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SOME REFINEMENTS OF ENESTROM-KAKEYA THEOREM

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Abstract. In this paper we present certain interesting refinements of a well-known Enestrom-Kakeya theorem in the theory of distribution of zeros of polynomials which among other things also improve upon some results of Aziz and Mohammad, Govil and Rehman and others.

Key words: *polynomial, modulius of zeros,* Enestrom-Kakeya *theorem*
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1 Introduction and Statement of Results

The following sesult due to Enestrom and Kakeya[8] is well known in the theory of the distribution of zeros of polynomials.

Theorem A. *If*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

is a polynomial of degree n, such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_0 > 0,$$

then $p(z)$ does not vanish in $|z| > 1$.

There already exist in the literature[2–4,7] some extensions of Enestron-Kakeya theorem. Govil and Rehman[6, Theorems 2,4] generalized this theorem to polynomials with complex coefficients, first by considering the moduli of the coefficients to the monotonically increasing and then by assuming the real parts of the coefficients to be monotonically increasing.

Recently, Aziz and Zargar[3] obtained the following generalization of Theorem A.

Theorem B. *If*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

is a polynomial of degree n, such that for some $k \geq 1$

$$ka_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0 > 0,$$

then $p(z)$ has all its zeros in

$$|z+k-1| \le k.$$

Aziz and Mohammad[1] used an old theorem of Gereshgorian (see also [9]) and proved among other things the following generalization of Enestrom-Kakeya Theorem.

Theorem C. *If*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

is a polynomial of degree n with complex coefficients such that

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \qquad j = 0, 1, 2, \cdots, n$$

for some real β and for some $t > 0$,

$$0 < t^n |a_n| \le t^{n-1} |a_{n-1}| \le \cdots \le t^k |a_k|,$$
$$t^k |a_k| \ge t^{k-1} |a_{k-1}| \ge \cdots \ge t|a_1| \ge |a_0| > 0,$$

where $0 \le k \le n$, then $p(z)$ has all its zeros in the circle

$$|z| \le t \left\{ \left(\frac{2t^k |a_k|}{t^n |a_n|} - 1 \right) \cos\alpha + \sin\alpha \right\} + 2\sin\alpha \sum_{j=0}^{n} \frac{|a_j|}{|a_n| t^{n-j-1}}. \tag{1}$$

For $k = n, \alpha = \beta = 0$ and $t = 1$ this reduces to Theorem A. Also for $k = n$ and this reduces to a result proved by Govil and Rehman [6, Theorem 2].

Theorem D. *If*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

is a polynomial of degree n and if

$$\begin{aligned} &\text{Re } a_j = \alpha_j, \quad \text{Im } a_j = \beta_j, \quad j = 0, 1, 2, \cdots, n \quad \text{and for some } t > 0, \\ &0 < t^n \alpha_n \le t^{n-1} \alpha_{n-1} \le \cdots \le t^k \alpha_k, \\ &t^k |a_k| \ge t^{k-1} \alpha_{k-1} \ge \cdots \ge t\alpha_1 \ge \alpha_0 > 0, \end{aligned} \tag{2}$$

where $0 \le k \le n$, then all the zeros of $P(z)$ lie in the circle

$$|z| \le t \left(\frac{2t^k \alpha_k}{t^n \alpha_n} - 1 \right) + \frac{2}{\alpha_n} \sum_{j=0}^{n} \frac{|\beta_j|}{t^{n-j-1}}. \tag{3}$$

In this paper, we shall first present the following result which among other things provides interesting refinements of Theorem C for $0 \le k \le n-2$.

