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On the Zeros of a Certain Class of Polynomials and Related Analytic Functions

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1. The following well-known result is due to Enestrom and Kakeya [6].

THEOREM A. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n such that*

$$a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_1 \geqslant a_0 \geqslant 0,$$

then all the zeros of $P(z)$ *lie in* $|z| \leqslant 1$.

From this theorem one can easily deduce

THEOREM B. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n with real and positive coefficients, then all the zeros of* $P(z)$ *lie in*

$$|z| \leqslant t = \operatorname{Max}(a_0/a_1, a_1/a_2, \ldots, a_{n-1}/a_n).$$

It is clear that Theorem B gives better information regarding the zeros of $P(z)$ if the coefficients are strictly decreasing, i.e. if $a_n > a_{n-1} > \cdots > a_1 > a_0 > 0$, because then $t < 1$. In the literature [1–3, 5] there exist some extensions of the Enestrom–Kakeya theorem. In this paper we generalize Theorems A and B by using Schwarz' lemma and thereby give an independent proof of Theorem B. Also, we show that the polynomial $P(z)$ of Theorem A cannot have a zero of order $\geqslant 2$ of modulus $\geqslant n/n+1$. W also obtain certain zero-free regions of a polynomial $P(z)$ having nonnegative coefficients, and as a consequence present a refinement of the Enestrom–Kakeya theorem. Finally we study the zeros of a class of analytic functions.

We start by proving the following:

THEOREM 1. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real positive coefficients. If* $t_1 > t_2 \geqslant 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geqslant 0, \qquad r = 1, 2, \ldots, n+1, \quad (a_{-1} = a_{n+1} = 0),$$

then all the zeros of $P(z)$ *lie in* $|z| \leqslant t_1$.

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For $t_2 = 0$, this reduces to Theorem B, and for $t_1 = 1$, $t_2 = 0$, this reduces to the Enestrom–Kakeya theorem.

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned} F(z) &= (t_2 + z)(t_1 - z) P(z) \\ &= (t_1 t_2 + (t_1 - t_2) z - z^2)(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0) \\ &= -a_n z^{n+2} + (a_n(t_1 - t_2) - a_{n-1}) z^{n+1} + (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}) z^n \\ &\quad + \cdots + (a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0) z^2 + (a_1 t_1 t_2 + a_0(t_1 - t_2)) z + a_0 t_1 t_2 . \end{aligned}$$

We have

$$\begin{aligned} G(z) &= z^{n+2} F(1/z) \\ &= -a_n + (a_n(t_1 - t_2) - a_{n-1}) z + (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}) z^2 + \cdots \\ &\quad + (a_1 t_1 t_2 + a_0(t_1 - t_2)) z^{n+1} + a_0 t_1 t_2 z^{n+2} \\ &= -a_n + \hat{D}(z) \qquad \text{(say)}. \end{aligned}$$

Then $\hat{D}(0) = 0$ and $\operatorname{Max}_{|z|=1/t_1} |\hat{D}(z)| = \hat{D}(1/t_1) = a_n$. Therefore, by Schwarz' lemma,

$$|\hat{D}(z)| \leqslant a_n |z| t_1 \qquad \text{for } |z| \leqslant 1/t_1 .$$

Hence

$$|G(z)| \geqslant a_n - |\hat{D}(z)| \geqslant a_n(1 - |z| t_1) \qquad \text{for } |z| \leqslant 1/t_1 .$$

Thus in $|z| \leqslant 1/t_1$, $|G(z)| > 0$ if $|z| < 1/t_1$.

Consequently, all the zeros of $G(z)$ lie in $|z| \geqslant 1/t_1$. As $F(z) = z^{n+2}G(1/z)$, we conclude that all the zeros of $F(z)$, and hence, all the zeros of $P(z)$ lie in $|z| \leqslant t_1$. This proves the theorem.

