

Bounds for the Zeros of a Polynomial with Restricted Coefficients

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ABSTRACT

In this paper we shall obtain some interesting extensions and generalizations of a well-known theorem due to Enestrom and Takeya according to which all the zeros of a polynomial $P(z) = a_n z^n + \dots + a_1 z + a_0$ satisfying the restriction $a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 > 0$ lie in the closed unit disk.

Keywords: Polynomial; Bounds; Zeros

1. Introduction and Statement of Results

The following results which is due to Enestrom and Takeya [1] is well known in the theory of the location of the zeros of polynomials.

THEOREM A. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

be a polynomial of degree n , such that

$$a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 > 0, \quad (1)$$

then $P(z)$ does not vanish in $|z| > 1$.

In the literature [2-5] there exist some extensions and generalization of Enestrom-Takeya Theorem. Joyal, Labelle and Rahman [6] extended this theorem to polynomials whose coefficients are monotonic but not necessarily non-negative by proving the following result:

THEOREM B. Let

$$a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0$$

then the polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

of degree n has all its zeros in

$$|z| \leq \frac{1}{|a_n|} \{a_n - a_0 + |a_0|\} \quad (2)$$

Recently Aziz and Zargar [7] relaxed the hypothesis in several ways and among other things proved the following results:

THEOREM C. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

be the polynomial of degree n , such that for some $k \geq 1$,

$$k a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 > 0 \quad (3)$$

then $P(z)$ has all its zeros in

$$|z + k - 1| \leq k \quad (4)$$

The aim of this paper is to prove some extensions of Enestrom-Takeya Theorem (Theorem-A) by relaxing the hypothesis in various ways. Here we shall first prove the following generalization of Theorem C which is an interesting extension of Theorem A.

2. Main Results

THEOREM 1.1. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

be a polynomial of degree n . If for some positive numbers k and ρ with $k \geq 1$, and $0 < \rho \leq 1$

$$k a_n \geq a_{n-1} \geq \dots \geq \rho a_0 \geq 0 \quad (5)$$

then all the zeros of $P(z)$ lie in the closed disk

$$|z + k - 1| \leq k + \frac{2a_0}{a_n} (1 - \rho) \quad (6)$$

If we take $k = \frac{a_{n-1}}{a_n} \geq 1$, in Theorem 1.1 we obtain the following result which is a generalization of Corollary 2 ([7]).

COROLLARY 1. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

be a polynomials of degree n . If for some positive real

number $\rho, 0 < \rho \leq 1$

$$a_n \leq a_{n-1} \geq a_{n-2} \geq \dots \geq \rho a_0 > 0 \tag{7}$$

then all zeros of $P(z)$ lie in

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{a_{n-1}}{a_n} + \frac{2a_0}{a_n} (1 - \rho)$$

REMARK 1. Theorem 1.1 is applicable to situations when Enestrom-Kakeya Theorem gives no information. To see this consider the polynomial.

$$P(z) = \alpha z^n + (\alpha - 1)z^{n-1} + \dots + (\alpha - 1)z + \alpha,$$

with $\alpha > 1$ is a positive real number. Here Enestrom-Kakeya Theorem is not applicable to $P(z)$ where as Theorem 1.1 is applicable with $k = 1, \rho = \frac{n-1}{n}$ and according to our result, all the zeros of $P(z)$ lie in the disk.

$$|z| \leq 1 + \frac{1}{\alpha}, \alpha > 1.$$

which is considerably better than the bound obtained by a classical result of Cauchy ([4]) which states that all the zeros of $P(z)$ lie in

$$|z| \leq 1 + A$$

where

$$A = \max_{1 \leq j \leq n} \left| \frac{a_{n-j}}{a_n} \right|,$$

Next, we present the following generalization of corollary 1 which includes Theorem 4 of [6] as a special case and considerably improves the bound obtained by Dewan and Bidkham ([8], Theorem1) for $t = 0$ and $0 \leq k \leq n - 1$.