Theorem 1. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial of degree $n \geq 2$ with complex coefficients. If for some real β

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2} \quad for \quad 0 \leq j \leq n$$

and for some $t > 0$

$$\begin{aligned} &0 < |a_0| \leq t|a_1| \leq \cdots \leq t^{k-1}|a_{k-1}| \leq t^k|a_k|, \\ &t^k|a_k| \geq t^{k+1}|a_{k+1}| \geq \cdots \geq t^{n-1}|a_{n-1}| \geq t^n|a_n| > 0, \end{aligned} \tag{4}$$

where $0 \leq k \leq n-2$, then all the zeros of $P(z)$ lie in the circle

$$\left|z + \frac{a_{n-1}}{a_n} - t\right| \leq \frac{t}{|a_n|}\left\{\frac{2|a_k|\cos\alpha}{t^{n-k}} - \frac{|a_{n-1}|}{t}(\cos\alpha + \sin\alpha)\right\} + 2\sin\alpha \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|t^{n-j-1}}. \tag{5}$$

Remark 1. In general Theorem 1 gives much better result than Theorem A. To see that Theorem 1 is an improvement of Theorem C for $0 \leq k \leq n-2$, we show that the circle defined by (5) is contained in the circle defined by (1). Let $z = w$ be any point belonging to the circle defined by (5) then

$$\left|w + \frac{a_{n-1}}{a_n} - t\right| \leq \frac{t}{|a_n|}\left\{\frac{2|a_k|\cos\alpha}{t^{n-k}} - \frac{|a_{n-1}|}{t}(\cos\alpha + \sin\alpha)\right\} + 2\sin\alpha \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|t^{n-j-1}}.$$

This implies

$$\begin{aligned} |w| &= \left|w + \frac{a_{n-1}}{a_n} - t + t - \frac{a_{n-1}}{a_n}\right| \leq \left|w + \frac{a_{n-1}}{a_n} - t\right| + \left|t - \frac{a_{n-1}}{a_n}\right| \\ &\leq \frac{|ta_n - a_{n-1}|}{|a_n|} + \frac{t}{|a_n|}\left(\frac{(2|a_k|\cos\alpha)}{t^{n-k}} - \frac{|a_{n-1}|}{t}(\cos\alpha + \sin\alpha)\right) + 2\sin\alpha \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|t^{n-j-1}}, \end{aligned} \tag{6}$$

since

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \qquad j = 0, 1, 2, \cdots, n.$$

It can be easily verified (see Lemma) that

$$\begin{aligned} |ta_n - a_{n-1}| &\leq |t|a_n| - |a_{n-1}||\cos\alpha + (t|a_n| + |a_{n-1}|)\sin\alpha \\ &= [|a_{n-1}| - t|a_n|]\cos\alpha + [t|a_n| + |a_{n-1}|]\sin\alpha \quad \text{(by (4))} \end{aligned}$$

using this in (6) we have

$$\begin{aligned} |w| &\leq \left[\frac{|a_{n-1}|}{|a_n|} - t\right]\cos\alpha + \left[t + \frac{|a_{n-1}|}{|a_n|}\right]\sin\alpha + t\left[\frac{2t^k|a_k|\cos\alpha}{t^n|a_n|}\right] \\ &\quad - \frac{|a_{n-1}|}{|a_n|}(\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|t^{n-j-1}} \\ &= t\left\{\left[\frac{2t^k|a_k|}{t^n|a_n|} - 1\right]\cos\alpha + \sin\alpha\right\} + 2\sin\alpha \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|t^{n-j-1}}. \end{aligned}$$

This shows that the point $z = w$ belongs to the circle defined by (1). Hence the circle defined by (5) is contained in the circle defined by (1).

Next we present the following result which considerably improves upon the Theorem for $0 \leq k \leq n-1$.

Theorem 2. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial of degree n with complex coefficients. If $\mathrm{Re}a_j = \alpha_j$ *and* $\mathrm{Im}a_j = \beta_j$ *for* $j = 0, 1, 2, \cdots, n$ *and for some* $t > 0$,

$$\begin{aligned} &0 < t^n \alpha_n \leq t^{n-1} \alpha_{n-1} \leq \cdots \leq t^k \alpha_k, \\ &t^k |a_k| \geq t^{k-1} \alpha_{k-1} \geq \cdots \geq t\alpha_1 \geq \alpha_0 > 0, \end{aligned} \tag{7}$$

where $0 \leq k \leq n-1$, *then all the zeros of* $P(z)$ *lie in the circle*

$$\left| z + \frac{\alpha_{n-1} - t\alpha_n}{a_n} \right| \leq \frac{1}{|a_n|} \left\{ \left(\frac{2\alpha_k t^{k+1}}{t^n} - \alpha_{n-1} \right) + t|\beta_n| + \frac{2}{t^{n-1}} \sum_{j=0}^{n-1} |\beta_j| t^j \right\}. \tag{8}$$

