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On the coefficients of concave univalent functions

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Let D denote the open unit disc and $f : D \to \overline{\mathbb{C}}$ be meromorphic and injective in D. We assume that f is holomorphic at zero and has the expansion

$$f(z) \;=\; z + \sum_{n=2}^{\infty} a_n z^n \,.$$

Especially, we consider f that map D onto a domain whose complement with respect to $\overline{\mathbb{C}}$ is convex. We call these functions concave univalent functions and denote the set of these functions by Co.

We prove that the sharp inequalities $|a_n| \geq 1$, $n \in \mathbb{N}$, are valid for all concave univalent functions. Furthermore, we consider those concave univalent functions which have their pole at a point $p \in (0,1)$ and determine the precise domain of variability for the coefficients a_2 and a_3 for these classes of functions.

1 Results

One of the oldest and most famous results on the Taylor coefficients of functions of special classes in geometric function theory is the following result due to K. Löwner (compare [8]):

Let D denote the open unit disc and let

$$g(z) \;=\; z + \sum_{n=2}^{\infty} b_n z^n$$

be holomorphic and injective in D. If $g(D)$ is a convex set, then

$$|b_n| \;\leq\; 1\,, \quad n \in \mathbb{N}\,.$$

Equality is attained for one $n \in \mathbb{N} \setminus \{1\}$ if and only if $g(z) = z/(1 - \exp(i\tau)z)$, $\tau \in [0, 2\pi)$.

The present article is devoted mainly to the proof of a perhaps surprising appendage to this result on convex functions. We consider functions $f : D \to \overline{\mathbb{C}}$ that are meromorphic and univalent in D, holomorphic at zero and have the expansion

$$f(z) \;=\; z + \sum_{n=2}^{\infty} a_n z^n \,.$$

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The class of those functions that, among these, map D onto a domain whose complement with respect to $\overline{\mathbb{C}}$ is convex will be denoted by Co and we call it the class of concave univalent functions. In [1] it was conjectured that for $f \in Co$ the inequalities $|a_n| \geq 1$, $n \in \mathbb{N}$, are valid. In fact, we prove here:

Theorem 1.1 *Let $f \in Co$. Then*

$$|a_n| \;\geq\; 1\,, \quad n \in \mathbb{N}\,.$$

Equality is attained for one $n \in \mathbb{N} \setminus \{1\}$ if and only if $f(z) = z/(1 - \exp(i\tau)z)$, $\tau \in [0, 2\pi)$.

In [1], it had been proved only that $|a_n| \geq 1$ for $n = 2, 3, 4$ and that $|a_n| \geq 1/2$ for $n \in \mathbb{N} \setminus \{1\}$.

More can be said if one fixes the pole of $f \in Co$. We denote by $Co(p)$ the class of those concave univalent functions that have the pole at the point p. To avoid unnecessary complications we assume $p \in (0, 1)$. Likewise, we denote by $Co(1)$ the class of concave univalent functions that are holomorphic in D and satisfy $f(1) = \infty$. These subclasses of Co for $p \in (0, 1)$ have been considered by several authors. The functions in $Co(p)$, $p \in (0, 1)$, have been called convex meromorphic functions (compare [7] and [9]) and those classes have been denoted by $C(p)$ there. We prefer the present notation which refers to the shape of $f(D)$ rather than to the shape of its complement and avoids confusion with the class of close-to-convex functions. Furthermore, we include the case of functions $f \in Co(1)$, which are holomorphic in D.

It has been proved by Miller in [9] (see also [1]) that

$$\left| a_2 - \frac{1 + p^2 + p^4}{p(1 + p^2)} \right| \;\leq\; \frac{p}{1 + p^2} \quad \text{for} \quad f \in Co(p)$$

and by Livingston in [7] that

$$\operatorname{Re} a_3 \;\geq\; \frac{1 + p^6}{p^2(1 + p^2)} \quad \text{for} \quad f \in Co(p)\,.$$

Livingston conjectured in [7] that

$$\operatorname{Re} a_n \;\geq\; \frac{1 + p^{2n}}{p^{n-1}(1 + p^2)} \quad \text{for} \quad f \in Co(p)\,, \;\; n \in \mathbb{N}\,.$$

Obviously, the truth of this conjecture would imply Theorem 1.1.

In the present article, we determine the precise domain of variability of a_2 and a_3 for $f \in Co(p)$, $p \in (0, 1)$.