THEOREM 2. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_1 \geqslant a_0 > 0;$$

then $P(z)$ cannot have a zero of order $\geqslant 2$ of modulus $\geqslant n/n + 1$. In other words, all the zeros of $P(z)$ of modulus $\geqslant n/n + 1$ are simple.

Proof of Theorem 2. We have $P'(z) = n a_n z^{n-1} + (n - 1) a_{n-1} z^{n-2} + \cdots + a_1$. By Theorem B, all the zeros of $P'(z)$ lie in

$$\begin{aligned} |z| &\leqslant \operatorname{Max}(a_1/2a_2, 2a_2/3a_3, \ldots, (n - 1) a_{n-1}/n a_n) \\ &\leqslant \operatorname{Max}(\tfrac{1}{2}, \tfrac{2}{3}, \ldots, (n - 1)/n) \qquad \text{(By hypothesis)} \\ &= (n - 1)/n. \end{aligned}$$

This implies that all the zeros of $P(z)$ of modulus $>(n-1)/n$ are simple. Since $n/(n+1) > (n-1)/n$, the conclusion follows immediately.

COROLLARY 1. *All the zeros of $P(z)$ of Theorem 2 of unit modulus are simple.*

Next we prove

THEOREM 3. *Let $P(z) = \sum_{j=1} a_j z^j$ be a polynomial of degree $n \geqslant 1$, such that $a_j \geqslant 0, j = 0, 1,..., n-1$ and $a_n > 0$. Then for every real $a > 0$, $P(z)$ does not vanish in the circle $|z - a| < a/n$.*

For the proof of this theorem we need the following lemma [7] which is an immediate consequence of Bernstein's theorem on the derivative of a trigonometric polynomial.

LEMMA. *Let $P(z)$ be a polynomial of degree $n \geqslant 1$; then*

$$\operatorname*{Max}_{|z|=R} |P'(z)| \leqslant (n/R) \operatorname*{Max}_{|z|=R} |P(z)| .$$

Proof of Theorem 3. Let a be a positive real number; then it is clear that $P^k(a) > 0$, $k = 0, 1, 2,..., n$. Consider the polynomial

$$\begin{aligned} F(z) &= P\left(\frac{az}{n} + a\right) \\ &= P(a) + (az/n)\,P'(a) + (a/n)^2 \frac{P''(a)}{2!} z^2 + \cdots + (a/n)^n \frac{P^n(a)}{n!} z^n, \end{aligned}$$

so that

$$\begin{aligned} G(z) = z^n F(1/z) &= P(a)\,z^n + (a/n)\,P'(a)\,z^{n-1} + \cdots + (a/n)^n \frac{P^n(a)}{n!} \\ &= \sum_{k=0}^{n} (a/n)^k \frac{P^k(a)}{k!} z^{n-k}. \end{aligned}$$

Then $G(z)$ has real and positive coefficients. Now $P^k(z)$, $k = 0, 1,..., n-1$ is a polynomial of degree $(n-k)$ with real and nonnegative coefficients. Therefore, applying the lemma above to the polynomial $P^k(z)$ we get

$$\operatorname*{Max}_{|z|=a} |P^{k+1}(z)| \leqslant (n-k)/a \operatorname*{Max}_{|z|=a} |P^k(z)| , \qquad k = 0, 1, 2,..., n-1.$$

That is,

$$\begin{aligned} P^{k+1}(a) &\leqslant \frac{n-k}{a} P^k(a), \qquad k = 0, 1, 2,..., n-1 \\ &\leqslant \frac{n(1+k)}{a} P^k(a) \end{aligned}$$

or

$$(a/n)^{k+1} \frac{P^{k+1}(a)}{(k+1)!} \leqslant (a/n)^k \frac{P^k(a)}{k!}, \qquad k = 0, 1,..., n-1.$$

This shows that the polynomial $G(z)$ satisfies the conditions of Theorem A, and therefore it follows that all the zeros of $G(z)$ lie in $|z| \leqslant 1$. Since $G(z) = z^n F(1/z)$, we conclude that all the zeros of $F(z)$ lie in $|z| \geqslant 1$. That is, $F(z) = P((a/n) z + a)$ does not vanish in $|z| < 1$. Replacing z by $(n/a)(z - a)$, it follows that $P(z)$ does not vanish in $|z - a| < a/n$. This completes the proof of the theorem.

