall need:

Theorem 3.1. *Let G be a connected semi-simple Lie group. Then every projective representation of G (say with multiplier m) is a direct integral of irreducible projective representations (all with the same multiplier m) of G.*

Proof. Let π be a projective representation of G. Let $\tilde{G}$ be the universal cover of G and let $p:\tilde{G}\to G$ be the covering homomorphism. Define a projective representation π_0 of $\tilde{G}$ by $\pi_0(\tilde{x}) = \pi(x)$ where $x = p(\tilde{x})$. A trivial computation shows that π_0 is indeed a projective representation of $\tilde{G}$ and its multiplier m_0 is given by $m_0(\tilde{x},\tilde{y}) = m(x,y)$, where $x = p(\tilde{x}),\ y = p(\tilde{y})$.

However, since $\tilde{G}$ is a connected and simply connected semi-simple Lie group, $H^2(\tilde{G},\mathbb{T})$ is trivial. (This is an easy and well-known consequence of Theorem 7.37 in [19] in conjunction with the Levy–Malcev theorem.) Therefore, m_0 is exact. That is, there is a Borel function $\gamma:\tilde{G}\to\mathbb{T}$ such that

$$m(x,y) = m_0(\tilde{x},\tilde{y}) = \gamma(\tilde{x})\gamma(\tilde{y})/\gamma(\tilde{x}\tilde{y}) \tag{3.3}$$

for all $\tilde{x},\tilde{y}$ in $\tilde{G}$, and $x = p(\tilde{x}),\ y = p(\tilde{y})$. Now define the ordinary representation $\tilde{\pi}$ of $\tilde{G}$ (equivalent to π_0) by: $\tilde{\pi}(\tilde{x}) = \gamma(\tilde{x})\pi_0(\tilde{x})$, for $\tilde{x}$ in $\tilde{G}$. Now, since $\tilde{G}$ is a locally compact and second countable group, by Theorem 2.9 in [9], the ordinary representation $\tilde{\pi}$ of $\tilde{G}$ may be written as a direct integral of (ordinary) irreducible representations $\tilde{\pi}_t$ of $\tilde{G}:\tilde{\pi}(\tilde{x}) = \int^{\oplus}\tilde{\pi}_t(\tilde{x})\,dP(t), \tilde{x}\in\tilde{G}$. Replacing $\tilde{\pi}$ by its definition in terms of π, we get that for each $x\in G$, $\pi(x) = \int^{\oplus}\gamma(\tilde{x})^{-1}\tilde{\pi}_t\,dP(t)$ for any $\tilde{x}$ such that $x = p(\tilde{x})$. So we would like to define $\pi_t: G\to\mathcal{U}(\mathcal{H})$ by $\pi_t(x) = \gamma(\tilde{x})^{-1}\tilde{\pi}_t(\tilde{x})$ for any $\tilde{x}$ as above and verify that π_t, thus defined, is an irreducible projective representation of G with multiplier m. But first we must show that π_t is well defined. That is, if $\tilde{x}$ and $\tilde{y}$ are elements of $\tilde{G}$ mapping into the same element x of G under p then we need to show

$$\gamma(\tilde{x})^{-1}\tilde{\pi}_t(\tilde{x}) = \gamma(\tilde{y})^{-1}\tilde{\pi}_t(\tilde{y}). \tag{3.4}$$

Let $\tilde{Z}$ be the kernel of the covering map p. Since $\tilde{Z}$ is a discrete normal subgroup of the connected topological group $\tilde{G}$, $\tilde{Z}$ is a central subgroup of $\tilde{G}$. Since for each t, $\tilde{\pi}_t$ is irreducible, it follows by Schur's lemma that there is a Borel function (indeed a continuous character of $\tilde{Z}$) $\gamma_t:\tilde{Z}\to\mathbb{T}$ such that $\tilde{\pi}_t(\tilde{z}) = \gamma_t(\tilde{z})I$ for all $\tilde{z}\in\tilde{Z}$. Also, we have $\tilde{\pi}(\tilde{z}) = \gamma(\tilde{z})\pi_0(\tilde{z}) = \gamma(\tilde{z})\pi(1) = \gamma(\tilde{z})I$ for all $\tilde{z}\in\tilde{Z}$. Therefore, evaluating $\tilde{\pi}(\tilde{z})$ using its direct integral representation, we find $\gamma(\tilde{z})I = \int^{\oplus}\gamma_t(\tilde{z})I\,dP(t)$ and hence $\gamma_t(\tilde{z}) = \gamma(\tilde{z})$ for all t in a set of full P-measure and all $\tilde{z}\in\tilde{Z}$ (Note that, being a discrete subgroup of the Lie group $\tilde{G}$, $\tilde{Z}$ is countable.) Replacing the domain of integration by this subset if need be, we may assume that $\gamma_t = \gamma$ for all t. Thus,

$$\tilde{\pi}_t(\tilde{z}) = \gamma(\tilde{z})I \tag{3.5}$$

for all $\tilde{z}$ in $\tilde{Z}$ and for all t. Also, for $\tilde{x}\in\tilde{G}$ and $\tilde{z}\in\tilde{Z}$, we have $\gamma(\tilde{x})\gamma(\tilde{z})/\gamma(\tilde{x}\tilde{z}) = m_0(\tilde{x},\tilde{z}) = m(x,1) = 1$ (where $x = p(\tilde{x})$) and hence

$$\gamma(\tilde{x}\tilde{z}) = \gamma(\tilde{x})\gamma(\tilde{z}). \tag{3.6}$$

Now we come back to the proof of Eq. (3.4). Since $p(\tilde{x}) = p(\tilde{y})$ there is a $\tilde{z}\in\tilde{Z}$ such that $\tilde{y} = \tilde{x}\tilde{z}$. Therefore, $\gamma(\tilde{y})^{-1}\tilde{\pi}_t(\tilde{y}) = \gamma(\tilde{x})^{-1}\gamma(\tilde{z})^{-1}\tilde{\pi}_t(\tilde{x})\tilde{\pi}_t(\tilde{z})$ (by Eq. (3.6)) $= \gamma(\tilde{x})^{-1}\tilde{\pi}_t(\tilde{x})$ (by Eq. (3.5)). This proves Eq. (3.4) and hence shows that π_t is well defined.