Theorem 1.2 *Let $p \in (0, 1)$ and $f \in Co(p)$. For $n = 2$ and $n = 3$ the inequality*

$$\left| a_n - \frac{1 - p^{2n+2}}{p^{n-1}(1 - p^4)} \right| \;\leq\; \frac{p^2\big(1 - p^{2n-2}\big)}{p^{n-1}\big(1 - p^4\big)} \tag{1.1}$$

is valid. Any point in the disc described by (1.1) *is attained as second, resp. third coefficient of a function $f \in Co(p)$, for example by the functions*

$$f_c(z) \;=\; \big(1 - p^2 c\big)^{-1} \, \frac{z(1 - p^2 c) - z^2(1 - c)p}{\big(1 - \frac{z}{p}\big)(1 - zp)}\,, \quad c \in \overline{D}\,. \tag{1.2}$$

The points on the boundary of the disc described by (1.1) *are attained if and only if $f(z) = f_c(z)$, $c = \exp(i\theta)$, $\theta \in [0, 2\pi)$.*

2 Proofs

For the proof of Theorem 1.1 we need the following lemma which may be of independent interest and is a slightly generalized version of Theorem 2.2 in [13]. Concerning the proof compare [11] and [12], too.

Lemma 2.1 *Let the functions F and G be meromorphic in D and holomorphic in a neighbourhood of the origin, where they have expansions*

$$F(z) \;=\; \sum_{n=0}^{\infty} A_n z^n \quad \textit{and} \quad G(z) \;=\; \sum_{n=0}^{\infty} B_n z^n \,.$$

If there exist two functions ϕ and ω holomorphic in D with $|\phi(z)| \le 1$ and $|\omega(z)| \le |z|$ for $z \in D$ (which implies $\omega(z) = z\omega_1(z)$, where ω_1 is holomorphic at $z = 0$) such that

$$F(z) \;=\; \phi(z) G(\omega(z)) \tag{2.1}$$

then for all $n \in \mathbb{N} \cup \{0\}$ the following inequalities are valid:

$$\sum_{k=0}^{n} |A_k|^2 \;\le\; \sum_{k=0}^{n} |B_k|^2 \,. \tag{2.2}$$

Proof. We fix $n \in \mathbb{N} \cup \{0\}$. From (2.1) we conclude that in a neighbourhood of the origin

$$\sum_{k=n+1}^{\infty} A_k z^k - \phi(z) \sum_{k=n+1}^{\infty} B_k\, \omega(z)^k \;=\; \phi(z) \sum_{k=0}^{n} B_k\, \omega(z)^k - \sum_{k=0}^{n} A_k z^k \,.$$

If we expand the difference on the left side of this equation in a Taylor series in a neighbourhood of the origin, we see that the coefficients of $z^k,\ 0 \le k \le n$, in this series are zero. Therefore, if we denote this function by H, we get an expansion

$$H(z) \;=\; \sum_{k=n+1}^{\infty} C_k z^k$$

in a neighbourhood of the origin. On the other hand, it is immediately clear that the difference on the right side is holomorphic in the whole unit disc. This implies that the identity

$$\sum_{k=0}^{n} A_k z^k + \sum_{k=n+1}^{\infty} C_k z^k \;=\; \phi(z) \sum_{k=0}^{n} B_k\, \omega(z)^k$$

is valid for $|z| < 1$, too.

Now, we may proceed as in the proof of Theorem 2.2 in [13] to get by use of Parseval's formula and Littlewood's theorem

$$\sum_{k=0}^{n} |A_k|^2 \;\le\; \sum_{k=0}^{n} |A_k|^2 + \sum_{k=n+1}^{\infty} |C_k|^2 \;\le\; \sum_{k=0}^{n} |B_k|^2 \,.$$

□

Remark 2.2 Following the notation of [11] and [13] it seems appropriate to call the above relation between the functions F and G quasi-subordination of meromorphic functions. Following the ideas of Goluzin (see [5] and compare also [4], Theorem 6.3) it is easy to see that the following generalization of Lemma 2.1 is valid.

Corollary 2.3 *Let $n \in \mathbb{N} \cup \{0\}$, F and G be as above and $\lambda_0 \ge \lambda_1 \ge \dots \lambda_n \ge 0$. Then the following inequality is valid:*

$$\sum_{k=0}^{n} \lambda_k\, |A_k|^2 \;\le\; \sum_{k=0}^{n} \lambda_k\, |B_k|^2 \,.$$

This inequality in the case of subordinate meromorphic functions was used in [2].