Remark 2. To verify that Theorem 2 is also an improvement of Theorem B for $0 \leq k \leq n-1$, again let $z = w$ be any point belonging to the circle defined by (8), then we have

$$\left| w + \frac{\alpha_{n-1} - t\alpha_n}{a_n} \right| \leq \frac{1}{|a_n|} \left\{ \left(\frac{2\alpha_k}{t^{n-k-1}} - \alpha_{n-1} \right) + t|\beta_n| + \frac{2}{t^{n-1}} \sum_{j=0}^{n-1} |\beta_j| t^j \right\}.$$

This implies

$$\begin{aligned} |w| &\leq \frac{1}{|a_n|} \left\{ |\alpha_{n-1} - t\alpha_n| + \frac{2\alpha_k}{t^{n-k-1}} - \alpha_{n-1} + t|\beta_n| + 2 \sum_{j=0}^{n-1} \frac{|\beta_j|}{t^{n-j-1}} \right\} \\ &\leq \frac{1}{|\alpha_n|} \left\{ |\alpha_{n-1} - t\alpha_n| + \frac{2\alpha_k}{t^{n-k-1}} - \alpha_{n-1} + t|\beta_n| + 2 \sum_{j=0}^{n-1} \frac{|\beta_j|}{t^{n-j-1}} \right\} \\ &= \frac{1}{|\alpha_n|} \left\{ \alpha_{n-1} - t\alpha_n + \frac{2\alpha_k}{t^{n-k-1}} - \alpha_{n-1} + t|\beta_n| + 2 \sum_{j=0}^{n-1} \frac{|\beta_j|}{t^{n-j-1}} \right\} \qquad \text{(by (7))} \\ &\leq t \left(\frac{2\alpha_k}{\alpha_n t^{n-k}} - 1 \right) + \frac{1}{\alpha_n} \left[t|\beta_n| + 2 \sum_{j=0}^{n-1} \frac{|\beta_j|}{t^{n-j-1}} \right] \\ &\leq t \left(\frac{2t^k \alpha_k}{\alpha_n t^n} - 1 \right) + \frac{2}{\alpha_n} \sum_{j=0}^{n} \frac{|\beta_j|}{t^{n-j-1}}. \end{aligned}$$

Hence the point $z = w$ belongs to the circle defined by (3) and therefore, the circle defined by (8) is contained in the circle defined by (3).

We need the following lemma to prove Theorems 1 and 2 which follows immediately from the lemma due to Aziz and Mohammad[1].

Lemma. *Let*

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$$

be a polynomial of degree n, if for some real β.

$$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2},\quad a_j\ne 0,\quad j=0,1,2,\cdots,n,$$

then for each $t>0$,

$$|ta_j-a_{j-1}|\le|t|a_j|-|a_{j-1}||\cos\alpha+[t|a_j|+|a_{j-1}|]\sin\alpha.$$

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned}F(z)&=(t-z)p(z)\\&=(t-z)(a_nz^n+a_{n-1}z^{n-1}+\cdots+a_{k+1}z^{k+1}+a_kz^k+a_{k-1}z^{k-1}+\cdots+a_1z+a_0)\\&=-a_nz^{n+1}+(ta_n-a_{n-1})z^n+\cdots+(ta_{k+1}-a_k)z^{k+1}+(ta_k-a_{k-1})z^k\\&\quad+(ta_{k-1}-a_{k-2})z^{k-1}+\cdots+(ta_2-a_1)z^2+(ta_1-a_0)z+ta_0\\&=-a_nz^{n+1}+(ta_n-a_{n-1})z^n+\sum_{j=0}^{n-1}(ta_j-a_{j-1})z^j,\qquad(a_{-1}=0).\end{aligned}$$