THEOREM 1.2. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

be a polynomial of degree n . If for some positive number $\rho, 0 < \rho \leq 1$ and for some non-negative integer $\lambda, 0 \leq \lambda \leq n - 1$

$$a_n \leq a_{n-1} \leq \dots \leq a_{\lambda+1} \leq a_\lambda \geq a_{\lambda-1} \geq \dots \geq a_1 \geq \rho a_0, \tag{8}$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|} \{ 2a_\lambda - a_{n-1} + (2 - \rho)|a_0| - \rho a_0 \} \tag{9}$$

Applying Theorem 1.2 to $P(tz)$, we get the following result:

$$\begin{aligned} F(z) &= (1 - z)P(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + \dots + (a_1 - a_0)z + a_0, \\ &= -a_n z^{n+1} + a_n z^n - k a_n z^n + (k a_n - a_{n-1})z^n + \dots + (a_1 - \rho a_0)z + (\rho - 1)a_0 z + a_0 \end{aligned}$$

COROLLARY 2. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

be a polynomial of degree n . If for some positive numbers t and ρ with $0 < \rho \leq 1$,

$$t^n a_n \leq t^{n-1} a_{n-1} \leq \dots \leq t^\lambda a_\lambda \geq \dots \geq t a_1 \geq t \rho a_0$$

where $\lambda, 0 \leq \lambda \leq n - 1$ is a non negative integer then all the zeros of $P(z)$, lie in

$$\left| z + \frac{a_{n-1}}{a_n} - t \right| \leq \frac{t}{|a_n|} \left\{ \left(\frac{2a_\lambda}{t^{n-\lambda}} - \frac{a_{n-1}}{t} \right) + \frac{1}{t^n} ((2 - \rho)|a_0| + a_0) \right\}$$

If we assume $a_0 > 0$, in Theorem 1.2, we obtain.

COROLLARY 3. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

be a polynomial of degree n . If for some positive numbers $\rho, 0 < \rho < 1$ and for some non-negative integer $\lambda, 0 \leq \lambda \leq n - 1$

$$a_n \leq a_{n-1} \leq \dots \leq a_\lambda \geq \dots \geq a_1 \geq \rho a_0 > 0$$

then all the zeros of $P(z)$ lie in,

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{a_n} (2a_\lambda - a_{n-1} + 2(1 - \rho)a_0) \tag{10}$$

Finally we present all following generalization of Theorem B due to Joyal, Labelle and Rahman which includes Theorem A as a special case.

THEOREM 1.3. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

be a polynomial of degree n , It for some positive number $\rho, 0 < \rho \leq 1$ and for some non-negative integer $\lambda, 0 \leq \lambda \leq n - 1$

$$a_n \leq a_{n-1} \leq \dots \leq a_\lambda \geq \dots \geq a_1 \geq \rho a_0$$

then all the zeros of $P(z)$ lie in

$$|z| \leq \frac{2a_\lambda - a_n + (2 - \rho)|a_0| + \rho a_0}{|a_n|} \tag{11}$$

REMARK 2. For $\rho = 1$, Theorem 1.3 reduces to Theorem B.

3. Proofs of the Theorems

PROOF OF THEOREM 1.1. Consider

then for $|z| > 1$, we have

$$\begin{aligned}
 |F(z)| &= \left| -a_n z^{n+1} + a_n z^n - ka_n z^n + (ka_n - a_{n-1})z^n + \dots + (a_1 - \rho a_0)z + (\rho - 1)a_0 z + a_0 \right| \\
 &\geq |a_n| |z|^n \left[|z+k-1| - \frac{1}{|a_n|} \left\{ (ka_n - a_{n-1}) + (a_{n-1} - a_{n-2}) \frac{1}{z} + \dots + (a_0 - \rho a_0)z(\rho - 1) \frac{a_0}{z^n} + \frac{a_0}{z^n} \right\} \right] \\
 &\geq |a_n| |z|^n \left[|z+k-1| - \frac{1}{|a_n|} \{ ka_n - \rho a_0 + (1-\rho)|a_0| + |a_0| \} \right] \\
 &= |a_n| |z|^n \left[|z+k-1| - \frac{1}{a_n} \{ ka_n - \rho a_0 + a_0 + (1-\rho)a_0 \} \right] > 0, \\
 \text{if } |z+k-1| &> \frac{ka_n + 2(1-\rho)a_0}{a_n}
 \end{aligned}$$

this shows that if $|z| > 1$ then $|F(z)| > 0$, if

$$|z+k-1| > k + 2(1-\rho) \frac{a_0}{a_n}$$

therefore all the zeros of $F(z)$, whose modulus is greater than 1 lie in the closed disk

$$|z+k-1| \leq k + 2(1-\rho) \frac{a_0}{a_n}$$

But those zeros of $F(z)$ whose modules is less than or equal to 1 already satisfy the Inequality (6).

Since all the zeros of $P(z)$ are also the zeros of $F(z)$. therefore it follows that all the zeros of $P(z)$ lie in the circle defined by (6). Which completes the proof of Theorem 1.1.