P r o o f of Theorem 1.1. We use that according to [1] there exists for any $f \in Co$ a function ϕ holomorphic in the unit disc such that $|\phi(z)| \leq 1$ for $z \in D$ and that

$$z\,f''(z) + 2f'(z) \;=\; \phi(z)\,f''(z) \tag{2.3}$$

in D. Now we apply Lemma 2.1 with

$$F(z) \;=\; z\,f''(z) + 2f'(z) \quad \text{and} \quad G(z) \;=\; f''(z)\,,$$

resp.

$$A_k \;=\; (k+1)(k+2)a_{k+1} \quad \text{and} \quad B_k \;=\; (k+1)(k+2)a_{k+2}$$

for $k \in \mathbb{N} \cup \{0\}$, where $a_1 = 1$. This together with (2.2) implies

$$\sum_{k=0}^{n-1} (k+1)^2(k+2)^2\,|a_{k+1}|^2 \;\leq\; \sum_{k=0}^{n-1} (k+1)^2(k+2)^2\,|a_{k+2}|^2$$

for $n \in \mathbb{N}$. This is equivalent to

$$\sum_{k=1}^{n} k^2(k+1)^2\,|a_k|^2 \;\leq\; \sum_{k=2}^{n+1} (k-1)^2k^2\,|a_k|^2 \tag{2.4}$$

for $n \in \mathbb{N}$. Now we proceed by mathematical induction. The case $n = 1$ of (2.4) proves $|a_2| \geq 1$ since $|a_1| = 1$. This was already proved in [1] and it was shown there that the only functions for which equality is attained here are those stated in Theorem 1.1. Furthermore, we conclude from (2.4) that

$$4\sum_{k=1}^{n} k^3\,|a_k|^2 \;\leq\; n^2(n+1)^2\,|a_{n+1}|^2\,. \tag{2.5}$$

The identity $4\sum_{k=1}^{n} k^3 = n^2(n+1)^2$ together with (2.5) immediately shows by mathematical induction that $|a_n| \geq 1$ for $n \in \mathbb{N}$ and $f \in Co$. Since equality is attained here if and only if $|a_2| = \ldots = |a_{n-1}| = 1$, we see, again by mathematical induction and using the results from [1] that the bound 1 is reached only in the above mentioned cases. $\square$.

Remark 2.4 By the same method one may get an alternative proof of the theorem of Löwner for the coefficients of convex functions cited above.

P r o o f of Theorem 1.2. In this proof we again need the results of [1], especially formula (2.3). It was proved there that (2.3) is valid for $f \in Co(p)$, $p \in (0, 1)$, if and only if the conditions

(i) $|\phi(z)| \leq 1$ for $z \in D$,
(ii) $\phi(p) = p,\ \phi'(p) = \phi''(p) = 0$,
(iii) ϕ has no fixed point in $D \setminus \{p\}$,

are fulfilled. Now it is an obvious consequence of the Schwarz lemma that a holomorphic selfmap of D has at most one fixed point in D unless it is the identity. Hence (iii) follows from (i) and (ii). On the other hand (i) and (ii) are fulfilled if and only if there exists a function ψ holomorphic in D such that $|\psi(z)| \leq 1$ for $z \in D$ and

$$\phi(z) \;=\; \frac{p + \left(\frac{z-p}{1-zp}\right)^3 \psi(z)}{1 + \left(\frac{z-p}{1-zp}\right)^3 p\,\psi(z)}\,. \tag{2.6}$$

The results of Pfaltzgraff and Pinchuk in [10] and of Livingston in [7] may be used to give an independent proof of this relation between unimodular bounded functions and concave univalent functions.

Since any unimodular bounded function ψ corresponds to a function $f \in Co(p)$ via (2.3) and (2.6), we may use these formulae together with the initial values $f(0) = 0$, $f'(0) = 1$ to construct examples for functions in

$Co(p)$. The simplest task is the solution of this initial value problem in the case $\psi(z) \equiv c$, $c \in \overline{D}$. The solutions are the functions given in (1.2), for $|c| = 1$ the set $\overline{\mathbb{C}} \setminus f_c(D)$ is a line segment, $\overline{\mathbb{C}} \setminus f_0(D)$ is a disc. One may find these functions in [9] as the extremal functions for the the set of variability of a_2 in the case $|c| = 1$. The correspondence between the functions dicussed by Miller there and the functions f_c is given by the identity $\lambda = \big(p^2 - c\big)/\big(1 - p^2c\big)$ (compare [9] again). The Taylor coefficients of these functions are given by

$$a_n \;=\; \frac{1 - p^{2n}c}{p^{n-1}(1 - p^2c)}\,, \quad n \in \mathbb{N}\,.$$

Hence, we get for the coefficients of f_c

$$\frac{1 - a_n p^{n-1}}{p^{n-1} - a_n} \;=\; p^{n+1}c\,, \quad n \in \mathbb{N} \setminus \{1\}\,. \tag{2.7}$$

A little computation reveals that (2.7) is equivalent to the fact that the n-th Taylor coefficients of the functions f_c, $c \in \overline{D}$, fill the disc described by (1.1). This implies that this disc is at least a subset of the domain of variability of a_n, $f \in Co(p)$. For the following we observe the trivial fact that the points on the boundary of the above mentioned disc are represented by the functions $\psi(z) \equiv \exp(i\theta)$, $\theta \in [0, 2\pi)$.