Let $|z|>t$, then

$$\begin{aligned}|F(z)|&\ge|a_n||z^n|\left(\left|z+\frac{a_{n-1}}{a_n}-t\right|-\frac{1}{|a_n|}\sum_{j=0}^{n-1}|ta_j-a_{j-1}|\frac{1}{|z|^{n-j}}\right)\\&>|a_n||z|^n\left(\left|z+\frac{a_{n-1}}{a_n}-t\right|-\frac{1}{|a_n|}\sum_{j=0}^{n-1}|ta_j-a_{j-1}|\frac{1}{t^{n-j}}\right)\\&=|a_n||z|^n\left(\left|z+\frac{a_{n-1}}{a_n}-t\right|-\frac{1}{|a_n|t^n}\sum_{j=0}^{n-1}|ta_j-a_{j-1}|t^j\right).\end{aligned}\tag{9}$$

Now by the Lemma after a short calculation, we have

$$\begin{aligned}\sum_{j=0}^{n-1}|ta_j-a_{j-1}|t^j&\le\sum_{j=0}^{n-1}|t|a_j|-|a_{j-1}||t^j\cos\alpha+\sum_{j=0}^{n-1}(t|a_j|+|a_{j-1}|)t^j\sin\alpha\\&=\sum_{j=0}^{k}|t|a_j|-|a_{j-1}||t^j\cos\alpha+\sum_{j=k+1}^{n-1}(|a_{j-1}|-t|a_j|)t^j\cos\alpha\end{aligned}$$

$$+\sum_{j=0}^{n-1}(t|a_j|+|a_{j-1}|)t^j\sin\alpha \qquad (0\le k\le n-1)$$

$$= 2|a_k|t^{k+1}\cos\alpha - t^n|a_{n-1}|\cos\alpha - t^n|a_{n-1}|\sin\alpha + 2t\sin\alpha\sum_{j=0}^{n-1}|a_j|t^j$$

$$= 2|a_k|t^{k+1}\cos\alpha - t^n|a_{n-1}|(\cos\alpha+\sin\alpha) + 2t\sin\alpha\sum_{j=0}^{n-1}|a_j|t^j.$$

Hence

$$\frac{1}{|a_n|t^n}\sum_{j=0}^{n-1}|ta_j-a_{j-1}|t^j \le \frac{t}{|a_n|}\left\{\frac{2|a_k|\cos\alpha}{t^{n-k}} - \frac{|a_{n-1}|}{t}(\cos\alpha+\sin\alpha)\right\} + 2\sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}.$$

Using this in (9), we obtain for $|z|>t$,

$$|F(z)| \ge |a_n||z|^n\left[\left|z+\frac{a_{n-1}}{a_n}-t\right|\right.$$
$$\left.-\frac{t}{|a_n|}\left\{\frac{2|a_k|\cos\alpha}{t^{n-k}} - \frac{|a_{n-1}|}{t}(\cos\alpha+\sin\alpha)\right\} - 2\sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}\right] > 0,$$

whenever

$$\left|z+\frac{a_{n-1}}{a_n}-t\right| > \frac{t}{|a_n|}\left\{\frac{2|a_k|\cos\alpha}{t^{n-k}} - \frac{|a_{n-1}|}{t}(\cos\alpha+\sin\alpha)\right\} + 2\sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}.$$

Hence all the zeros of $F(z)$ whose modulus is greater than t lie in the circle

$$\left|z+\frac{a_{n-1}}{a_n}-t\right| \le \frac{t}{|a_n|}\left\{\frac{2|a_k|\cos\alpha}{t^{n-k}} - \frac{|a_{n-1}|}{t}(\cos\alpha+\sin\alpha)\right\} + 2\sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}.$$

We now show that all those zeros of $F(z)$ whose modulus is less than or equal to t also satisfy (5) for $0\le k\le n-2$. Let $|z|\le t$, then by using the above Lemma and the hypothesis (4), we get

$$\begin{aligned}
\left|z+\frac{a_{n-1}}{a_n}-t\right| &\le t+\frac{|ta_n-a_{n-1}|}{|a_n|}\\
&\le t+\left(\left|\frac{a_{n-1}}{a_n}\right|-t\right)\cos\alpha+\left(t+\left|\frac{a_{n-1}}{a_n}\right|\right)\sin\alpha\\
&\le \frac{t}{|a_n|}\left\{\frac{2|a_k|\cos\alpha}{t^{n-k}} - \frac{|a_{n-1}|}{t}(\cos\alpha+\sin\alpha)\right\} + 2\sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}
\end{aligned}$$

whenever

$$t(1-\cos\alpha)+t\sin\alpha+2\left|\frac{a_{n-1}}{a_n}\right|\cos\alpha \le 2\left|\frac{a_k}{a_n}\right|\frac{1}{t^{n-k-1}}\cos\alpha+2\sin\alpha\sum_{j=0}^{n-2}\left|\frac{a_j}{a_n}\right|\frac{t^{j+1}}{t^n}. \qquad (10)$$

