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Extremal Functions for Functionals on some Classes of Analytic Functions*

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It is shown that the theorem of Carathéodory and Toeplitz on the characterization of the Taylor coefficients of analytic functions with positive real part can be applied to extremal problems in several classes of analytic functions.

Let U be the open unit disk$\{z: |z| < 1\}$ in the complex plane. As usual, we denote by $H(U)$ the set of analytic functions on U equipped with the topology of uniform convergence on compact subsets of U. Let P denote the set of functions f in $H(U)$ such that $\operatorname{Re} f(z) > 0$ for $z \in U$ and $f(0) = 1$.

Several familiar subclasses of $H(U)$ such as the classes of starlike, convex, typically real functions and the class of functions f in $H(U)$ such that $f(z) \neq 0$, $|f(z)| \leqslant 1$ for every $z \in U$, can be expressed in terms of the class P. Therefore, extremal problems for functionals on these classes can be transformed into extremal problems for corresponding functionals on P.

The purpose of this note is to show that the theorem of Carathéodory [2] and Toeplitz [7] on the characterization of the Taylor coefficients of elements of P implies that many functionals on P attain their extreme values on functions f in P of the form

$$f(z) = \sum_{j=1}^{n} \lambda_j \frac{\xi_j + z}{\xi_j - z},$$

where $\lambda_j \geqslant 0$, $|\xi_j| = 1$, $j = 1, 2,..., n$, and $\sum_{j=1}^{n} \lambda_j = 1$.

Consequently we obtain, in an elementary way, solutions to extremal problems for the classes of analytic function described above. These solutions have been obtained previously by other authors, by means of elaborate variational methods (see [1, 3, 4]).

In order to state our results we need the following notation. For $\nu + 1$ complex numbers $b_0, b_1, ..., b_\nu$ we denote by $D_\nu(b_0, b_1, ..., b_\nu)$ the determinant of the $(\nu + 1) \times (\nu + 1)$ Toeplitz matrix $(b_{j-k})_{j,k=0}^{\nu}$, where we put $b_{-j} = \bar{b}_j$,

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$j = 1, 2, \ldots, \nu$. For a point $x = (s_1, t_1, \ldots, s_n, t_n)$ in $\mathbb{R}^{2n}$ let $c_j(x) = s_j + it_j$, $j = 1, 2, \ldots, n$, and $r_\nu(x) = D_\nu(2, c_1(x), \ldots, c_\nu(x))$, $\nu = 1, 2, \ldots, n$. Let M_n be the set of points x in $\mathbb{R}^{2n}$ such that $r_\nu(x) \geqslant 0$, $\nu = 1, 2, \ldots, n$. It is easily verified that M_n is a compact subset of $\mathbb{R}^{2n}$ which is contained in the cube $[-2, 2]^{2n}$ and that its boundary consists of all points x in $\mathbb{R}^{2n}$ such that $r_\nu(x) = 0$ for some ν in $\{1, 2, \ldots, n\}$. For a positive integer n let P_n denote the set of functions f in P of the form

$$f(z) = \sum_{j=1}^{n} \lambda_j \frac{\xi_j + z}{\xi_j - z},$$

where $\lambda_j \geqslant 0$, $|\xi_j| = 1$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} \lambda_j = 1$. Clearly $P_n \subset P_{n+1}$, $n = 1, 2, \ldots$. Finally, for a function $f(z) = 1 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots$ in P, let $T_n(f)$ denote the point $(c_1, c_1, \ldots, c_n)$ in C^n.

Our first result is the following:

THEOREM 1. *Let F be a real-valued continuous function on M_n which attains its maximum on the boundary of M_n only. Let Φ be the functional on P which is defined by $\Phi(f) = F(x)$, $f \in P$ where $f(z) = 1 + c_1 z + \cdots + c_n z^n + \cdots$, $s_j = \operatorname{Re} c_j$, $t_j = \operatorname{Im} c_j$, $j = 1, 2, \ldots, n$, and $x = (s_1, t_1, \ldots, s_n, t_n)$. Then Φ attains its maximum over P by an element of P_n only.*

For applications of Theorem 1 the following is useful:

COROLLARY 1. *Let F be a continuous function on M_n which is not constant and which satisfies one of the following conditions:*

(a) *F has partial derivatives of the first order in the interior of M_n, and* $\operatorname{grad} F \neq 0$ *there.*

(b) *F is harmonic in the interior of M_n.*

(c) $F(x) = \operatorname{Re} G(c_1(x), c_2(x), \ldots, c_n(x))$, $x \in M_n$, *where G is an analytic function of n complex variables in a neighborhood of the domain in C^n, which is the image of M_n under the* map $x \to (c_1(x), c_2(x), \ldots, c_n(x))$; *Then the functional Φ associated with F, as in Theorem* 1, *attains its maximum over P by an element of P_n only.*

Proof of Corollary 1. In all of the three cases F attains its maximum only on the boundary of M_n. In case (a) this is clear, in case (b) this follows from the well-known maximum principle for harmonic functions, and case (c) is a particular case of case (b).

As stated above, Theorem 1 is a consequence of the theorem of Carathéodory [2] and Toeplitz [7], which we now recall.

THEOREM 2 (Carathéodory–Toeplitz). (I) *Let* $(c_1, c_2, ..., c_n)$ *be* n *complex numbers. A necessary and sufficient condition for the existence of a function* f *in* P *such that* $T_n(f) = (c_1, c_2, ..., c_n)$ *is that* $D_\nu(2, c_1, c_2, ..., c_\nu) \geqslant 0$ *for* $\nu = 1, 2, ..., n$.

