Advances in Mathematics **144**, 1–12 (1999)
Article ID aima.1998.1801, available online at http://www.idealibrary.com on IDEAL®

A Class of Invariant Unitary Operators

Luis Báez-Duarte

Departamento de Matemáticas, Instituto Venezolano de Investigaciones Científicas, Apartado 21827, Caracas 1020-A, Venezuela
E-mail: lbaez@ccs.internet.ve

Received September 11, 1998; accepted September 19, 1998

Let $\mathscr{H} = L_2((0, \infty), dx)$, and $K_\lambda f(x) = f(\lambda x)$, for $\lambda > 0$, $f \in \mathscr{H}$. An *invariant operator* on $\mathscr{H}$ is one commuting with all the K_λ. A *skew root* is a self-adjoint, unitary operator on $\mathscr{H}$ satisfying $T^2 = I$, and $TK_\lambda = K_\lambda^* T$, for all $\lambda > 0$. A *generator* g is an element of $\mathscr{H}$ such that the smallest, closed subspace containing $\{K_\lambda\, g\}_{\lambda > 0}$ is equal to $\mathscr{H}$. We show that for any skew root T and any real-valued generator g there is a unique, invariant, unitary operator W satisfying $Wg = Tg$. It turns out that $W^{-1} = TWT$. This construction is related to an approximation problem in $\mathscr{H}$ arising from a theorem due to A. Beurling (1955, *Proc. Nat. Acad. Sci. U.S.A.* **41**, 312–314) and B. Nyman (1950, "On Some Groups and Semigroups of Translations," Thesis, Uppsala) which shows the Riemann hypothesis is equivalent to a closure problem in Hilbert space.

Key Words: unitary operators; Riemann zeta function; Riemann hypothesis; Beurling–Nyman theorem; Hardy operator.

1. INTRODUCTION

Let $\mathscr{H}$ be the complex Hilbert space $L_2((0, \infty), dx)$. The operator $X_{1/2}$ given by

$$X_{1/2}\, f(x) = x^{1/2} f(x) \tag{1.1}$$

is an invertible isometry between $\mathscr{H}$ and $L_2((0, \infty), dx/x)$. The Fourier, or Fourier–Mellin, transform $\mathscr{F}$ on the multiplicative group $(0, \infty)$ with Haar measure dx/x is given by

$$(\mathscr{F} f)(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{it} f(x)\, \frac{dx}{x}, \tag{1.2}$$

valid for $f \in \mathscr{H}$ as $\int_{1/T}^{T} \xrightarrow{\mathscr{H}} \int_0^\infty$ when $T \to \infty$. Plancherel's theorem says that $\mathscr{F}$ is an invertible isometry between $L_2((0, \infty), dx/x)$ and $L_2((-\infty, \infty), dt)$. The inverse transform is given by

$$(\mathscr{F}^{-1} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{-it} f(t)\, dt, \tag{1.3}$$

0001-8708/99 $30.00

valid for $f \in \mathcal{H}$ as $\int_{-T}^{T} \xrightarrow{\mathcal{H}} \int_{-\infty}^{\infty}$ when $T \to \infty$. We define the Mellin transform $\mathcal{F}_{\mathcal{H}}$ on $\mathcal{H}$ by $\mathcal{F}_{\mathcal{H}} = \mathcal{F} X_{1/2}$.

The contraction/dilation operators K_λ, $\lambda > 0$, are defined on $\mathcal{H}$ by

$$K_\lambda f(x) = f(\lambda x). \tag{1.4}$$

Clearly

$$K_\lambda^* = \frac{1}{\lambda} K_{1/\lambda}, \tag{1.5}$$

so that $\{\sqrt{\lambda}\, K_\lambda\}_{\lambda > 0}$ is an abelian, unitary group. Note that

$$\mathcal{F}_{\mathcal{H}} K_\lambda f(t) = \lambda^{-(1/2 + it)} \mathcal{F}_{\mathcal{H}} f(t). \tag{1.6}$$

