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ON THE BOUNDARY BEHAVIOUR OF BLOCH AND NORMAL FUNCTIONS

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Criteria for an analytic function f defined in $|z| < 1$ to belong to B_0, the class of Bloch functions satisfying $\lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0$, and criteria for a meromorphic function g defined in $|z| < 1$ to belong to N_0, namely, to satisfy $\lim_{|z|\to 1} (1 - |z|^2)\frac{|g'(z)|}{1 + |g(z)|^2} = 0$, are obtained in terms of the area and the length of the images of hyperbolic disks and hyperbolic circles, respectively.

§1.

Let f be a holomorphic function in the unit disk $D = \{z \mid |z| < 1\}$ of the complex plane $\mathbb{C} = \{z \mid |z| < \infty\}$. Let B be the family of holomorphic functions f in D such that

$$\sup_{z\in D}(1 - |z|^2)|f'(z)| < \infty$$

and B_0 the family of holomorphic functions f in D such that

$$\lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0.$$

If $f \in B$, then f is said to be a Bloch function. In Theorem 1 we shall propose some criteria for f to belong to B_0. These criteria are

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immediate consequences of Yamashita's Theorem in [4].

Let

$$d(z,w) = \frac{1}{2} \log \frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|}$$

be the hyperbolic distance between z and w in D. For $0 < r < \infty$ and for $z \in D$, we set

$$U(z,r) = \{w \in D \mid d(w,z) < r\}$$

and

$$\Gamma(z,r) = \{w \in D \mid d(w,z) = r\}.$$

Let $\mathcal{A}_f(z,r)$ be the euclidean area of the Riemannian image $\mathcal{F}(z,r)$ of $U(z,r)$ by f, and let $A_f(z,r)$ be the euclidean area of the image $F(z,r)$ of $U(z,r)$ by f; we note that $F(z,r)$ is the projection of $\mathcal{F}(z,r)$ to $\mathbb{C}$. Let $\mathcal{L}_f(z,r)$ be the euclidean length of the Riemannian image of $\Gamma(z,r)$ by f, and $L_f(z,r)$ the euclidean length of the outer boundary of $F(z,r)$. The outer boundary of a bounded domain G in $\mathbb{C}$ means the boundary of $\mathbb{C} \setminus E$, where E is the unbounded component of the complement $\mathbb{C} \setminus G$ of G. The inequalities

$$\mathcal{A}_f(z,r) \geqq A_f(z,r) \quad \text{and} \quad \mathcal{L}_f(z,r) \geqq L_f(z,r)$$

hold for each $0 < r < \infty$ and each $z \in D$.

Yamashita proved the following:

THEOREM A. *Let f be non-constant and holomorphic in D. Then the following are mutually equivalent:*

(I) $f \in B$;

(II) *there exists* $0 < r < \infty$ *such that* $\sup_{z \in D} \mathcal{A}_f(z,r) < \infty$;

(III) *there exists* $0 < r < \infty$ *such that* $\sup_{z \in D} A_f(z,r) < \infty$;

(IV) *there exists* $0 < r < \infty$ *such that* $\sup_{z \in D} \mathcal{L}_f(z,r) < \infty$;

(V) *there exists* $0 < r < \infty$ *such that* $\sup_{z \in D} L_f(z,r) < \infty$.

From this theorem we obtain

THEOREM 1. *Let* f *be non-constant and holomorphic in* D. *Then the following are mutually equivalent:*

(I) $f \in B_0$;

(II) *there exists* $0 < r < \infty$ *such that* $\lim_{|z| \to 1} \mathrm{A}_f(z,r) = 0$;

(III) *there exists* $0 < r < \infty$ *such that* $\lim_{|z| \to 1} A_f(z,r) = 0$;

(IV) *there exists* $0 < r < \infty$ *such that* $\lim_{|z| \to 1} \mathrm{L}_f(z,r) = 0$;

(V) *there exists* $0 < r < \infty$ *such that* $\lim_{|z| \to 1} L_f(z,r) = 0$.

Proof. The assertions follow immediately from the proof of Theorem A in [4] by replacing the bounded term by a sequence of terms converging to zero.

§2.

The meromorphic analogue of a Bloch function is a normal meromorphic function. A function f, meromorphic in D, is said to be *normal in* D if $\sup_{z \in D} (1 - |z|^2) f^*(z) < \infty$ where $f^*(z) = |f'(z)|/(1 + |f(z)|^2)$ is the spherical derivative of f (cf. [3]). We denote by N the family of all normal meromorphic functions in D. Further, let N_0 be the family of meromorphic functions f in D such that

$$\lim_{|z| \to 1} (1 - |z|^2) f^*(z) = 0 .$$

The euclidean area and length used in the above theorems will be replaced by the spherical area and spherical length. We shall denote the spherical area of $F(z,r)$ by $B_f(z,r)$ and the spherical area of $F(z,r)$ by $B_f(z,r)$. Let $M_f(z,r)$ be the spherical length of the

Riemannian image of $\Gamma(z,r)$ by f, and let $M_f(z,r)$ be the length of the boundary of $F(z,r)$. The corresponding inequalities as above are valid, that is,

$$\mathrm{B}_f(z,r) \geqq B_f(z,r) \quad \text{and} \quad \mathrm{M}_f(z,r) \geqq M_f(z,r). \tag{1}$$

For normal meromorphic functions we cannot obtain results corresponding to those in Theorem A, as shown by Yamashita in [4]. For example, implication $(III) \Rightarrow (I)$ does not hold as Lappan has shown in [2] and the implication $(V) \Rightarrow (I)$ is still open. Therefore it is interesting to notice that the meromorphic analogue of Theorem 1 for the functions of N_0 is valid. For the proof of our theorem we shall make use of the following lemma [1, Lemma II]:

LEMMA. *For the function g meromorphic in D suppose that the spherical area $B_g(0,r)$ is strictly less than π. Then,*

$$g^*(0)^2 \leqq \frac{B_g(0,r)}{\pi x^2\left(1 - \frac{B_g(0,r)}{\pi}\right)},$$

where $x = (e^{2r} - 1)/(e^{2r} + 1)$.