Now, for $x, y\in G$,

$$\begin{aligned}\pi_t(xy) &= \gamma(\tilde{x}\tilde{y})\tilde{\pi}_t(\tilde{x}\tilde{y})\\ &= \gamma(\tilde{x}\tilde{y})\tilde{\pi}_t(\tilde{x})\tilde{\pi}_t(\tilde{y})\\ &= (\gamma(\tilde{x})\gamma(\tilde{y})/\gamma(\tilde{x}\tilde{y}))\pi_t(x)\pi_t(y)\\ &= m_0(\tilde{x},\tilde{y})\pi_t(x)\pi_t(y)\\ &= m(x,y)\pi_t(x)\pi_t(y),\end{aligned}$$

where $\tilde{x},\tilde{y}$ in $\tilde{G}$ are such that $x = p(\tilde{x}),\ y = p(\tilde{y})$. This shows that π_t is indeed a projective representation of G with multiplier m. Since from the definition of π_t it is clear that π_t and $\tilde{\pi}_t$ have the same invariant subspaces, and since the latter is irreducible, it follows that each π_t is irreducible. Thus, we have the required decomposition of π as a direct integral of irreducible projective representations π_t with the same multiplier: $\pi = \int^{\oplus} \pi_t\, dP(t)$. □

As a consequence of (the proof of) Theorem 3.1, we have the following corollary. Here, as above, $\tilde{G}$ is the universal cover of G, and $p: \tilde{G}\to G$ is the covering map. Fix a Borel section $s: G\to\tilde{G}$ for p (that is, s is a Borel function such that $p\circ s$ is the identity function on G) such that $s(1) = 1$. Note that the kernel $\tilde{Z}$ of p is naturally identified with the fundamental group $\pi^1(G)$ of G. Define the map $\alpha: G\times G\to\tilde{Z}$ by

$$\alpha(x,y) = s(xy)s(y)^{-1}s(x)^{-1}, \quad x,y\in G. \tag{3.7}$$

For any character (i.e., continuous homomorphism into the circle group $\mathbb{T}$) χ of $\pi^1(G) = \tilde{Z}$, define $m_\chi: G\times G\to\mathbb{T}$ by

$$m_\chi(x,y) = \chi(\alpha(x,y)), \quad x,y\in G. \tag{3.8}$$

Since $\tilde{Z}$ is a central subgroup of $\tilde{G}$, it is easy to verify that α satisfies the multiplier identity (3.2). Hence, m_χ is a multiplier on G for each character χ of $\tilde{Z}$.

Let $H_1(G)$ denote the first (singular) homology (with integer coefficients) of G as a manifold. Since $H_1(G) = \tilde{Z}$—in general, it is the abelianisation of $\pi^1(G)$, but in our case $\pi^1(G) = \tilde{Z}$ is abelian—the group of characters χ of $\pi^1(G)$ may be identified with the Pontryagin dual $\widehat{H_1(G)} = \mathrm{Hom}(H_1(G),\mathbb{T})$.

Finally, for any multiplier m on G, let $[m]$ denote its image in $H^2(G,\mathbb{T})$ under the quotient map. In terms of these notations, we have:

Corollary 3.1. *Let G be a connected semi-simple Lie group. Then the multipliers* m_χ *are mutually inequivalent, and every multiplier on G is equivalent to* m_χ *for a unique character* χ. *In other words,* $\chi \mapsto [m_\chi]$ *defines a group isomorphism*

$$H^2(G, \mathbb{T}) \equiv \mathrm{Hom}(H_1(G), \mathbb{T}).$$

Proof. Let m be any multiplier on G. Define a projective representation π of G on the Hilbert space $L^2(G)$ by

$$(\pi(g)F)(x) = m(x,g)F(xg), \quad g, x \in G, \; F \in L^2(G).$$

Then, using the defining Eq. (3.2) for a multiplier, it is easy to verify that π is indeed a projective representation of G and the multiplier associated with π is m. Therefore, the calculations done in proving Theorem 3.1 apply to m. Let χ denote the restriction to $\tilde{Z}$ of the Borel map γ which occurs in this proof. Eq. (3.6) implies, in particular, that χ is a character of $\tilde{Z}$. Define the Borel map $f : G \to \mathbb{T}$ by $f = \gamma \circ s$. Then, for $x, y \in G$, $s(xy)s(y)^{-1}s(x)^{-1} \in \tilde{Z}$ and hence Eq. (3.6) gives $f(xy) = \gamma(s(x)s(y))m_\chi(x,y)$. Also, Eq. (3.3) (with the choice $\tilde{x} = s(x), \tilde{y} = s(y)$) gives $m(x,y) = f(x)f(y)/\gamma(s(x)s(y))$. Hence $m(x,y) = \frac{f(x)f(y)}{f(xy)} m_\chi(x,y)$. Thus the multiplier m is equivalent to m_χ.

Finally, since $\chi \mapsto m_\chi$ is a group homomorphism, to show that the multipliers m_χ are mutually inequivalent, it suffices to show that $m_\chi \equiv 1$ implies that χ is the trivial character. So let χ be a character of $\tilde{Z}$ such that m_χ is exact. Hence, there is a Borel function $g : G \to \mathbb{T}$ such that $m_\chi(x,y) = g(x)g(y)/g(xy)$ for $x, y \in G$. Hence we have

$$m_\chi(p(\tilde{x}), p(\tilde{y})) = h(\tilde{x})h(\tilde{y})/h(\tilde{x}\tilde{y})$$

for $\tilde{x}, \tilde{y} \in \tilde{Z}$. Here the Borel function $h : \tilde{G} \to \mathbb{T}$ is given by $h = g \circ p$. But Eq. (3.3) (with $m = m_\chi, \; x = p(\tilde{x}), \; y = p(\tilde{y})$) shows that

$$m_\chi(p(\tilde{x}), p(\tilde{y})) = \gamma(\tilde{x})\gamma(\tilde{y})/\gamma(\tilde{x}\tilde{y})$$

for $\tilde{x}, \tilde{y} \in \tilde{G}$. Comparing these two equations we see that γ/h is a character of $\tilde{G}$. But there is no non-trivial character of $\tilde{G}$. (A semi-simple Lie group is its own commutator, so there is no non-trivial homomorphism from such a group into any abelian group.) Therefore, $\gamma = h = g \circ p$. But $g \circ p$ is a constant function on the kernel $\tilde{Z}$ of p, while the restriction of γ to $\tilde{Z}$ is the character χ. Thus χ is trivial. □

Remark 3.1. (a) The isomorphism $\chi \mapsto [m_\chi]$ in Corollary 3.1 appears to depend on the choice of the section s. But it is quite easy to prove that actually there is no such dependence. Thus, the isomorphism of this corollary is a natural one.

(b) The beginning of the proof of Corollary 3.1 shows that any multiplier m on a locally compact second countable group G is actually associated with ('comes from') some projective representation of G. In conjunction with Theorem 3.1, it then follows that if G is a connected semi-simple Lie group then any multiplier of G comes from an irreducible projective representation.

(c) Let G be a connected semi-simple Lie group and let $\tilde{G}$ be its universal cover. Also, let $\tilde{Z}$ be as above. Finally, let χ be a character of $\tilde{Z}$. Let us say that an ordinary representation $\tilde{\pi}$ of $\tilde{G}$ is pure of type χ if $\tilde{\pi}(\tilde{z}) = \chi(\tilde{z})I$ for all $\tilde{z} \in \tilde{Z}$. The proof of Theorem 3.1 shows that there is a natural bijection $\pi \mapsto \tilde{\pi}$ between the (equivalence classes of) projective representations of G and the (equivalence classes of) pure ordinary representations of $\tilde{G}$. Further, under this bijection, the projective representations with multiplier m_χ correspond to the representations of pure type χ. Finally (since in general π and $\tilde{\pi}$ have the same invariant subspaces, and since by Schur's Lemma the irreducible representations of $\tilde{G}$ are pure), the irreducible projective representations of G are in bijection with the irreducible representations of $\tilde{G}$ under this map.

3.2. The irreducible representations of the Möbius group

In view of Theorem 3.1, to understand all the projective representations of Möb it suffices to know its irreducible projective representations. Most of these representations happen to arise out of the following construction.