A function g in $\mathcal{H}$ is said to be a *generator* if the linear hull $\mathcal{A}(g)$ of the set of its "translates" $\{K_\lambda g\}_{\lambda > 0}$ is dense in $\mathcal{H}$. By Wiener's L_2-Tauberian theorem (see, for example, [9]) g is a generator if and only if $\mathcal{F}_{\mathcal{H}} g \neq 0$ a.e. This together with (1.6) shows that the $K_\lambda g$ are linearly independent when g is a generator, since for different $\lambda_1, \lambda_2, ..., \lambda_n$

$$\mathcal{F}_{\mathcal{H}} \left(\sum_{j=1}^{n} c_j K_{\lambda_j} g \right)(t) = \sum_{j=1}^{n} c_j \lambda_j^{-(1/2 + it)} \mathcal{F}_{\mathcal{H}}\, g(t), \tag{1.7}$$

is zero a.e. if and only if the entire function of t, $\sum_{j=1}^{n} c_j \lambda_j^{-(1/2+it)} = 0$ a.e.; hence it is identically zero, and taking it instead of t, it is easy to see that each $c_j = 0$. We record this as a proposition for future use.

PROPOSITION 1.1. *If g is a generator then $\{K_\lambda g\}_{\lambda > 0}$ is a vector basis for $\mathcal{A}(g)$.*

Two important generators are $\chi = \chi_{(0,1]}$, the indicator function of the interval $(0, 1]$, and ρ_1, defined by $\rho_1(x) = \rho(1/x)$, where $\rho(x)$ denotes, as in [1], [2], the fractional part of the real number x. The fact that χ is a generator is quite trivial; on the other hand, one can see that ρ_1 is a generator taking $s = \frac{1}{2} + it$ in the identity

$$\frac{\zeta(s)}{-s} = \int_0^\infty x^s \rho\left(\frac{1}{x}\right) \frac{dx}{x} \qquad (0 < \Re s < 1), \tag{1.8}$$

which appears in Titchmarsh's monograph on the zeta function [10, Eq. (2.1.5)]. Now consider $L_2((0, 1), dx)$ as imbedded canonically in $\mathcal{H}$. A minor re-working of a theorem by A. Beurling [2] and B. Nyman [7] yields the following:

THEOREM 1.1. *The Riemann hypothesis is true if and only if the closed subspace generated by the contractions* $\{K_\lambda \rho_1\}_{\lambda>1}$ *contains* $L_2((0,1), dx)$.

Remark 1.1. The necessary and sufficient condition for the Riemann hypothesis in the above theorem may be substituted by the requirement that χ be in the closed subspace generated by the $\{K_\lambda \rho_1\}_{\lambda>1}$. Other functions may also take the place of χ.

Remark 1.2. Note that χ is more than a generator, let's say it is a *strong generator*, in that $\{K_\lambda \chi\}_{\lambda>1}$ generates $L_2((0,1), dx)$. The important question is whether ρ_1 is also a strong generator.

Looking through the prism, as it were, of unitary operators on $\mathscr{H}$ one hopes to achieve a better understanding of the implications of the above theorem. Some results already obtained in this sense in relation with Theorem 1.1 shall appear under separate cover. In principle, bounded invertible operators should be equally useful for the task, in practice, however, one needs to calculate the effect of the operator on $\mathscr{A}(\rho_1)$, which makes it rather convenient to require that it commute with the K_λ, and the quest for such an operator led us naturally to the isometries constructed in Section 2. We shall say that a linear operator W on $\mathscr{H}$ is *invariant* if $WK_\lambda = K_\lambda W$ for all $\lambda>0$, and *skew* if $WK_\lambda = K_\lambda^* W$ for all $\lambda>0$. A *skew root* T shall be a skew, self-adjoint, unitary operator satisfying $T^2 = I$. It is clear that two invariant operators are equal if and only if they coincide for at least one, and hence for all generators. The analogous statement obviously holds true for skew operators. The product of two invariant operators is invariant, while that of two skew operators is invariant. Important skew roots are the inversion or substitution operator S on $\mathscr{H}$ given by

$$Sf(x) = \frac{1}{x} f\left(\frac{1}{x}\right), \tag{1.9}$$

and the sine and cosine tranforms $\mathfrak{S}$, $\mathfrak{C}$ defined on $\mathscr{H}$ by

$$\mathfrak{S}f(x) = 2\int_0^\infty f(t)\sin(2\pi xt)\,dt, \tag{1.10}$$

$$\mathfrak{C}f(x) = 2\int_0^\infty f(t)\cos(2\pi xt)\,dt, \tag{1.11}$$

as principal values $\int_0^T \xrightarrow{\mathscr{H}} \int_0^T$ when $T \to \infty$.