As an immediate consequence of Theorem 3, we present the following refinement of the Enestrom–Kakeya theorem:

COROLLARY 2. *If* $a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_0 \geqslant 0$, *then the polynomial* $P(z) = \sum_{j=0}^{n} a_j z^j \not\equiv 0$ *has all its zeros in the region*

(i) $|z| \leqslant 1 \bigcap |z - a| \geqslant a/n$, *where* $0 < a \leqslant 1$;

(ii) $|z| \leqslant 1 \bigcap_{k=1}^{m} |z - 1/k| \geqslant 1/kn$, where m is a finite positive integer.

THEOREM 4. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* $n \geqslant 1$ *such that* $a_j \geqslant 0, j = 0, 1,..., n-1$, *and* $a_n > 0$. *If* $a > 0$ *can be found such that* $nP(a) \geqslant 2aP'(a)$, *then* $P(z)$ *does not vanish in* $|z - a| \leqslant 2a/n$.

Proof of Theorem 4. Consider the polynomial

$$F(z) = P((2a/n) z + a) = \sum_{k=0}^{n} (2a/n)^k \frac{P^k(a)}{k!} z^k, \qquad a > 0,$$

so that

$$G(z) = z^n F(1/z) = \sum_{k=0}^{n} (2a/n)^k \frac{P^k(a)}{k!} z^{n-k}.$$

Clearly $G(z)$ is a polynomial with real and positive coefficients. Since $P^k(z)$, $k = 0, 1,..., n-1$ is a polynomial of degree $(n-k)$ with nonnegative coefficients, therefore, as in the proof of Theorem 3, we have

$$P^{k+1}(a) \leqslant (n-k)/a \, P^k(a), \qquad k = 0, 1,..., n-1.$$

Now for $k = 1, 2,..., n-1$

$$P^k(a) \geqslant \frac{a}{n-k} P^{k+1}(a) \geqslant (2a/n) \frac{P^{k+1}(a)}{k+1}.$$

Also by hypothesis, $P(a) \geqslant (2a/n) P'(a)$; therefore,

$$(2a/n)^k \frac{P^k(a)}{k!} \geqslant (2a/n)^{k+1} \frac{P^{k+1}(a)}{(k+1)!}, \qquad k = 0, 1, 2,..., n-1.$$

Hence the polynomial $G(z)$ satisfies the conditions of Theorem A and therefore, as a consequence, all the zeros of $G(z)$ lie in $|z| \leqslant 1$. As $G(z) = z^n F(1/z)$ we conclude that $F(z) = P((2a/n) z + a)$ does not vanish in $|z| < 1$. Replacing z by $(n/2a)(z - a)$, it follows that $P(z)$ does not vanish in $|z - a| < 2a/n$, which is the desired result.

It was conjectured by Erdös and proved by Lax [4], that if $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then $\text{Max}_{|z|=1} |P'(z)| \leqslant (n/2) \text{Max}_{|z|=1} |P(z)|$. Applying this result to the polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ $a_j \geqslant 0$, $j = 0, 1,..., n - 1$ and $a_n > 0$, which does not vanish in $|z| < 1$, we get $P'(1) \leqslant (n/2) P(1)$ or $nP(1) \geqslant 2P'(1)$. Keeping this in mind, the following corollary can be easily deduced from Theorem 4.