For φ in Möb, φ' is a non-vanishing analytic function on $\bar{\mathbb{D}}$. Hence, there is an analytic branch of $\log \varphi'$ on $\bar{\mathbb{D}}$. For the rest of this paper, fix such a branch for each φ such that (a) for $\varphi = 1, \log \varphi' \equiv 0$ and (b) the map $(\varphi, z) \mapsto \log \varphi'(z)$ from Möb $\times$ $\bar{\mathbb{D}}$ into $\mathbb{C}$ is a Borel function. With such a determination of the logarithm, we define the functions $(\varphi')^{\lambda/2}$ (for any fixed complex number λ) and $\arg \varphi'$ on $\bar{\mathbb{D}}$ by $\varphi'(z)^{\lambda/2} = \exp(\frac{\lambda}{2} \log \varphi'(z))$ and $\arg \varphi'(z) = \operatorname{Im} \log \varphi'(z)$. (If our determination of the logarithms are changed then—it is easy to see—the representations of Möb introduced below as well as the multipliers on Möb defined in the next subsection remain unchanged modulo equivalence.)

Let λ be a real parameter, and μ be a complex parameter. Further, let $I = \mathbb{Z}$ or $I = \mathbb{Z}^+$. Let $\mathscr{H} = \mathscr{H}^{\lambda,\mu}(I)$ be the Hilbert space spanned by the orthogonal set $\{f_n\colon n \in I\}$ where

$$||f_n||^2 = \frac{\Gamma(1 - \mu + n)}{\Gamma(\lambda + \bar{\mu} + n)}, \quad n \in I. \tag{3.9}$$

Of course, the parameters λ and μ must be such that the expression on the right of this formula is a positive real number for every n in I. (This requires, in particular, that either μ is real or $\mu + \bar{\mu} = 1 - \lambda$.) Henceforth, we assume that λ and μ satisfy this requirement.

Let $\mathscr{M} = \mathscr{M}(I)$ be the linear space of all functions on $\mathbb{T}$ which have analytic continuation to some neighbourhood (depending on the function) of $\mathbb{T}$ in case $I = \mathbb{Z}$ and to some neighbourhood of $\bar{\mathbb{D}}$ in case $I = \mathbb{Z}^+$. Identify the basis elements f_n of $\mathscr{H}$ with the elements $w \mapsto w^n$ of $\mathscr{M}$. Since the elements of $\mathscr{M}$ are infinitely smooth, their Fourier coefficients decay faster than any rational function. Also, $||f_n||$ grows at most like a polynomial in $|n|$ as $|n| \to \infty$. Therefore, $\mathscr{M}$ is identified with a dense linear subspace of $\mathscr{H}$. In order to ensure that $\mathscr{M}(I)$ is invariant under the operators $R_{\lambda,\mu}(\varphi^{-1})$ defined below, we also assume that, in case $I = \mathbb{Z}^+$, we have $\mu = 0$.

For $\varphi \in$ Möb, define the operator $R_{\lambda,\mu}(\varphi^{-1}) : \mathscr{M} \to \mathscr{M}$ by

$$(R_{\lambda,\mu}(\varphi^{-1})f)(z) = \varphi'(z)^{\lambda/2}|\varphi'(z)|^{\mu} f(\varphi(z)), \quad z \in \mathbb{T}, \quad f \in \mathscr{M}, \quad \varphi \in \text{Möb}.$$

Of course, there is no a priori guarantee that this is a unitary operator. But, when it is, then clearly it extends to a unitary operator on $\mathscr{H}^{\lambda,\mu}(I)$. When that is the case, it is easy to see that $R_{\lambda,\mu}$ is then a projective representation of Möb (see the proof of Theorem 3.2 below). Thus, the description of the representation is complete if we specify I and the two parameters λ, μ. It turns out that all the non-trivial irreducible projective representations of Möb have this form (excepting the anti-holomorphic Discrete series representations which are of the form $R^{\#}_{\lambda,\mu}$).

By Remark 3.1(c), there is a natural bijection between the irreducible projective representations of Möb and the irreducible (ordinary) representations of its universal cover. But a complete list of the irreducible representations (up to equivalence) of the universal cover of Möb was obtained by Pukanszky (see [16], for instance). Hence, one obtains a complete list of the irreducible projective representations of Möb. This is as follows. (However, see the remark following the list.)

List 3.1. (1) *Holomorphic Discrete series representations* D^+_λ. Here $\lambda > 0, \mu = 0$, $I = \mathbb{Z}^+$ and the representation space may be identified with the functional Hilbert space $\mathscr{H}^{(\lambda)}$ of analytic functions on $\mathbb{D}$ with reproducing kernel $(1 - z\bar{w})^{-\lambda}, z, w \in \mathbb{D}$.

(2) *Anti-holomorphic Discrete series representations* D^-_λ, $\lambda > 0$: D^-_λ may be defined as the composition of D^+_λ with the automorphism $*$ of Eq. (2.1). Thus, $D^-_\lambda = (D^+_\lambda)^{\#}$ (recall Eq. (2.4)). This may be realized on a functional Hilbert space of anti-holomorphic functions on $\mathbb{D}$, in a natural way.

(3) *Principal series representations* $P_{\lambda,s}, -1 < \lambda \leqslant 1$, *s purely imaginary*: Here $\lambda = \lambda,\ \mu = \frac{1-\lambda}{2} + s,\ I = \mathbb{Z}$. Notice that in this case the representation space is $L^2(\mathbb{T})$.

(4) *Complementary series representation* $C_{\lambda,\sigma}$, $-1 < \lambda < 1,\ 0 < \sigma < \frac{1}{2}(1 - |\lambda|)$: Here $\lambda = \lambda,\ \mu = \frac{1}{2}(1 - \lambda) + \sigma,\ I = \mathbb{Z}$.

Remark 3.2. All the projective representations in List 3.1 are mutually inequivalent with the exception that $P_{\lambda,s}$ and $P_{\lambda,-s}$ are equivalent representations for each s. Moreover, all these representations are irreducible with the sole exception of $P_{1,0}$ for which we have the decomposition $P_{1,0} = D^+_1 \oplus D^-_1$. Every non-trivial irreducible projective representation of Möb is equivalent to a member of List 3.1.

3.3. The multipliers of the Möbius group

Next we describe the multipliers of Möb up to equivalence. Let us define the Borel function $\boldsymbol{n}$: Möb $\times$ Möb $\to \mathbb{Z}$ by

$$\boldsymbol{n}(\varphi_1^{-1}, \varphi_2^{-1}) = \frac{1}{2\pi}(\arg(\varphi_2\varphi_1)'(0) - \arg\varphi_1'(0) - \arg\varphi_2'(\varphi_1(0))). \tag{3.10}$$

The chain rule implies that this is indeed an integer valued function. For any $\omega \in \mathbb{T}$, define $m_\omega :$ Möb $\times$ Möb $\to \mathbb{T}$ by

$$m_\omega(\varphi_1, \varphi_2) = \omega^{\boldsymbol{n}(\varphi_1,\varphi_2)}. \tag{3.11}$$

Then we have:

Theorem 3.2. (a) m_ω *is a multiplier of* Möb *for each* $\omega \in \mathbb{T}$. *Up to equivalence*, $m_\omega, \omega \in \mathbb{T}$, *are all the multipliers on* Möb; *further*, *these are mutually inequivalent multipliers. In other words*, $H^2(\text{Möb}, \mathbb{T})$ *is naturally isomorphic to* $\mathbb{T}$ *via the map* $\omega \mapsto [m_\omega]$.