2. CONSTRUCTION OF INVARIANT UNITARY OPERATORS

2.1. *Existence and Uniqueness Theorem.* The main result of this note is the following simple theorem:

THEOREM 2.1. *For any skew root T and any real-valued generator g there is a unique, bounded, invariant operator $W=\mathfrak{W}(T,g)$ with $Wg=Tg$. W is unitary and satisfies $(TW)^2=(WT)^2=I$, $W^{-1}=TWT$.*

Proof. Since $\{K_\lambda g\}_{\lambda>0}$ is a vector basis of $\mathscr{A}(g)$ by Proposition 1.1, W is uniquely determined as a linear operator on $\mathscr{A}(g)$ as follows. Let

$$f=\sum_{j=1}^{n} c_j K_{\lambda_j} g. \tag{2.1}$$

Then,

$$\begin{aligned} Wf &= \sum_{j=1}^{n} c_j K_{\lambda_j} Tg \\ &= T\left(\sum_{j=1}^{n} c_j K^*_{\lambda_j} g\right). \end{aligned} \tag{2.2}$$

Since the K_λ commute and g is real-valued we have

$$\begin{aligned} \left\| \sum_{j=1}^{n} c_j K^*_{\lambda_j} g \right\|^2_{\mathscr{H}} &= \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} \langle K^*_{\lambda_j} g, K^*_{\lambda_k} g \rangle_{\mathscr{H}} \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} \langle K_{\lambda_j} g, K_{\lambda_k} g \rangle_{\mathscr{H}} \\ &= \|f\|^2_{\mathscr{H}}. \end{aligned} \tag{2.3}$$

But T is an isometry, so $\|Wf\|^2_{\mathscr{H}}=\|f\|^2_{\mathscr{H}}$, i.e., W is an isometry on $\mathscr{A}(g)$. Therefore W extends uniquely as an isometry to all of $\mathscr{H}$. Clearly the resulting operator W is invariant, and therefore $(TW)^2$ is also invariant, so to prove the equality $(TW)^2=I$ we need only prove $(TW)^2 g=g$. This is immediate: $TWTWg=TWTTg=TWg=TTg=g$. Therefore $TWTW=I$, and $WTW=T$. Hence also $(WT)^2=(WTW)T=I$, and thus $(TWT)W=W(TWT)=I$, that is, $W^{-1}=TWT$. Thus W is unitary. ∎

Remark 2.1. It might be of some interest to study the formal properties of the functor $\mathfrak{W}$. For example, continuing with the notation employed in the proof of Theorem 2.1, since both WT and TW are new skew roots one should ask what new invariant operators can be constructed from them. It is quite easy to see then that $\mathfrak{W}(WT,g)=W^2$, whereas $\mathfrak{W}(TW,g)=I$.

2.2. *A Discontinuous Möbius Inversion Process.* We shall explore below some natural operators that occur in conjunction with Theorem 1.1. Thus one wishes to apply Theorem 2.1 to construct $W = \mathfrak{W}(T, \rho_1)$ for suitable skew roots T. Once this is done it is reasonable to consider W as a known quantity only after it is possible to ascertain what $W\chi$ is. A paradox arises here. Since both ρ_1 and χ are generators it should in principle be equally feasible to write each one of them as a limit in the sense of $\mathscr{H}$ of finite linear combinations of translates of the other. It is quite clear how one should do this to achieve $\rho_1 = \lim f_n$ with $f_n \in \mathscr{A}(\chi)$. Namely, one begins with

$$\left[\frac{1}{x}\right] = \sum_{j \leqslant 1/x} 1 = \sum_{j=1}^{\infty} \chi_{(0,1]}(jx), \tag{2.4}$$

whence it is quite simple to obtain

$$\rho_1 \chi_{(1/n, n]} = S\chi_{(1/n, n]} - \sum_{j=1}^{n} j\chi_{(1/j+1, 1/j]} \xrightarrow{\mathscr{H}} \rho_1. \tag{2.5}$$