Since by (4),

$$t^n|a_{n-1}| \le |a_k|t^{k+1}$$

it follows that (10) is true, if

$$t(1-\cos\alpha)+t\sin\alpha \le 2\sin\alpha\sum_{j=0}^{n-2}\left|\frac{a_j}{a_n}\right|\frac{t^{j+1}}{t^n}. \qquad (11)$$

Again using (4) and noting that $0 \le k \le n-2$, we get

$$\begin{aligned}\sum_{j=0}^{n-2}\left|\frac{a_j}{a_n}\right|\frac{t^{j+1}}{t^n} &= \sum_{j=0}^{k-1}\left|\frac{a_j}{a_n}\right|\frac{t^{j+1}}{t^n}+\sum_{j=k}^{n-2}\left|\frac{a_j}{a_n}\right|\frac{t^{j+1}}{t^n} \ge \sum_{j=k}^{n-2}\left|\frac{a_j}{a_n}\right|\frac{t^{j+1}}{t^n} \\ &= t\sum_{j=k}^{n-2}\left|\frac{a_j}{a_n}\right|\frac{t^j}{t^n} \ge t\sum_{j=k}^{n-2}1 = t(n-k-1) \ge t.\end{aligned}$$

Hence (11) holds, whenever

$$t(1-\cos\alpha)+t\sin\alpha \le 2t\sin\alpha, \qquad 0 \le \alpha \le \frac{\pi}{2}$$

or

$$\cos\alpha+\sin\alpha \ge 1, \qquad \text{where} \quad 0 \le \alpha \le \frac{\pi}{2}. \qquad (12)$$

Note that when $0 \le \alpha \le \frac{\pi}{2}, \cos\alpha+\sin\alpha = \sqrt{2}\sin\left(\alpha+\frac{\pi}{4}\right) \ge \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1$, so (12) holds.

Thus we have shown that if $|z| \le t$, then for $0 \le k \le n-2$

$$\left|z+\frac{a_{n-1}}{a_n}-t\right| \le \frac{t}{|a_n|}\left\{\frac{2|a_k|\cos\alpha}{t^{n-k}}-\frac{|a_{n-1}|}{t}(\cos\alpha+\sin\alpha)\right\}+2\sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}.$$

Hence all the zeros of $F(z)$ lie in the circle defined by (5). But all the zeros of $P(z)$ are also the zeros of $F(z)$, we conclude that all the zeros of $P(z)$ lie in the circle defined by (5). This proves the Theorem 1.

Proof of Theorem 2. Consider the polynomial

$$\begin{aligned}F(z) &= (t-z)P(z) \\ &= -a_nz^{n+1}+(ta_n-a_{n-1})z^n+(ta_{n-1}-a_{n-2})z^{n-1}+\cdots+(ta_{k+1}-a_k)z^{k+1} \\ &\quad +(ta_k-a_{k-1})z^k+\cdots+(ta_2-a_1)z^2+(ta_1-a_0)z+ta_0 \\ &= -a_nz^{n+1}+(ta_n-a_{n-1})z^n+\sum_{j=0}^{n-1}(ta_j-a_{j-1})z^j, \qquad (a_{-1}=0).\end{aligned}$$

Let $|z| > t$, then we have

$$\begin{aligned} |F(z)| &\geq |z|^n \left\{ |a_n z + a_{n-1} - t a_n| - \sum_{j=0}^{n-1} |t a_j - a_{j-1}| \frac{1}{|z|^{n-j}} \right\} \\ &\geq |z|^n \left\{ |a_n z + \alpha_{n-1} - t\alpha_n| - |\beta_{n-1}| - t|\beta_n| - \sum_{j=0}^{n-1} |t a_j - a_{j-1}| \frac{1}{t^{n-j}} \right\}. \end{aligned} \tag{13}$$