(b) *For each of the representations of* Möb *in List* 3.1, *the associated multiplier is* m_ω, *where* (*in terms of the parameter* λ *of the representation*) $\omega = e^{i\pi\lambda}$ *in each case*, *except for the anti-holomorphic Discrete series representation* (*s*) *for which* $\omega = e^{-i\pi\lambda}$.

Proof. We first prove Part (b). Let $\pi = R_{\lambda,\mu}$ be a representation in List 3.1. Thus, π is not in the anti-holomorphic Discrete series. From the definition of $R_{\lambda,\mu}$, one calculates that the associated multiplier m is given by

$$m(\varphi_1^{-1}, \varphi_2^{-1}) = \frac{((\varphi_2\varphi_1)'(z))^{\lambda/2}}{(\varphi_1'(z))^{\lambda/2}(\varphi_2'(\varphi_1(z)))^{\lambda/2}}, \quad z \in \mathbb{T}$$

for any two elements φ_1, φ_2 of Möb. Notice that the right-hand side of this equation is an analytic function of z for z in $\bar{\mathbb{D}}$ and it is of constant modulus 1 in view of the chain rule for differentiation. Therefore, by the maximum modulus principle, this formula is independent of z for z in $\bar{\mathbb{D}}$. Hence we may take $z = 0$ in this formula. This yields $m = m_\omega$ with $\omega = e^{i\pi\lambda}$. Notice that if m is the multiplier associated with the representation π then the multiplier associated with $\pi^\#$ is $\bar{m}$. Since $D_\lambda^- = D_\lambda^{+\#}$, it follows that if $\pi = D_\lambda^-$ is in the anti-holomorphic Discrete series, then its multiplier is m_ω where $\omega = e^{-i\pi\lambda}$.

This argument also shows that m_ω is indeed a multiplier of Möb for each $\omega \in$ Möb. Further, since these multipliers include all the multipliers of Möb associated with irreducible projective representations, Remark 3.1(b) shows that modulo equivalence these are all the multipliers on Möb. Unfortunately, it seems very hard to see directly that the multipliers $m_\omega, \omega \in \mathbb{T}$, are mutually inequivalent. (Since $\omega \mapsto [m_\omega]$ is clearly a group homomorphism from $\mathbb{T}$ onto $H^2(\text{Möb}, \mathbb{T})$, this amounts to verifying that m_ω is never exact for $\omega \neq 1$.) This fact may be deduced from Corollary 3.1 as follows.

Identify Möb with $\mathbb{T} \times \mathbb{D}$ via $\varphi_{\alpha,\beta} \mapsto (\alpha, \beta)$. The group law on $\mathbb{T} \times \mathbb{D}$ is given by

$$(\alpha_1, \beta_1)(\alpha_2, \beta_2) = \left(\alpha_1\alpha_2 \frac{1 + \bar{\alpha}_2\beta_1\bar{\beta}_2}{1 + \alpha_2\bar{\beta}_1\beta_2}, \frac{\beta_1 + \alpha_2\beta_2}{\alpha_2 + \beta_1\bar{\beta}_2}\right).$$

The identity in $\mathbb{T} \times \mathbb{D}$ is $(1, 0)$ and the inverse map is $(\alpha, \beta)^{-1} = (\bar{\alpha}, -\alpha\beta)$.

Then the universal cover $\widetilde{\text{Möb}}$ is naturally identified with $\mathbb{R} \times \mathbb{D}$. Taking the covering map $p : \mathbb{R} \times \mathbb{D} \to \mathbb{T} \times \mathbb{D}$ to be $p(t, \beta) = (e^{2\pi it}, \beta)$, the group law on $\mathbb{R} \times \mathbb{D}$ is determined by (continuity and) the requirement that p be a group homomorphism, as follows:

$$(t_1, \beta_1)(t_2, \beta_2) = \left(t_1 + t_2 + \frac{1}{\pi}\operatorname{Im}\operatorname{Log}(1 + e^{-2\pi it_2}\beta_1\bar{\beta}_2), \quad \frac{\beta_1 + e^{2\pi it_2}\beta_2}{e^{2\pi it_2} + \beta_1\bar{\beta}_2}\right),$$

where 'Log' denotes the principal branch of the logarithm on the right half-plane. The identity in $\mathbb{R} \times \mathbb{D}$ is $(0, 0)$ and the inverse map is $(t, \beta)^{-1} = (-t, -e^{2\pi it}\beta)$. The kernel $\tilde{Z}$ of the covering map p is identified with the additive group $\mathbb{Z}$ via $n \mapsto (n, 0)$.

Let's choose a Borel branch $\arg : \mathbb{T} \to \mathbb{R}$ of the argument function satisfying $\arg(\bar{z}) = -\arg(z)$, $z \in \mathbb{T}$. Let us then make an explicit choice of the Borel function $(\varphi, z) \mapsto \arg(\varphi'(z))$ (which occurs in the definition of $\boldsymbol{n}$ in Eq. (3.10)) as follows:

$$\arg \varphi'_{\alpha,\beta}(z) = \arg(\alpha) - 2\operatorname{Im}\operatorname{Log}(1 - \bar{\beta}z).$$

Let us also choose the section $s : \mathbb{T} \times \mathbb{D} \to \mathbb{R} \times \mathbb{D}$ as follows: $s(\alpha, \beta) = (\frac{1}{2\pi}\arg(\alpha), \beta)$. An easy computation shows that, for these choices, we have $s(\varphi_1\varphi_2)s(\varphi_2)^{-1}s(\varphi_1)^{-1} = -\boldsymbol{n}(\varphi_1, \varphi_2)$ for φ_1, φ_2 in Möb. Hence we get that, for $\omega \in \mathbb{T}$, $m_\omega = m_\chi$ where $\chi = \chi_\omega$ is the character $n \mapsto \omega^{-n}$ of $\mathbb{Z}$. Thus, the map $\omega \mapsto [m_\omega]$ is but a special case of the isomorphism $\chi \mapsto [m_\chi]$ of Corollary 3.1. □

As a simple but important consequence of this theorem, we have:

Corollary 3.2. *Take any two representations from List* 3.1. *Their multipliers are either equal or inequivalent. If both or neither of these two representations are from the anti-holomorphic Discrete series, then they have the same multiplier iff their λ parameters differ by an even integer. If, on the other hand, exactly one of them is from the anti-holomorphic Discrete series, then they have the same multiplier iff their λ parameters add to an even integer.*

3.4. The simple representations of the Möbius group

Let $\mathbb{K}$ be the maximal compact subgroup of Möb given by $\mathbb{K} = \{\varphi_{\alpha,0} : \alpha \in \mathbb{T}\}$. Of course, $\mathbb{K}$ is isomorphic to the circle group $\mathbb{T}$ via $\alpha \mapsto \varphi_{\alpha,0}$.

Definition 3.4. Let π be a projective representation of Möb. We shall say that π is normalised if $\pi|_{\mathbb{K}}$ is an ordinary representation of $\mathbb{K}$.