On the other hand, however, there is no known constructible, explicit sequence $f_n \in \mathscr{A}(\rho_1)$ satisfying $f_n \xrightarrow{\mathscr{H}} \chi$. A heuristic way out is this: one finds the Möbius inverse of the relation (2.4), namely

$$\chi_{(0,1]}(x) = \sum_{j=1}^{\infty} \mu(j) \left[\frac{1}{jx}\right], \tag{2.6}$$

then writing $[1/x] = 1/x - \rho_1(x)$ in the above equation, and using the well-known identity

$$\sum_{j=1}^{\infty} \frac{\mu(j)}{j} = 0, \tag{2.7}$$

one obtains

$$-\chi = \sum_{j=1}^{\infty} \mu(j)\, K_j \rho_1, \tag{2.8}$$

which is true pointwise and even in $L_1((0,1), dx)$ (see [1]), but is definitely not true as an equation in $\mathscr{H}$. This is a new result which we shall prove below (Theorem 2.2). Nevertheless, one may, as we shall do immediately below, use (2.8) formally to ascertain what $W\chi$ should be, then use the inverse path (2.5) to actually establish the equality.

2.3. *Examples of Invariant Unitary Operators.* From the preceding discussion it is quite natural to start with the operator

$$U = \mathfrak{W}(S, \rho_1). \tag{2.9}$$

Note that

$$U\rho_1(x) = \frac{1}{x}\,\rho(x). \tag{2.10}$$

With help from this operator one immediately obtains what was announced in the previous subsection:

THEOREM 2.2. *The series defined in Eq.* (2.8) *diverges in* $\mathcal{H}$.

Proof. Take the partial sum $s_n = \sum_{j=1}^{n} \mu(j)\, K_j\, \rho_1$ of the series on the right of Eq. (2.8) and apply U to obtain

$$Us_n(x) = \frac{1}{x} \sum_{j=1}^{n} \frac{\mu(j)}{j}\, \rho(jx). \tag{2.11}$$

Thus, on the interval $(0, 1/n)$ we have $Us_n(x) \equiv \sum_{j=1}^{n} \mu(j)$, so that

$$\int_0^{1/n} |Us_n(x)|^2\, dx = \frac{1}{n} \left(\sum_{j=1}^{n} \mu(j) \right)^2, \tag{2.12}$$

which does not go to zero as $n \to \infty$ according to a theorem by Titchmarsh (see [10, Theorem 14.26(B)]). This is impossible if s_n converges in $\mathcal{H}$. ∎

Remark 2.2. From Odlyzko's disproof of the Merten's hypothesis [8] it is easy to see that there is a subsequence $n_k \to \infty$ such that $\sum_{j=1}^{n_k} \mu(j) = 0$, which gives support to the hope, not shared by the author at this point, that at least a subsequence of s_n may still converge in $\mathcal{H}$ to χ.

On the other hand the summation on the right side of (2.11) converges uniformly and boundedly to $-(1/\pi) \sin 2\pi x$, a fact proven long ago by H. Davenport [3, 4]. The identification of the limit is very easy to see formally as follows: first recall the well-known Fourier series

$$\rho(x) - \frac{1}{2} = \sum_{\nu=1}^{\infty} \frac{\sin 2\pi \nu x}{-\pi \nu}, \tag{2.13}$$

then use the identity (2.7) together with the fundamental relation for the Möbius coefficients to write

$$\sum_{j=1}^{\infty} \frac{\mu(j)}{j} \rho(jx) = -\frac{1}{\pi} \sum_{j=1}^{\infty} \sum_{v=1}^{\infty} \frac{\mu(j)}{jv} \sin(2\pi jvx)$$
$$= -\frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{j|m} \mu(j) \frac{\sin(2\pi mx)}{m}$$
$$= -\frac{1}{\pi} \sin 2\pi x. \tag{2.14}$$