Corollary 3. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with non-negative coefficients, all of whose zeros lie in $|z| \geqslant 1$. Then $P(z)$ does not vanish in $|z - 1| < 2/n$.*

If $a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n > 0$, then it can be easily deduced from Theorem A that all the zeros of $P(z) = \sum_{j=0}^{n} a_j z^j$ lie in $|z| \geqslant 1$. Then we have the following:

Corollary 4. *If $a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n > 0$, then all the zeros of $P(z) = \sum_{j=0}^{n} a_j z^j$ lie in $|z| \geqslant 1 \cap |z - 1| \geqslant 2/n$.*

2. We now turn to the study of the zeros of a class of analytic functions. We prove

Theorem 5. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ be analytic in $|z| \leqslant t$. If*

$$a_j > 0 \quad \text{and} \quad a_{j-1} - t a_j \geqslant 0, \quad j = 1, 2, 3,... \tag{1}$$

then $f(z)$ does not vanish in $|z| < t$.

Proof of Theorem 5. It is easy to observe that $\lim_{k\to\infty} a_k t^k = 0$. Now consider the function

$$\begin{aligned} F(z) &= (z - t) f(z) = (z - t)(a_0 + a_1 z + a_2 z^2 + \cdots) \\ &= -ta_0 + z_j \sum_{j=1}^{\infty} (a_{,-1} - ta_j) z^{j-1} \\ &= -ta_0 + G(z), \qquad \text{(say)}. \end{aligned}$$

Then $G(0) = 0$ and by (1) $\text{Max}_{|z|=t} |G(z)| = t \sum_{j=1}^{\infty} (a_{j-1} - ta_j) t^{j-1} = ta_0$. Therefore, by Schwarz' lemma $|G(z)| \leqslant a_0 |z|$ for $|z| \leqslant t$. Hence $|F(z)| = |-ta_0 + G(z)| \geqslant ta_0 - |G(z)| \geqslant a_0(t - |z|)$ for $|z| \leqslant t$. Thus in $|z| \leqslant t$, $|F(z)| > 0$ if $|z| < t$. Consequently, $f(z)$ does not vanish in $|z| < t$. This proves the theorem.

THEOREM 6. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ be analytic in $|z| \leqslant t$. If $|\arg a_j| \leqslant \alpha \leqslant \frac{1}{2}/2$, $j = 0, 1, 2,\ldots$ and for some finite nonnegative integer k, $|a_0| \leqslant t|a_1| \leqslant \cdots \leqslant t^k |a_k| \geqslant t^{k+1}|a_{k+1}| \geqslant \cdots$, then $f(z)$ does not vanish in*

$$|z| < t\Big/\left((2t^k |a_k/a_0| - 1)\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_0|}\sum_1^{\infty} t^j |a_j|\right).$$

Proof of Theorem 6. It is obvious that $\lim_{j\to\infty} t^j a_j = 0$. As before, we consider the function

$$\begin{aligned} F(z) = (z-t)f(z) &= -ta_0 + z\sum_{j=1}^{\infty}(a_{j-1} - ta_j)z^{j-1} \\ &= -ta_0 + G(z), \qquad \text{(say)}. \end{aligned} \tag{2}$$

Since $|\arg a_j| \leqslant \alpha \leqslant \frac{1}{2}/2$, $j = 0, 1, 2,\ldots$, it can be easily verified that $|ta_j - a_{j-1}| \leqslant |t|a_j| - |a_{j-1}||\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha$. Hence for $|z| = t$