To prove the remaining assertions of Theorem 1.2 we solve (2.6) with respect to ψ:

$$\psi(z) \;=\; \frac{(1 - pz)^3(F(z) - pG(z))}{(z - p)^3(G(z) - pF(z))}\,, \tag{2.8}$$

where F and G stand in the same relation to $f \in Co(p)$ as in the proof of Theorem 1.1. Now, let

$$\psi(z) \;=\; \sum_{k=0}^{\infty} c_k z^k\,.$$

It remains to compute the coefficients c_k of ψ from (2.8) as functions of the coefficients a_k of $f \in Co(p)$ and to use the precise knowledge on the coefficient body of unimodular bounded functions which we have by the work of J. Schur (see [14]). This computation delivers

$$c_0 \;=\; \frac{1 - pa_2}{p^3(p - a_2)}\,. \tag{2.9}$$

The inequality $|c_0| \le 1$ in which equality is attained if and only if $\psi(z) \equiv \exp(i\theta)$ implies everything we have asserted concerning the second coefficient.

In the case $n = 3$ we get

$$c_1 \;=\; 3\big(1 - p^2\big)\,\frac{pa_3 - (1 + p^2)a_2 + p}{p^4(p - a_2)^2}\,.$$

Using this and (2.9) we see that the inequality

$$|c_1| \;\le\; 1 - |c_0|^2$$

is equivalent to

$$3p^2\,\big|pa_3 - \big(1 + p^2\big)a_2 + p\big| \;\le\; p^4 - \big|p\big(1 + p^2\big)a_2 - \big(1 + p^2 + p^4\big)\big|^2\,. \tag{2.10}$$

Because of the identity

$$p\,\big(pa_3 - \big(1 + p^2\big)a_2 + p\big) \;=\; p^2a_3 - \big(1 + p^4\big) - \big(p\big(1 + p^2\big)a_2 - \big(1 + p^2 + p^4\big)\big)$$

it is convenient to define

$$x \;:=\; \big|p\big(1 + p^2\big)a_2 - \big(1 + p^2 + p^4\big)\big|$$

and to remember that $x \in [0, p^2]$, where $x = p^2$ if and only if $\psi(z) \equiv \exp(i\theta)$, $\theta \in [0, 2\pi)$. This is just the assertion in the case $n = 2$. With this abbreviation (2.10) implies

$$3p\left|p^2 a_3 - \left(1 + p^4\right)\right| \le p^4 - x^2 + 3px =: h(x)\,.$$

Because of

$$\max\left\{h(x) \mid x \in \left[0, p^2\right]\right\} = h\left(p^2\right) = 3p^3$$

the last inequality together with the above proves our assertions in the case $n = 3$. □

An immediate consequence of Theorem 1.2 considering the limiting case $p \to 1$ as in [1] is the following

Corollary 2.5 *Let $f \in Co(1)$. Then for $n = 2, 3$ the inequalities*

$$\left|a_n - \frac{n+1}{2}\right| \le \frac{n-1}{2} \tag{2.11}$$

are valid.

The observations made above led us to the following generalization of Livingston's conjecture (see [7])

Conjecture 2.6 *Let $f \in Co(p)$. If $p \in (0, 1)$ then (1.1) holds for all n, and if $p = 1$ then (2.11) holds for all n. In particular $\operatorname{Re} a_n \ge 1$ holds for all n.*

Remark 2.7 It should be mentioned here that the upper bounds for $\operatorname{Re} a_n$ which would follow from the validity of the conjecture are the bounds known from de Branges' proof of the Bieberbach conjecture and Jenkins' observation concerning univalent meromorphic functions (compare [3] and [6]).

Note added in proof Meanwhile, it has been proved by the first and the third author that for $f \in Co(p)$ and $n \in \mathbb{N}$ the inequalities

$$\operatorname{Re} a_n \ge \frac{1 + p^{2n}}{p^{n-1}(1 + p)^2}$$

are valid (see F. G. Avkhadiev and K.-J. Wirths, On a conjecture of Livingston, Mathematica (Cluj) **46** (69), 19–23 (2004)).

Further , the inequality (1.1) has been proved in the cases $n = 4$ and $n = 5$ by the third author (see K.-J. Wirths, A proof of the Livingston conjecture for the fourth and the fifth coefficient of concave univalent functions, Ann. Pol. Math. **83.1**, 87–93 (2004)).

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