Now, *if* (2.8) were true in the sense of $\mathscr{H}$ along a subsequence of partial sums, a statement that would imply the Riemann hypothesis, then a further subsequence of the partial sums would have to converge a.e., and one would be forced to admit that

$$U\chi(x) = \frac{\sin 2\pi x}{\pi x}. \tag{2.15}$$

Now, despite the shaky allegations, it is a fact that this equation is true, and furthermore that, having recognized on its right-hand side the cosine transform of χ leads us, by our main Theorem 2.1, to surmise, and prove something stronger, namely,

THEOREM 2.3.

$$U = \mathfrak{W}(\mathfrak{C}, \chi). \tag{2.16}$$

LEMMA 2.1. *If f is a continuous function of compact support defined in* $(0, \infty)$, *then*

$$\mathfrak{W}(\mathfrak{C}, \chi)\, f(x) = \int_0^{\infty} f(\lambda) \frac{d}{d\lambda} \frac{\sin 2\pi x/\lambda}{\pi x/\lambda}\, d\lambda. \tag{2.17}$$

Remark 2.3. This lemma is considerably strengthened below in Corollary 2.1.

Proof of Lemma 2.1. The integral representation in (2.17) is clearly true when f is the characteristic function of an interval, *a fortiori* when f is a step function. The rest is a trivial application of uniform convergence. ∎

Proof of Theorem 2.3. Let $U_1 = \mathfrak{W}(\mathfrak{C}, \chi)$. According to the existence and uniqueness Theorem 2.1 the desired conclusion would follow from

$$U_1 \rho_1 = S\rho_1. \tag{2.18}$$

To prove this we bring in the limit relation (2.5). Applying U_1 to its left-hand side, then using Lemma 2.1, and integration by parts, we get

$$U_1(\rho_1\chi_{(1/n,\,n]})(x)=\frac{1}{x}\sum_{j=1}^{n}\frac{\sin(2\pi jx)}{-\pi j}+\frac{1}{\pi x}\int_{1/n}^{n}\frac{\sin(2\pi x\lambda)}{\lambda}\,d\lambda+\frac{1}{\pi x}\sin\frac{2\pi x}{n}. \tag{2.19}$$

It is enough to take the pointwise limit as $n\to\infty$ of this assuredly $\mathscr{H}$-convergent expression to obtain, using the series expansion (2.13) and the classical integral $\int_0^\infty \sin x\,dx/x=\pi/2$, the expected limit $(1/x)\,\rho(x)$. ▮

The dual characterization of the invariant operator

$$U=\mathfrak{W}(S,\rho_1)=\mathfrak{W}(\mathfrak{C},\chi) \tag{2.20}$$

allows us to prove some further interesting facts about it. In the first place we have

$$U^{-1}=SUS=\mathfrak{C}U\mathfrak{C}. \tag{2.21}$$

Now we relate U to Hardy's well known averaging operator $\mathfrak{M}$ defined by

$$\mathfrak{M}f(x)=\frac{1}{x}\int_0^x f(t)\,dt. \tag{2.22}$$

This is a linear bounded operator in $L_p((0,\infty),dx)$, but here we see it only as an operator on $\mathscr{H}$. Clearly $\mathfrak{M}$ is a bounded invariant operator. We now factorize U as follows.

THEOREM 2.4.

$$U=(\mathfrak{M}-I)\,\mathfrak{C}S. \tag{2.23}$$

COROLLARY 2.1. *For any $f\in\mathscr{H}$ we have the following integral representation,*

$$Uf(x)=\int_0^\infty f(\lambda)\,\frac{d}{d\lambda}\,\frac{\sin 2\pi x/\lambda}{\pi x/\lambda}\,d\lambda, \tag{2.24}$$

understood this sense: $\int_{1/T}^\infty \xrightarrow{\mathscr{H}} \int_0^\infty$ *when* $T\to\infty$.