Lemma 3.1. *Any projective representation of* Möb *is equivalent to a normalised representation.*

Proof. Take any projective representation σ of Möb. Then $\sigma|_{\mathbb{K}}$ is a projective representation of $\mathbb{K}$, say, with multiplier m. But $H^2(\mathbb{K}, \mathbb{T})$ is trivial (see [19, Theorem 7.41]). So, there exists a Borel function $f : \mathbb{K} \to \mathbb{T}$ such that

$$m(x, y) = \frac{f(x)f(y)}{f(xy)}, \quad x, y \in \mathbb{K}.$$

Extend f to a Borel function g: Möb$\to \mathbb{T}$. Define π by $\pi(x) = g(x)\sigma(x)$, $x \in$Möb. Then π is normalised and equivalent to σ. $\square$

Notation 3.1. For $n \in \mathbb{Z}$, let χ_n be the character of $\mathbb{T}$ given by $\chi_n(x) = x^{-n}$, $x \in \mathbb{T}$. For any normalised projective representation π of Möb on some Hilbert space $\mathscr{H}$ and $n \in \mathbb{Z}$, let

$$V_n(\pi) = \{v \in \mathscr{H} : \pi(x)v = \chi_n(x)v, \quad \forall\ x \in \mathbb{T}\}.$$

Then $\mathscr{H} = \oplus_{n \in \mathbb{Z}} V_n(\pi)$ (an orthogonal direct sum). The subspaces $V_n(\pi)$ are usually called the $\mathbb{K}$-isotypic subspaces of $\mathscr{H}$. Put

$$d_n(\pi) = \dim V_n(\pi) \quad \text{and} \quad \mathscr{T}(\pi) = \{n \in \mathbb{Z} : d_n(\pi) \neq 0\}.$$

Definition 3.5. (a) A subset A of $\mathbb{Z}$ is said to be connected if for any three elements $a < b < c$ in $\mathbb{Z}$, $a, c \in A$ implies $b \in A$. If B is any subset of $\mathbb{Z}$, a connected component of B is a maximal connected subset of B (with respect to set inclusion). Since the union of two intersecting connected sets is clearly connected, the connected components of a set partition the set.

(b) Let π be a normalised projective representation of Möb. We shall say that π is connected if $\mathscr{T}(\pi)$ is connected. π will be called simple if π is connected and, further, $d_n(\pi) \leqslant 1$ for all $n \in \mathbb{Z}$. More generally, a projective representation is connected/simple if it is equivalent to a connected/simple (normalised) representation.

Remark 3.3. (a) Notice that if π and σ are equivalent normalised representations then there is an integer h such that $V_n(\sigma) = V_{n+h}(\pi)$ for all n in $\mathbb{Z}$. Consequently, $\mathscr{T}(\sigma)$ is an additive translate of $\mathscr{T}(\pi)$. Hence σ is connected/simple if and only if π is. Thus, the definitions given above are consistent.

(b) Let π be one of the representations $R_{\lambda,\mu}$ in List 3.1. Then, after a suitable normalisation, we have $\mathscr{T}(\pi) = I$ and $V_n(\pi) = \langle f_n \rangle$, $n \in I$, in the notation of Section 3.2. Thus all the representations in this list are simple.

Lemma 3.2. *Let π be any normalised projective representation of* Möb. *Then each connected component of $\mathscr{T}(\pi)$ is unbounded.*

Proof. By Theorem 3.1, we may write

$$\pi = \int^{\oplus} \pi_t \, dP(t),$$

where P is a regular measure and π_t is an irreducible projective representation of Möb for all t. In view of Remark 3.3(b), we have that $\mathscr{T}(\pi_t)$ is connected and unbounded for each t. So it suffices to show that the same must be true of their direct integral π. To this end, we claim that, for each n in $\mathbb{Z}$,

$$V_n(\pi) = \int^{\oplus} V_n(\pi_t) \, dP(t). \tag{3.12}$$

Indeed, the inclusion $\supseteq$ is trivial. To prove the inclusion $\subseteq$, take $v \in V_n(\pi)$. Then $v = \int^{\oplus} v_t \, dP(t)$ for some $v_t \in \mathscr{H}_t$ (:= the space on which π_t acts). Consequently,

$$\begin{aligned}\int^{\oplus} \chi_n(x) v_t \, dP(t) &= \chi_n(x) v \\ &= \pi(x) v \\ &= \int^{\oplus} \pi_t(x) v_t \, dP(t).\end{aligned}$$

This implies that $\chi_n(x) v_t = \pi_t(x) v_t$ for almost all t. Therefore, $v_t \in V_n(\pi_t)$ for almost all t. This proves Claim(3.12)

Therefore, $n \in \mathscr{T}(\pi)$ if and only if $n \in \mathscr{T}(\pi_t)$ for all t in a set of positive P measure. Now suppose some component of $\mathscr{T}(\pi)$ is bounded. Then there exists $a<b<c$ in $\mathbb{Z}$ such that b is in $\mathscr{T}(\pi)$ but a and c are not in $\mathscr{T}(\pi)$. It follows that a and c are not in $\mathscr{T}(\pi_t)$ for almost all t but b is in $\mathscr{T}(\pi_t)$ for all t in a set of positive measure. Therefore, there is a t for which $b \in \mathscr{T}(\pi_t)$ but $a, c \notin \mathscr{T}(\pi_t)$. Then the component of $\mathscr{T}(\pi_t)$ containing b is bounded. Contradiction. □

Theorem 3.3. *Up to equivalence, the only simple projective representations of* Möb *are the irreducible projective representations of* Möb *and the representations* $D_\lambda^+ \oplus D_{2-\lambda}^-$, $0<\lambda<2$.

Proof. Let π be a simple representation of Möb. If π is irreducible then we have nothing to prove. So, assume $\pi = \pi_1 \oplus \pi_2$. By Eq. (3.12), we have $V_n(\pi) = V_n(\pi_1) \oplus V_n(\pi_2)$. Hence $d_n(\pi_1) + d_n(\pi_2) = d_n(\pi) \leqslant 1$. Therefore, we have $\mathscr{T}(\pi) = \mathscr{T}(\pi_1) \cup \mathscr{T}(\pi_2), \mathscr{T}(\pi_1) \cap \mathscr{T}(\pi_2) = \emptyset$. Therefore, by Lemma 3.2, the connected components of $\mathscr{T}(\pi_i)$, $i = 1, 2$ together form a collection of pairwise disjoint unbounded connected sets. Since three unbounded connected subsets of $\mathbb{Z}$ cannot be pairwise disjoint, it follows that this collection contains at most (and hence exactly) two sets. Thus both π_1 and π_2 are (connected and hence) simple. Since the connected set $\mathscr{T}(\pi)$ is the disjoint union of the two unbounded connected sets $\mathscr{T}(\pi_1)$ and $\mathscr{T}(\pi_2)$, it follows that $\mathscr{T}(\pi) = \mathbb{Z}$. In consequence, (up to interchanging of π_1 and π_2)

the connected sets $\mathcal{T}(\pi_1)$ (respectively $\mathcal{T}(\pi_2)$) must be bounded below (respectively bounded above).