Now by the hypothesis,

$$\begin{aligned} \sum_{j=0}^{n-1} |t a_j - a_{j-1}| t^j &\leq \sum_{j=0}^{n-1} |t\alpha_j - \alpha_{j-1}| t^j + \sum_{j=0}^{n-1} (|\beta_{j-1}| + t|\beta_j|)\, t^j \\ &= \sum_{j=0}^{k} |t\alpha_j - \alpha_{j-1}| t^j + \sum_{j=k+1}^{n-1} |t\alpha_j - \alpha_{j-1}| t^j + \sum_{j=0}^{n-1} (|\beta_{j-1}| + t|\beta_j|)\, t^j \\ &\leq t\alpha_0 + (t\alpha_1 - \alpha_0)t + \cdots + (t\alpha_{k-1} - \alpha_{k-2})t^{k-1} \\ &\quad + (t\alpha_k - \alpha_{k-1})t^k + (\alpha_k - t\alpha_{k+1})t^{k+1} \\ &\quad + \cdots + (\alpha_{n-2} - t\alpha_{n-1})t^{n-1} + t|\beta_0| + (|\beta_0| + t|\beta_1|)\,t + (|\beta_1| + t|\beta_2|)\, t^2 \\ &\quad + \cdots + (|\beta_{n-3}| + t|\beta_{n-2})\, t^{n-2} + (|\beta_{n-2}| + t|\beta_{n-1}|)\, t^{n-1} \\ &= 2\alpha_k t^{k+1} - \alpha_{n-1} t^n + 2t \sum_{j=0}^{n-1} |\beta_j| t^j - |\beta_{n-1}| t^n. \end{aligned}$$

Using this in (13), we get for $|z| > t$,

$$|F(z)| \geq |z|^n \left\{ |a_n z + \alpha_{n-1} - t\alpha_n| - \frac{2\alpha_k t^{k+1}}{t^n} - \alpha_{n-1} - \frac{2t}{t^n} \sum_{j=0}^{n-1} |\beta_j| t^j - |\beta_n| t \right\} > 0,$$

whenever

$$|a_n z + \alpha_{n-1} - t\alpha_n| > \frac{2\alpha_k t^{k+1}}{t^n} + (t|\beta_n| - \alpha_{n-1}) + \frac{2}{t^{n-1}} \sum_{j=0}^{n-1} |\beta_j| t^j.$$

Hence all the zeros of $F(z)$ whose modulus is greater than t lie in the circle

$$\left| z + \frac{\alpha_{n-1} - t\alpha_n}{a_n} \right| \leq \frac{1}{|a_n|} \left\{ \left(\frac{2\alpha_k t^{k+1}}{t^n} - \alpha_{n-1} \right) + t|\beta_n| + \frac{2}{t^{n-1}} \sum_{j=0}^{n-1} |\beta_j| t^j \right\}. \tag{14}$$

If $|z| \le t$, then we have

$$\begin{aligned}
|a_n z + \alpha_{n-1} - t\alpha_n| &\le |a_n|t + |\alpha_{n-1} - t\alpha_n| \\
&\le t\alpha_n + |\beta_n|t + \alpha_{n-1} - t\alpha_n \qquad \text{(by (7))} \\
&= t|\beta_n| + \alpha_{n-1} = t|\beta_n| + 2\alpha_{n-1} - \alpha_{n-1} \\
&\le t|\beta_n| + \frac{2t^{k+1}}{t^n}\alpha_k - \alpha_{n-1} \qquad \text{(by (7))} \\
&\le \frac{2t^{k+1}}{t^n}\alpha_k - \alpha_{n-1} + t|\beta_n| + \frac{2}{t^{n-1}}\sum_{j=0}^{n-1}|\beta_j|t^j.
\end{aligned}$$

This shows that all ther zeros of $F(z)$ whose modulus is less than or equal to t also satisfy the inequality (14). Thus we conclude that all the zeros of $F(z)$ lie in the circle defined by (14). Since all the zeros of $P(z)$ are also the zeros of $F(z)$, this completes the proof of Theorem 2.

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