PROOF OF THEOREM 1.2. Consider

$$\begin{aligned}
 F(z) &= (1-z)P(z) \\
 &= (1-z)(a_n z^n + a_{n-1} z^{n-1} + \dots + a_\lambda z^\lambda + \dots + a_1 z + a_0), \\
 &= -a_n z^{n+1} + (a_{n-\lambda} a_{n-1})z^n + \dots + (a_{\lambda-1} - a_\lambda)z^{\lambda+1} + (a_\lambda - z_{\lambda-1})z^\lambda + \dots + a_1 z - a_0 z + a_0
 \end{aligned}$$

therefore, for $|z| > 1, \leq \lambda \leq n-1$, and $0 < \rho < 1$, we have

$$\begin{aligned}
 |F(z)| &\geq \left| a_n z^{n+1} - a_n z^n + a_{n-1} z^n - (a_{n-1} - a_{n-2})z^{n-1} + \dots + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda \right. \\
 &\quad \left. + \dots + (a_1 - \rho a_0)z + (\rho a_0 - a_0)z + a_0 \right| \\
 &\geq |a_n| |z|^n \left[\left| z + \frac{a_{n-1}}{a_n} - 1 \right| - |z|^n \left\{ |a_{n-1} - a_{n-2}| \frac{1}{|z|} + \dots + |a_{\lambda-1} - a_\lambda| \frac{1}{|z|^{n-\lambda-1}} + |a_{\lambda-1} - a_\lambda| \frac{1}{|z|^{n-\lambda}} \right. \right. \\
 &\quad \left. \left. + \dots + |a_1 - \rho a_0| \frac{1}{|z|^{n-1}} + |1-\rho| |a_0| \frac{1}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \right\} \right] \\
 &> |a_n| |z|^n \left[\left| z + \frac{a_{n-1}}{a_n} - 1 \right| - \frac{1}{|a_n|} \{ (a_{n-2} - a_{n-1}) + (a_{n-3} - a_{n-2}) + (a_\lambda - a_{\lambda+1}) \right. \right. \\
 &\quad \left. \left. + (a_\lambda - a_{\lambda-1}) + \dots + (a_1 - \rho a_0) + (1-\rho)|a_0| + |a_0| \} \right] \\
 &= |a_n| |z|^n \left[\left| z + \frac{a_{n-1}}{a_n} - 1 \right| - \frac{1}{|a_n|} \{ 2a_\lambda - a_{n-1} + (2-\rho)|a_0| - \rho a_0 \} \right] > 0, \\
 \text{if } \left| z + \frac{a_{n-1}}{a_n} - 1 \right| &> \frac{2a_\lambda - a_{n-1} + (2-\rho)|a_0| - \rho a_0}{a_n}
 \end{aligned}$$

Therefore all the zeros of $F(z)$ whose modulus is greater than 1 lie in the circle.

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{2a_\lambda - a_{n-1} + (2-\rho)|a_0| - \rho a_0}{a_n}$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the Inequality (9).

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= (1-z)(a_n z^n + a_{n-1} z^{n-1} + \dots + a_\lambda z^\lambda + \dots + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \dots + (a_{\lambda+1} a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + \dots + a_1 z - a_0 z + a_0 \end{aligned}$$

therefore, for $|z| > 1$, $0 \leq \lambda \leq n-1$ and $0 < \rho \leq 1$, we have

$$|F(z)| \geq |a_n z^{n+1}| - |(a_n - a_{n-1})z^n + \dots + (a_\lambda - a_{\lambda-1})z^\lambda + a_1 z - a_0 z + a_0|$$

Proceeding similarly as in the proof of Theorem 1.2, we have

$$|F(z)| > \frac{|a_n||z|^n \{ |z| - 2a_\lambda - a_n + (2-\rho)|a_0| - \rho a_0 \}}{a_n} > 0,$$

$$\text{if } |z| > \frac{2a_\lambda - a_n + (2-\rho)|a_0| + \rho a_0}{|a_n|}$$

therefore all the zeros of $F(z)$ whose modules is greater than 1 lie in the circle

$$|z| \leq \frac{(2a_\lambda - a_n) + (2-\rho)|a_0| + \rho a_0}{a_n}$$

But those zeros of $F(z)$ whose modulus is ≤ 1 already satisfy the (11). Since all the zeros of $P(z)$ are also the zero of $F(z)$, therefore it follows that all the zeros of $P(z)$ lie in circle defined by (11) and hence Theorem 1.3 is proved completed.

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Since all the zeros of $P(z)$ are also the zeros of $F(z)$, therefore it follows that all the zeros of $P(z)$ lie in the circle defined by (9). This completes the proof of Theorem 1.2.

PROOF OF THEOREM 1.3. Consider

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