The argument so far shows, in particular, that whenever a simple projective representation π is reducible, $\mathcal{T}(\pi) = \mathbb{Z}$ is forced. Since π_1 and π_2 are simple but $\mathcal{T}(\pi_i)$ is a proper subset of $\mathbb{Z}$, it follows that π_1 and π_2 are irreducible and hence are members of List 3.1 (modulo equivalence). But, by Remark 3.3, the only members π_1 (respectively π_2) of List 3.1 for which $\mathcal{T}(\pi_1)$ (respectively $\mathcal{T}(\pi_2)$) is bounded below (respectively above) are the holomorphic (respectively anti-holomorphic) Discrete series representations. Therefore, there are positive real numbers λ and μ such that π_1 and π_2 are equivalent to D_λ^+ and D_μ^-, respectively. Since D_λ^+ and D_μ^- occur as subrepresentations of a common projective representation (viz. an equivalent copy of π), they must have a common multiplier. In view of Corollary 3.2, this implies that $\lambda + \mu$ is an even integer. Now, a computation shows that (up to additive translation) for $\pi = D_\lambda^+ \oplus D_\mu^-$, we have

$$\mathcal{T}(\pi) = \{n \in \mathbb{Z}:\ n \geqslant 0\} \cup \{n \in \mathbb{Z}:\ n \leqslant -(\lambda + \mu)/2\}.$$

Since $\mathcal{T}(\pi) = \mathbb{Z}$, we must have $\lambda + \mu = 2$. Thus, up to equivalence, $\pi = D_\lambda^+ \oplus D_{2-\lambda}^-,\ 0 < \lambda < 2.$ $\square$

4. Examples of homogeneous weighted shifts

Now we present a list of homogeneous weighted shifts. Later in this paper we shall see that this list is exhaustive.

List 4.1. (1) *The Principal series example*: The unweighted bilateral shift B (i.e., the bilateral shift with weight sequence $w_n = 1,\ n \in \mathbb{Z}$) is homogeneous. To see this, apply Theorem 2.3 to any of the Principal series representations of Möb. Being normal (in fact unitary) this operator is far from irreducible. By construction, all the Principal series representations are associated with it.

(2) *The Holomorphic Discrete series examples*: For any real number $\lambda > 0$, the unilateral shift $M^{(\lambda)}$ with weight sequence $\sqrt{\frac{n+1}{n+\lambda}},\ n \in \mathbb{Z}^+$, is homogeneous. To see this, apply Theorem 2.3 to the Discrete series representation D_λ^+. These are irreducible, and, by construction, the representation associated with $M^{(\lambda)}$ is D_λ^+.

(3) *The anti-holomorphic Discrete series examples*: Being adjoints of homogeneous operators, the operators $M^{(\lambda)^*}$, $\lambda > 0$, are homogeneous. Since $D_\lambda^{+\#} = D_\lambda^-$, Proposition 2.1 implies that the representation associated with $M^{(\lambda)^*}$ is D_λ^-. It was shown in [10] that these operators are the only homogeneous operators in the Cowen–Douglas class $B_1(\mathbb{D})$.

(4) *The Complementary series examples*: For any two distinct real numbers a and b in the open unit interval $(0,1)$, the bilateral shift $K_{a,b}$ with weight sequence $\sqrt{\frac{n+a}{n+b}},\ n \in \mathbb{Z}$, is homogeneous. To see this in case $0 < a < b < 1$, apply Theorem 2.3 to the Complementary series representation $C_{\lambda,\sigma}$ with $\lambda = a + b - 1$ and $\sigma = (b - a)/2$.

If $0<b<a<1$ then $K_{a,b}$ is the adjoint inverse of the homogeneous operator $K_{b,a}$, and hence is homogeneous. By Proposition 2.1, $K_{a,b}$ and $K_{b,a}$ have the same associated representation. Thus, for any two numbers $0<a\neq b<1$, the representation associated with the irreducible operator $K_{a,b}$ is $C_{\lambda,\sigma}$ with $\lambda = a+b-1$, $\sigma = |a-b|/2$.

(5) *The Constant Characteristic examples*: For any strictly positive real number $x\neq 1$, the bilateral shift B_x with weight sequence $\ldots, 1, 1, 1, x, 1, 1, 1, \ldots$, ($x$ in the zeroth slot, 1 elsewhere) is homogeneous. Indeed, if $0<x<1$ then B_x is a completely non unitary contraction with constant characteristic function $-x$; hence it is homogeneous because of Theorem 2.10 in [4]. (In [4] we show that apart from the unweighted unilateral shift $M^{(1)}$ and its adjoint, these are the only irreducible contractions with a constant characteristic function.) If $x>1$, B_x is the adjoint inverse of the homogeneous operator B_y with $y = x^{-1}$, hence it is homogeneous. (In [4] we presented an unnecessarily convoluted argument to show that B_x is homogeneous for $x>1$ as well.) It was shown in [4] that the representation $P_{1,0} = D_1^+ \oplus D_1^-$ is associated with each of the operators B_x, $x>0$. This also follows from the proof of Theorem 5.2 given below.

Remark 4.1. Using well-known criteria (see [17]) for unitary equivalence (or similarity) of weighted shifts, it is easy to see that the operators listed above are unitarily inequivalent. In fact, no two of the unilateral weighted shifts in this list are similar. Each operator in the Constant Characteristic family is similar to B. Further, the complementary series operators $K_{a,b}$ and $K_{c,d}$ are similar iff $a-b = c-d$.

5. Classification

Theorem 5.1. *If T is an irreducible homogeneous operator then T is a block shift. If π is a normalised representation associated with T then the blocks of T are precisely the $\mathbb{K}$-isotypic subspaces $V_n(\pi)$, $n\in\mathcal{T}(\pi)$.*

Proof. By Theorem 2.2 and Lemma 3.1, there is a normalised representation π associated with T, and it is unique up to equivalence. Because of Lemma 2.2, it suffices to show that

$$T(V_n(\pi))\subseteq V_{n+1}(\pi) \quad \text{for all } n\in\mathcal{T}(\pi). \tag{5.1}$$

Indeed, since T is irreducible, (5.1) shows that π is connected (if there were $a<b<c$ in $\mathbb{Z}$ with $a, c\in\mathcal{T}(\pi)$ and $b\notin\mathcal{T}(\pi)$ then (5.1) would imply that $\oplus_{n<b} V_n(\pi)$ is a non-trivial reducing subspace of T). Since $\mathcal{T}(\pi)$ is also unbounded by Lemma 3.2 it follows that (replacing π by an equivalent normalised representation, if necessary) the indexing set $\mathcal{T}(\pi)$ can be taken to be either $\mathbb{Z}$ or $\mathbb{Z}^+$ or $\mathbb{Z}^-$. Therefore, T is a block shift.