$$\begin{aligned} |G(z)| &= \left|z\sum_{j=1}^{\infty}(a_{j-1} - ta_j)z^{j-1}\right| \leqslant t\sum_{j=1}^{\infty}|a_{j-1} - ta_j|t^{j-1} \\ &\leqslant t\left(\sum_{j=1}^{\infty}|t|a_j| - |a_{j-1}||t^{j-1}\cos\alpha + \sum_{j=1}^{\infty}(t|a_j| + |a_{j-1}|)t^{j-1}\sin\alpha\right) \\ &= t\left(\cos\alpha\sum_1^k(t|a_j| - |a_{j-1}|)t^{j-1} + \sum_{k+1}^{\infty}(|a_{j-1}| - t|a_j|)t^{j-1}\cos\alpha\right. \\ &\qquad \left. + 2\sin\alpha\sum_{j=1}^{\infty}t^j|a_j| + |a_0|\sin\alpha\right) \\ &= t|a_0|\left((2|a_k/a_0|t^k - 1)\cos\alpha + \text{sid}\,\alpha + \frac{2\sin\alpha}{|a_0|}\sum_1^{\infty}t^j|a_j|\right) \\ &= t|a_0|M, \qquad \text{(say)}. \end{aligned}$$

Since $G(0) = 0$, it follows by Schwarz' lemma that $|G(z)| \leqslant |a_0|M|z|$ for $|z| \leqslant t$. This gives, with the help of (2), $|F(z)| \geqslant ta_0 - |G(z)| \geqslant a_0(t - |z|M)$ for $|z| \leqslant t$. Hence in $|z| \leqslant t$, $|F(z)| > 0$ if $|z| < t/M$. Consequently, $F(z)$, and therefore $f(z)$ does not vanish in $|z| < t/M$, which is equivalent to the desired result.

The following corollary is obtained by taking $k = 0$ in Theorem 6.

COROLLARY 5. *Let $f(z) = \sum_1^{\infty} a_j z^j$ be analytic in $|z| \leqslant t$. If $|\arg a_j| \leqslant \alpha \leqslant \frac{1}{2}/2$, $j = 0, 1, 2,\ldots$ and $|a_0| \geqslant t|a_1| \geqslant t^2|a_2| \geqslant \cdots$, then $f(z)$ does not vanish in*

$$|z| < t\Big/\left(\cos\alpha + \sin\alpha + (2\sin\alpha/|a_0|)\sum_1^{\infty}t^j|a_j|\right).$$

For $t = 1$, this reduces to a result proved by Govil and Rahman [2].

THEOREM 7. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *be analytic in* $|z| \leqslant t$. *If* Re $a_j = \alpha_j$ *and* Im $a_j = \beta_j$, $j = 0, 1, 2, \ldots$, *and for some finite nonnegative integer* k

$$0 < \alpha_0 \leqslant t\alpha_1 \leqslant \cdots \leqslant t^k \alpha_k \geqslant t^{k+1}\alpha_{k+1} \geqslant \cdots, \tag{3}$$

then $f(z)$ *does not vanish in*

$$|z| < t\Big/\left((2\alpha_k/\alpha_0)\, t^k - 1 + (2/\alpha_0) \sum_{j=0}^{\infty} t^j |\beta_j|\right). \tag{4}$$

Proof of Theorem 7. *We have* $\lim_{j\to\infty} \alpha_j t^j = 0$, $\lim_{j\to\infty} \beta_j t^j = 0$. We proceed similarly as in the proof of Theorem 6 and use the fact

$$|a_{j-1} - ta_j| \leqslant |\alpha_{j-1} - t\alpha_j| + |\beta_{j-1}| + t|\beta_j|, \qquad j = 1, 2, \ldots$$

to get for $|z| = t$,

$$\begin{aligned} |G(z)| &\leqslant t \sum_{j=1}^{\infty} |a_{j-1} - ta_j| \, t^{j-1} \\ &\leqslant t\left(\sum_{j=1}^{\infty} |\alpha_{j-1} - t\alpha_j| \, t^{j-1} + \sum_{j=1}^{\infty} (|\beta_{j-1}| + t|\beta_j|)\right) \\ &= t\alpha_0 \left((2\alpha_k/\alpha_0)\, t^k - 1 + (2/\alpha_0) \sum_{0}^{\infty} t^j |\beta_j|\right) = t\alpha_0 B, \qquad \text{(say)}, \end{aligned}$$

so that by Schwarz' lemma $|G(z)| \leqslant \alpha_0 |z| B \leqslant |a_0| \, |z| B$ for $|z| \leqslant t$. Hence from (2) we get as before, $|F(z)| \geqslant |a_0| (t - |z| B)$ for $|z| \leqslant t$. Consequently, in $|z| \leqslant t$, $|F(z)| > 0$ if $|z| < t/B$. Thus $F(z)$ and therefore, $f(z)$ does not vanish in the region defined by (4).