Proof of Theorem 2.4. A simple computation yields

$$(\mathfrak{M}-I)\chi=S\chi, \tag{2.25}$$

then, applying Theorem 2.3 we obtain $US(\mathfrak{M}-I)\chi=\mathfrak{C}\chi$, and, as both sides are skew, also $US(\mathfrak{M}-I)=\mathfrak{C}$. Left multiplying by S the preceding equation, taking into account (2.21), we obtain $U^{-1}(\mathfrak{M}-I)=S\mathfrak{C}$. Now just use the fact that both S and $\mathfrak{C}$ are its own inverses. ▮

Proof of Corollary 2.1. We shall calculate $Uf(x)$ according to the factorization equation (2.23). Since $Sf\in\mathscr{H}$ we have for a.e. t

$$\mathfrak{C}Sf(t)=2\int_0^\infty \frac{1}{u} f\left(\frac{1}{u}\right)\cos(2\pi tu)\,du, \tag{2.26}$$

as a Cauchy principal value $\lim_{T\to\infty}\int_0^T$, this limit taking place in $\mathscr{H}$, thus also in L_1 of the finite interval $(0, x)$. It is easy then to justify the following interchange of iterated integrations

$$\begin{aligned}\mathfrak{M}\mathfrak{C}Sf(x)&=\frac{2}{x}\int_0^x\int_0^\infty \frac{1}{u} f\left(\frac{1}{u}\right)\cos(2\pi tu)\,du\,dt\\&=\frac{1}{x}\int_0^\infty \frac{1}{u} f\left(\frac{1}{u}\right)\frac{\sin(2\pi xu)}{\pi u}\,du.\end{aligned}$$

Therefore

$$\begin{aligned}(\mathfrak{M}-I)\,\mathfrak{C}Sf(x)&=\frac{1}{x}\int_0^\infty \frac{1}{u} f\left(\frac{1}{u}\right)\left(\frac{\sin(2\pi xu)}{\pi u}-2\cos(2\pi xu)\right)du\\&=\int_0^\infty f(u)\left(\frac{\sin(2\pi x/u)}{\pi x}-\frac{\cos(2\pi x/u)}{xu}\right)du,\end{aligned}$$

which is equal to the right side of (2.24). ▮

As an immediate consequence of the factorization Theorem 2.4 we see that $\mathfrak{M}-I$ is a unitary operator. But we can very easily get more information about this interesting operator, since we can identify it as our construct $\mathfrak{W}(S,\chi)$.

THEOREM 2.5. *The unitary operator $\mathfrak{M}-I$ satisfies the relations*

$$\mathfrak{M}-I=\mathfrak{W}(S,\chi), \tag{2.27}$$

$$(\mathfrak{M}-I)^{-1}=S(\mathfrak{M}-I)\,S=\mathfrak{C}(\mathfrak{M}-I)\,\mathfrak{C}, \tag{2.28}$$

$$(\mathfrak{M}-I)\;\rho_1(x)=\frac{1}{x}\left(-\gamma+\log x+\sum_{j\leqslant 1/x}\frac{1}{j}\right), \tag{2.29}$$

where γ is Euler's constant.

Remark 2.4. The first expression for the inverse in (2.28) means that the following two relations are equivalent for f and h in $\mathscr{H}$:

$$h(x) = f(x) - \frac{1}{x}\int_0^x f(t)\,dt, \tag{2.30}$$

$$f(x) = h(x) - \int_x^\infty h(t)\,\frac{dt}{t}. \tag{2.31}$$

R. J. Duffin had already proven this fact in [6] by direct analytic methods. The second expression for the inverse seems to be new.

Proof of Theorem 2.5. Since $\mathfrak{M} - I$ is an invariant operator, $\mathfrak{M} - I = \mathfrak{W}(S, \chi)$ follows from our main Theorem 2.1 in view of (2.25). This also yields the first expression for its inverse. The second expression follows from the factorization (2.23) of U, and from $U^{-1} = SUS$, namely,

$$(\mathfrak{M} - I)^{-1} = \mathfrak{C} S U^{-1} \tag{2.32}$$

$$= \mathfrak{C} U S$$

$$= \mathfrak{C}(\mathfrak{M} - I)\mathfrak{C}. \tag{2.33}$$

We shall omit the proof of the last statement of the theorem. It is a direct calculation based on (2.5). ∎

A further, rather interesting characterization of U related to the Riemann zeta function is obtained from the fundamental identity (1.8). It gives the *diagonalization* of U as follows:

THEOREM 2.6. *Let T_η be the operator of multiplication by the unimodular function η given by*

$$\eta(t) = \frac{1/2 + it}{1/2 - it}\,\frac{\zeta(1/2 - it)}{\zeta(1/2 + it)}$$

$$= -\frac{1}{\pi}(2\pi)^{1/2 - it}\left(\frac{1}{2} - it\right)\sin\left(\frac{\pi}{2}\left(\frac{1}{2} - it\right)\right)\Gamma\left(-\frac{1}{2} + it\right). \tag{2.34}$$

Then

$$U = \mathscr{F}_{\mathscr{H}}^{-1}\, T_\eta\, \mathscr{F}_{\mathscr{H}}. \tag{2.35}$$

Proof. The right-hand side of (2.35) is easily seen to be an invariant unitary operator. Then we check that both sides give the same result when applied to the generator ρ_1, which is equivalent to showing instead that

$\mathscr{F}_{\mathscr{H}} U\rho_1 = T_\eta \mathscr{F}_{\mathscr{H}} \rho_1$. But this last equation is nothing but a rewriting of the fundamental identity (1.8) defining Riemann's zeta function in the critical strip. ∎

Remark 2.5. The second expression for $\eta(t)$ arises from the functional equation for the Riemann zeta function (see, for example, [10]).

Remark 2.6. The above theorem connects the additive Fourier transform $\mathfrak{C}$ with the multiplicative Fourier transform $\mathscr{F}$.

It should be interesting to determine the *type set* of U (see [5]), that is, for which pairs $(1/p_1, 1/p_2)$ it is bounded as an operator from $L_{p_1}((0, \infty), dx)$ to $L_{p_2}((0, \infty), dx)$. One conjectures that the type set is simply $\{(\frac{1}{2}, \frac{1}{2})\}$.

A final example is given by $V = \mathfrak{W}(\mathfrak{S}, \chi)$. The following theorem shows it is, on the one hand, very much like its cousin U, as far as its factorization is concerned, yet with a much more complicated value for $V\rho_1$.

THEOREM 2.7. *Let* $V = \mathfrak{W}(\mathfrak{S}, \chi)$. *Then*

$$V\chi(x) = \frac{2}{\pi x} \sin^2 \pi x, \tag{2.36}$$

$$V\rho_1(x) = -\frac{1}{\pi x} \log \frac{|\sin \pi x|}{\pi x}, \tag{2.37}$$

$$V = (\mathfrak{M} - I)\, \mathfrak{S} S. \tag{2.38}$$

We omit a formal proof, but note the following: The calculation for $V\chi$ is trivial, while that for $V\rho_1$ is an involved, subjectively interesting calculation using (2.5). On the other hand the factorization for V is not as simple to prove as the corresponding one (2.23) for U, as we are missing, so far, a dual relation such as (2.20).

ACKNOWLEDGMENTS

The author expresses his thanks to M. Balazard, A. Octavio, and E. Saias for very useful conversations.

REFERENCES

1. L. Báez-Duarte, On Beurling's real variable reformulation of the Riemann hypothesis, *Adv Math.* **101**, No. 1 (1993), 10–30.
2. A. Beurling, A closure problem related to the Riemann zeta-function, *Proc. Nat. Acad. Sci. U.S.A.* **41** (1955), 312–314.

3. H. Davenport, On some infinite series involving arithmetical functions, *Quart. J. Math. Oxford Ser.* **8** (1937), 8–13.
4. H. Davenport, On some infinite series involving arithmetical functions, II, *Quart. J. Math. Oxford Ser.* **8** (1937), 313–320.
5. P. Donoghue, "Distributions and Fourier Transforms," Pure and Applied Mathematics, Vol. 32, Academic Press, San Diego, 1969.
6. R. J. Duffin, Some simple unitary transformations, *Ann. of Math.* **55** (1952), 531–537.
7. B. Nyman, "On Some Groups and Semigroups of Translations," Thesis, Uppsala, 1950.
8. A. M. Odlyzko and H. te Riele, Disproof of Mertens conjecture, *J. Reine Angew. Math.* **357** (1985), 138–160.
9. W. Rudin, "Fourier Analysis in Groups," Tracts in Pure and Applied Mathematics, Vol. 12, Interscience, New York, 1962.
10. E. C. Titchmarsh, "The Theory of the Riemann–Zeta Function," Clarendon, Oxford, 1951.