So, it only remains to prove (5.1). To do this, fix $n \in \mathscr{T}(\pi)$ and $v \in V_n(\pi)$. For $x \in \mathbb{K}$, we have $\pi(x)v = \chi_n(x)v$. Consequently,

$$\begin{aligned}
\pi(x)Tv &= \pi(x^{-1})^* Tv \\
&= \pi(x^{-1})^* T\pi(x^{-1})(\pi(x)v) \\
&= (x^{-1}T)(x^{-n}v) \\
&= x^{-(n+1)}Tv \\
&= \chi_{n+1}(x)Tv.
\end{aligned}$$

(For the first two equalities, recall that as π is normalised, its restriction to $\mathbb{K}$ is an ordinary representation of $\mathbb{K}$. In the third equality, we use the assumption that π is associated with T.) So, $Tv \in V_{n+1}(\pi)$. This proves (5.1). □

Lemma 5.1. *Let T be any homogeneous (scalar) weighted shift. Let π be the projective representation of* Möb *associated with T. Then up to equivalence, π is one of the representations in List* 3.1. *Further*,

(a) *if T is a forward shift then the associated representation is holomorphic Discrete series*,
(b) *if T is a backward shift then the associated representation is anti-holomorphic Discrete series*, *and*
(c) *if T is a bilateral shift then the associated representation is either Principal series or Complementary series.*

Proof. Let T be a homogeneous shift. If T is reducible, then by Theorem 2.1, $T = B$ and hence the associated representations are principal series. So assume T is irreducible. Notice that a scalar shift is by definition a block shift with one-dimensional blocks. But by Theorem 5.1, the subspaces $V_n(\pi)$, $n \in \mathscr{T}(\pi)$ are blocks of T. Therefore, by the uniqueness of the blocks (Lemma 2.2), we conclude that π is simple. Thus by Theorem 3.3, either π is irreducible or $\pi = D_\lambda^+ \oplus D_{2-\lambda}^-$ for some λ in the range $0 < \lambda < 2$. In the first case we are done since List 3.1 includes all irreducible projective representations.

In the latter case, let us write $\pi = \pi_1 \oplus \pi_2$ with $\pi_1 = D_{2-\lambda}^-$ and $\pi_2 = D_\lambda^+$. Thus, the space on which T acts is $\mathscr{H}^{(2-\lambda)} \oplus \mathscr{H}^{(\lambda)} = \mathscr{H}_1 \oplus \mathscr{H}_2$ (say). Let $e_{n,\lambda}$, $n \in \mathbb{Z}^+$, denote the standard orthonormal basis of $\mathscr{H}^{(\lambda)}$. (That is, $e_{n,\lambda}(z) = \binom{n+\lambda-1}{n}^{1/2} z^n$.) Define the orthonormal basis x_n, $n \in \mathbb{Z}$, of $\mathscr{H}$ by

$$x_n = \begin{cases} e_{-1-n,2-\lambda} & \text{if } n < 0, \\ e_{n,\lambda} & \text{if } n \geqslant 0. \end{cases}$$

Then a calculation shows that we have $V_n(\pi) = \langle x_n \rangle$ for n in $\mathbb{Z}$. Therefore, in view of Theorem 5.1, with respect to the decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, T looks like

$$\begin{pmatrix} T_1 & 0 \\ rE & T_2 \end{pmatrix}.$$

Here r is a non-zero scalar and the operator $E : \mathscr{H}_1 \to \mathscr{H}_2$ is given by $Ex_{-1} = x_0$, $Ex_n = 0$ for $n \neq -1$. Hence, for $\varphi = \varphi_{1,\beta}$ in Möb ($\beta \in \mathbb{D}$ arbitrary), we have

$$\pi(\varphi)^* T \pi(\varphi) = \begin{pmatrix} \pi_1(\varphi)^* T_1 \pi_1(\varphi) & 0 \\ r\pi_2(\varphi)^* E \pi_1(\varphi) & \pi_2(\varphi)^* T_2 \pi_2(\varphi) \end{pmatrix}$$

and

$$\varphi(T) = \begin{pmatrix} \varphi(T_1) & 0 \\ rX & \varphi(T_2) \end{pmatrix},$$

where $X = (1 - |\beta|^2)(I - \bar{\beta} T_2)^{-1} E (I - \bar{\beta} T_1)^{-1}$.

Therefore, equating $\pi(\varphi)^* T \pi(\varphi)$ and $\varphi(T)$, we get (a) $\varphi(T_i) = \pi_i(\varphi)^* T_i \pi_i(\varphi)$ for $i = 1, 2$, and (b) $X = \pi_2(\varphi)^* E \pi_1(\varphi)$. Since the elements $\varphi_{1,\beta}$, $\beta \in \mathbb{D}$, generate Möb, (a) holds for all φ in Möb, i.e., π_i is associated with T_i. But (we shall see during the proof of Theorem 5.2 below that) the only operators associated with $\pi_1 = D^-_{2-\lambda}$ is $M^{(2-\lambda)*}$ and the only operator associated with $\pi_2 = D^+_\lambda$ is $M^{(\lambda)}$. Thus, $T_1 = M^{(2-\lambda)*}$ and $T_2 = M^{(\lambda)}$. Now, substituting the value of X and simplifying, Eq. (b) becomes

$$(1 - |\beta|^2) \pi_2(\varphi)(I - \bar{\beta} T_2)^{-1} E = E \pi_1(\varphi)(I - \bar{\beta} T_1).$$

Since the left-hand side vanishes at $e_{n,2-\lambda}$ for $n > 0$, so must the right-hand side (with $T_1 = M^{(2-\lambda)*}$, $\pi_1 = D^-_{2-\lambda}$). But a calculation shows that this never happens unless $\lambda = 1$. Thus, in this case, π is equivalent to $D^+_1 \oplus D^-_1 = P_{1,0}$ which is in List 3.1.

Now, by Remark 3.3(b), if π is a normalised representation equivalent to one of the representations in List 3.1, then (up to additive translations) $\mathscr{T}(\pi) = \mathbb{Z}^+$ (respectively $\mathbb{Z}^-$) if π is holomorphic (respectively anti-holomorphic) Discrete series and $\mathscr{T}(\pi) = \mathbb{Z}$ if π is Principal or Complementary series. Therefore, statements (a), (b), (c) in the lemma follow. □

Notation 5.1. For $z \in \mathbb{C}$ and $n \in \mathbb{Z}^+$, $\binom{z}{n}$ will denote the coefficient of t^n in the power series representation of $(1+t)^z$ around $t = 0$. This coincides with the usual notation for binomial coefficients in case $z \in \mathbb{Z}^+$.

Theorem 5.2. *Up to unitary equivalence, the only homogeneous (scalar) weighted shifts (with non-zero weights) are the ones in List* 4.1.

Proof. Let T be a homogeneous weighted shift. If T is reducible, we are done by Theorem 2.1. So assume that T is irreducible. Then by Theorem 2.2 there is a projective representation π of Möb associated with T. By Lemma 5.1, π is one of the representations in List 3.1. Further, replacing T by T^* if necessary, we may assume that T is either a forward shift or a bilateral shift. Accordingly, π is either a holomorphic discrete series representation or a Principal/Complementary series representation. Hence $\pi = R_{\lambda,\mu}$ for some parameters λ, μ. Recall that the representations pace $\mathscr{H}_\pi$ is the closed linear span of the functions $f_n,\ n \in I$, where $f_n(z) = z^n,\ n \in I$, and $I = \mathbb{Z}^+$ in the former case and $I = \mathbb{Z}$ in the latter case. The elements $f_n,\ n \in I$, form a complete orthogonal set of vectors in $\mathscr{H}_\pi$ but these vectors are (in general) not unit vectors. Their norms are as given in (3.9) (depending on π). Since (by Theorem 5.1 and Lemma 2.2) T is a weighted shift with respect to the orthonormal basis of $\mathscr{H}_\pi$ obtained by normalising f_n's, there are scalars $a_n \neq 0,\ n \in I$, such that

$$Tf_n = a_n f_{n+1}, \quad n \in I.$$

Notice that since the f_n's are (in general) not normalised, the numbers a_n are not the weights of the weighted shift T. These weights are given by $w_n := a_n \|f_{n+1}\| / \|f_n\|,\ n \in I$. It follows that the adjoint T^* acts by

$$T^* f_n = \frac{\|f_n\|^2}{\|f_{n-1}\|^2} \bar{a}_{n-1} f_{n-1}, \quad n \in I,$$

where one puts $a_{-1} = 0$ in case $I = \mathbb{Z}^+$.