If we take $k = 0$ in Theorem 7, we get the following:

COROLLARY 6. *Let* $f(z) = \sum_{1}^{\infty} a_j z^j$ *be analytic in* $|z| \leqslant t$. *If* $a_j = \alpha_j + i\beta_j$, $j = 0, 1, 2, \ldots$ *and* $0 < \alpha_0 \geqslant \alpha t \geqslant \alpha_2 t^2 \geqslant \cdots$, *then* $f(z)$ *does not vanish in*

$$|z| < t\Big/\left(1 + (2/\alpha_0) \sum_{0}^{\infty} |\beta_j| \, t^j\right).$$

Finally we prove

THEOREM 8. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *be analytic in* $|z| \leqslant t$. *If* $a_j = \alpha_j + i\beta_j$, $j = 0, 1, 2, \ldots$ *and for some finite nonnegative integers* k *and* r $0 < \alpha_0 \leqslant t\alpha_1 \leqslant \cdots$

$\leqslant t^k\alpha_k \geqslant t^{k+1}\alpha_{k+1} \geqslant \cdots$ *and* $\beta_0 \leqslant t\beta_1 \leqslant \cdots \leqslant t^r\beta_r \geqslant t^{r+1}\beta_{r+1} \geqslant \cdots$, *then* $f(z)$ *does not vanish in*

$$|z| < \frac{t\,|a_0|}{\alpha_0+\beta_0} \Big/ \left(\frac{2(\alpha_k t^k + \beta_r t^r)}{\alpha_0+\beta_0} - 1\right).$$

Proof of Theorem 8. Since $|a_{j-1} - ta_j| \leqslant |\alpha_{j-1} - t\alpha_j| + |\beta_{j-1} - t\beta_j|$, $j = 1, 2, 3, \ldots$, we have from (2) for $|z| = t$,

$$\begin{aligned} |G(z)| &\leqslant t\left(\sum_1^{\infty} |\alpha_{j-1} - t\alpha_j|\, t^{j-1} + \sum_1^{\infty} |\beta_{j-1} - t\beta_j|\, t^{j-1}\right) \\ &= t(2\alpha_k t^k - \alpha_0 + 2\beta_r t^r - \beta_0) \qquad \text{(By hypothesis)} \\ &= t\left(\frac{2(\alpha_k t^k + \beta_r t^r)}{\alpha_0+\beta_0} - 1\right)(\alpha_0+\beta_0) = C\,|a_0|\,t, \qquad \text{(say).} \end{aligned}$$

Since $G(0) = 0$, it follows by Schwarz' lemma $|G(z)| \leqslant C\,|a_0|\,|z|$ for $|z| \leqslant t$. We can now complete the proof in the same way as the proof of Theorem 6.

The following corollary follows by taking $k = r = 0$ in Theorem 8 and noting that $\alpha_0 + \beta_0 \leqslant 2^{1/2}\,|a_0|$.

Corollary 7. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *be analytic in* $|z| \leqslant t$. *If* $a_j = \alpha_j + i\beta_j$, $j = 0, 1, 2, \ldots$ *and* $0 < \alpha_0 \geqslant t\alpha_1 \geqslant t^2\alpha_2 \geqslant \cdots$, $\beta_0 \geqslant t\beta_1 \geqslant t^2\beta_2 \geqslant \cdots$, *then* $f(z)$ *does not vanish in* $|z| < t/2^{1/2}$.

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