THEOREM 2. *Let f be non-constant and meromorphic in D. Then the following are mutually equivalent:*

(I) $f \in N_0$;

(II) there exists $0 < r < \infty$ *such that* $\lim_{|z|\to 1} \mathrm{B}_f(z,r) = 0$;

(III) there exists $0 < r < \infty$ *such that* $\lim_{|z|\to 1} B_f(z,r) = 0$;

(IV) there exists $0 < r < \infty$ *such that* $\lim_{|z|\to 1} \mathrm{M}_f(z,r) = 0$ *and* $\mathrm{B}_f(z,r) \leqq \alpha < \pi$ *for all* z, $r_0 < |z| < 1$;

(V) there exists $0 < r < \infty$ *such that* $\lim_{|z|\to 1} M_f(z,r) = 0$ *and* $B_f(z,r) \leqq \alpha < \pi$ *for all* z, $r_0 < |z| < 1$.

Proof. We prove first $(III) \Rightarrow (I)$; let

$$g(w) = f\left(\frac{w + z}{1 + \bar{z}w} \right) .$$

By the assumption there is a $r_0 > 0$ such that $B_f(z,r) < \pi$ for all z, $r_0 < |z| < 1$. Let $|z| > r_0$. Then by a simple calculation and the Lemma we have

$$(1 - |z|^2)f^*(z) = g^*(0) \leq \left\{ \frac{B_g(0,r)}{\pi x^2 \left(1 - \frac{B_g(0,r)}{\pi}\right)} \right\}^{1/2}$$

$$= \left\{ \frac{B_f(z,r)}{\pi x^2 \left(1 - \frac{B_f(z,r)}{\pi}\right)} \right\}^{1/2},$$

where $x = (e^{2r} - 1)/(e^{2r} + 1)$. Hence $(III) \Rightarrow (I)$. $(I) \Leftrightarrow (II)$ Yamashita has proved this result in [5]. $(II) \Rightarrow (IV)$ By the above equivalence it is sufficient to prove that $(I) \Rightarrow (IV)$. We choose a sequence of points (z_n) for which $|z_n| \to 1$ as $n \to \infty$. Let $r > 0$. We take the sequence of hyperbolic disks $(U(z_n,r))$ and form the functions

$$f_n(\zeta) = f\left\{ \frac{\zeta + z_n}{1 + \bar{z}_n \zeta} \right\} .$$

Let $\zeta_0 \in \Gamma(0,r)$ and let $z_n' = (\zeta_0 + z_n)/(1 + \bar{z}_n\zeta_0)$. The radius of D going through z_n intersects $\Gamma(z_n,r)$ in two points. We denote by z_n'' the point for which $|z_n''| < |z_n|$. Then we obtain for the spherical derivative

$$f_n^*(\zeta_0) = \frac{1}{1 - \delta(z_n,z_n')^2} \cdot (1 - |z_n'|^2) f^*(z_n')$$

$$\leq \frac{1}{\alpha}(1 - |z_n''|^2) \max_{z \in \Gamma(z_n,r)} f^*(z) = \frac{1}{\alpha} (1 - |z_n''|^2) f^*(z_n''') ,$$

where $z_n''' \in \Gamma(z_n,r)$ and $1 - \delta(z_n,z_n')^2 = 1 - \left|\frac{z_n - z_n'}{1 - \bar{z}_n' z_n}\right|^2 \geq \alpha > 0$, since $d(z_n,z_n') = d(0,\zeta_0) = r$. Now

$$M_f(z_n,r) = \int_{\Gamma(z_n,r)} f^*(z)\ |dz| = \int_{\Gamma(0,r)} f_n^*(\zeta)\ |d\zeta|$$

$$\leq \frac{1}{\alpha}(1 - |z_n''|^2) f^*(z_n''') \int_{\Gamma(0,r)} |d\zeta|$$

$$= \frac{\pi}{\alpha} \log\frac{1+r}{1-r}\cdot\frac{1 - |z_n''|^2}{1 - |z_n'''|^2}\ (1 - |z_n'''|^2) f^*(z_n''') \to 0 ,$$

since $|z_n'''| \to 1$ and $\dfrac{1 - |z_n''|^2}{1 - |z_n'''|^2} \to 1$.

The latter part of the assertion follows from the assumption *(II)*. *(IV)* $\Rightarrow$ *(V)* : This follows trivially from (1). *(V)* $\Rightarrow$ *(III)* : Let (z_n) be any sequence of points for which $|z_n| \to 1$ as $n \to \infty$. Then, for sufficiently large n, either the diameter of $F(z_n,r)$

$$\text{(2)} \qquad \operatorname{diam} F(z_n,r) \leq M_f(z_n,r)$$

or the complement $\hat{\mathbb{C}} \setminus F(z_n,r)$ is divided into the components $E_i(z_n,r)$, $i \in I$ (I an index set) for which

$$\sum_{i\in I} \operatorname{diam} E_i(z_n,r) \leq M_f(z_n,r).$$

When n is large enough, the latter alternative is not possible by the assumption $B_f(z,r) \leq \alpha < \pi$. The assertion follows by (2) and thus the theorem is proved.

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