Let M be the multiplication operator on $\mathscr{H}_\pi$ defined by $Mf_n = f_{n+1},\ n \in I$. Notice that for each multiplier representation $\pi = R_{\lambda,\mu}$ in List 3.1, the corresponding operator M is in List 4.1. Also, in case M is invertible, M^{*-1} is also in the latter list.

Let β be a fixed but arbitrary element of $\mathbb{D}$, and let $\varphi_\beta := \varphi_{-1,\beta} \in$ Möb (recall the notation in Eq. (1.1)). Note that φ_β is an involution, and this simplifies the computation of $\pi(\varphi_\beta)$ a little bit. Indeed, a straightforward calculation shows that, for $\pi = R_{\lambda,\mu}$, we have

$$\langle \pi(\varphi_\beta) f_m, f_n \rangle = c(-1)^n \bar{\beta}^{n-m} \|f_n\|^2 \sum_{k \geqslant (m-n)^+} C_k(m,n) r^k, \quad 0 \leqslant r \leqslant 1, \tag{5.2}$$

where we have put $r = |\beta|^2$, $c = \varphi'_\beta(0)^{\lambda/2} |\varphi'_\beta(0)|^\mu$ and

$$C_k(m,n) = \binom{-\lambda - \mu - m}{k + n - m} \binom{-\mu + m}{k}.$$

Since π is associated with T, from the defining Eq. (2.3) we have $T\pi(\varphi_\beta)(I - \bar{\beta}T) = \pi(\varphi_\beta)(\beta I - T)$. That is,

$$\bar{\beta} T \pi(\varphi_\beta) T + \beta \pi(\varphi_\beta) = T\pi(\varphi_\beta) + \pi(\varphi_\beta) T.$$

Fix m, n in I. Evaluate each side of the above equation at f_m and take the inner product of the resulting vectors with f_n. We have, for instance, $\langle T\pi(\varphi_\beta) T f_m, f_n \rangle =$

$\langle \pi(\varphi_\beta)Tf_m, T^*f_n \rangle = a_m a_{n-1}||f_n||^2/||f_{n-1}||^2 \langle \pi(\varphi_\beta)f_{m+1}, f_{n-1} \rangle$, and similarly for the other three terms. Now substituting from Eq. (5.2) and cancelling the common factor $c(-1)^{n-1}||f_n||^2\bar{\beta}^{n-m}$, we arrive at the following identity in the indeterminate r:

$$a_m a_{n-1} \sum_{k \geqslant (m-n+2)^+} C_k(m+1, n-1)r^k - \sum_{k \geqslant (m-n)^+} C_k(m,n)r^{k+1}$$
$$= a_{n-1} \sum_{k \geqslant (m-n+1)^+} C_k(m, n-1)r^k - a_m \sum_{k \geqslant (m-n+1)^+} C_k(m+1, n)r^k. \quad (5.3)$$

Taking $m = n$ in Eq. (5.3) and equating the coefficients of r^1, we obtain

$$(n + 1 - \mu)a_n = (n - \mu)a_{n-1} + 1, \quad n \in I. \quad (5.4)$$

First consider the case $\mu = 0$. Then Eq. (5.4) with $n = 0$ yields $a_0 = 1$ and $(n+1)a_n = na_{n-1} + 1, n \geqslant 1$. So we get $a_n = 1$ for all $n \geqslant 0$. Therefore, in case $I = \mathbb{Z}^+$, we get $T = M$, which is in List 4.1—as was to be shown. In case $I = \mathbb{Z}$, Eq. (5.4) with $n = -1$ yields $a_{-2} = 1$ and hence $a_n = 1$ for all $n \leqslant -2$. Thus, we get $a_n = 1$ for all n except for $n = -1$. The value of a_{-1} remains undetermined. Hence, T is unitarily equivalent to the operator B_x of List 4.1 with $x = |a_{-1}|$.

Next consider the case $\mu \neq 0$. Hence, as one sees from List 3.1, $I = \mathbb{Z}$. Also (one sees from List 3.1 that μ is not an integer, and hence) a_n is determined by the recurrence relation (5.4) once we fix the value of a_0. To determine the value of a_0, equate the coefficients of r^0 in Eq. (5.3) in case $m = 0$, $n = 2$. Then we get

$$(a_0 + \lambda + \mu)a_1 = (\lambda + \mu + 1)a_0.$$

Eliminating a_1 between this equation and the case $n = 1$ of Eq. (5.4), we get $(1 - \mu)a_0^2 - (\lambda + 1)a_0 + (\lambda + \mu) = 0$, and hence

$$a_0 = 1 \quad \text{or} \quad a_0 = (\lambda + \mu)/(1 - \mu).$$

In case $a_0 = 1$, we get $a_n = 1$ for all n and hence $T = M$, which is in List 4.1. So let us take $a_0 = (\lambda + \mu)/(1 - \mu)$. Then, by Eq. (5.4), we get $a_n = (n + \lambda + \mu)/(n + 1 - \mu)$, $n \in \mathbb{Z}$. In case π is in the Complementary series, a computation shows that the weights of T now agree with those of M^{*-1} and hence $T = M^{*-1}$, which is in List 4.1, so we are done. On the other hand, if π is in the Principal series, the weights of T become

$$w_n = a_n = \frac{n + (1 + \lambda)/2 + s}{n + (1 + \lambda)/2 - s}, \quad n \in \mathbb{Z} \quad (5.5)$$

with s purely imaginary. Hence $|w_n| = 1$ for all n, so that T is unitarily equivalent to the unweighted shift B. Since B is in List 4.1, we are done. □

Remark 5.1. The proof of Theorem 5.2 shows that with each irreducible Principal series representation $P_{\lambda,s}$ are associated two operators (which coincide iff $s = 0$), namely the unweighted shift B and its unitarily equivalent copy with weight sequence given by (5.5). This surely deserves an explanation, and here it is. Let $U_{\lambda,s} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ be the diagonal unitary given by

$$U_{\lambda,s}f_n = \frac{\Gamma(n + (1+\lambda)/2 - s)}{\Gamma(n + (1+\lambda)/2 + s)} f_n, \quad n \in \mathbb{Z}.$$

Then, as is well known, $U_{\lambda,s}$ intertwines $P_{\lambda,-s}$ and $P_{\lambda,s}$

$$P_{\lambda,-s}U_{\lambda,s} = U_{\lambda,s}P_{\lambda,s}.$$

Since $P_{\lambda,-s}$ is associated with the unweighted bilateral shift B, it then follows that $P_{\lambda,s}$ is associated with the operator $U^*_{\lambda,s}BU_{\lambda,s}$, which is nothing but the weighted shift with weight sequence (5.5).

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