(II) *If the numbers* $c_1, c_2, ..., c_n$ *satisfy the inequalities of part* (I) *and* $D_j(2, c_1, c_2, ..., c_j) = 0$ *for some* j *in* $\{1, 2, ..., n\}$, *then the function* f *in* P *such that* $T(f) = (c_1, ..., c_n)$ *is uniquely determined and belongs to* P_n.

Remark. In 1907 Carathéodory [2] proved a different form of this theorem, and in 1911 Toeplitz [7] proved that Carathéodory's theorem is equivalent to Theorem 2.

Proof of Theorem 1. Let B_n be the set of all points in C^n which are of the form $T_n(f)$ for f in P. By part (I) of Theorem 2 the mapping $x \to (c_1(x), c_2(x), ..., c_n(x))$ maps M_n onto B_n, and therefore Φ is defined over the whole class P. Since F is continuous on M_n, it follows that Φ is continuous on P (with respect to the topology of $H(U)$), and therefore, since P is a compact set in $H(U)$, there exists an element f_0 in P, such that $\Phi(f_0) = \max\{\Phi(f), f \in P\}$. Let x^0 be the point in M_n such that $(c_1(x^0), c_2(x^0), ..., c_n(x^0)) = T(f_0)$; clearly $\Phi(f_0) = F(x^0) = \max\{F(x), x \in M_n\}$, and therefore by the hypothesis on F, we infer that x^0 is a boundary point of M_n; that is, $r_j(x^0) = 0$ for some j in $\{1, 2, ..., n\}$. Consequently by part (II) of Theorem 2, f_0 belongs to P_n, and this completes the proof of Theorem 1.

Remark. In the application of Theorem 1 to the class of typically real functions we make use of the following remark. Let P_R denote the set of functions f in P such that $f^{(j)}(0)$ is real for $j \geqslant 1$, and for a positive integer n let $P_{R,n} = P_R \cap P_n$. Let $M_{\mathbb{R},n}$ be the set of points $x = (s_1, s_2, ..., s_n)$ in $\mathbb{R}^n$ such that $D_\nu(2, s_1, s_2, ..., s_\nu) \geqslant 0$, for $\nu = 1, 2, ..., n$. Using the fact that P_R is a closed subset of P one can easily verify that if the set M_n is replaced by the set $M_{n,R}$ in the hypothesis of Theorem 1, then its conclusion remains true if P_n is replaced by $P_{R,n}$.

APPLICATIONS. We give now some applications of Theorem 1 to several familiar classes of analytic functions. First we recall some standard definitions and facts, which can be found in [6, Chap. 2].

Let N denote the set of functions f in $H(U)$ such that $f(0) = 0$ and $f'(0) = 1$. We consider the following classes of functions:

$$S^* = \{f \in N; f(U) \text{ is starlike with respect to the origin}\},$$

$$K = \{f \in N: f(U) \text{ is convex}\},$$

$$T_{\mathbb{R}} = \{f \in N: (J_m z)(J_m f(z)) \geqslant 0, \forall z \in U\}.$$

These are the familiar normalized classes of starlike, convex, and typically real functions, respectively.

It is known (see [6, Chap. 2]) that the following mappings establish a one-to-one correspondence between the classes S^*, K, respectively, and the class P, and between the classes T_R and P_R:

$$f \to \frac{zf'}{f}, \qquad f \in S^*, \tag{1}$$

$$f \to 1 + \frac{zf''}{f'}, \qquad f \in K, \tag{2}$$

$$f \to \frac{(1-z^2)}{z} f, \qquad f \in T_{\mathbb{R}}. \tag{3}$$

For a positive integer n we adopt the notation

$$S_n^* = \left\{ f \in S^* : f(z) = z \prod_{j=1}^{n-1} (1 - \xi_j z)^{-\mu_j}, \mu_j \geqslant 0, |\xi_j| = 1, j = 1, 2,..., n-1, \right.$$
$$\left. \text{and } \sum_{j=1}^{n-1} \mu_j = 2 \right\}.$$

$$K_n = \left\{ f \in K, f(z) = \int_0^z \prod_{j=1}^{n-1} (1 - \eta_j \xi)^{-\mu_j} \, d\xi, \mu_j \geqslant 0, \right.$$
$$\left. |\eta_j| = 1, j = 1, 2,..., n, \text{ and } \sum_{j=1}^{n-1} \mu_j = 2 \right\}.$$

$$T_{\mathbb{R},n} = \left\{ f \in T_{\mathbb{R}}; f = \frac{z}{1-z^2} g, g \in P_{\mathbb{R},n-1} \right\}.$$

It is easily verified that by the above mappings, the classes S_n^* and K_n are mapped onto the class P_{n-1} and the class $T_{R,n}$ is mapped onto the class $P_{R,n-1}$. It is also apparent from these mappings that if $f(z) = z + a_2 z + \cdots + a_n z^n + \cdots$ belongs to one of the classes S^*, K, T_R and $g(z) = 1 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots$ is the corresponding function in the class P into which f is mapped, then for every $j \geqslant 2$ the coefficient a_j is a polynomial in the coefficients $c_1, c_2,..., c_{j-1}$.

THEOREM 3. *Let G and H be nonconstant analytic functions of $n-1$ complex variables in some neighborhoods of the sets $\{z = (z_1, z_2,..., z_{n-1}) \in C^{n-1}; |z_j| \leqslant j, j = 1, 2,..., n-1\}$ and $\{z = (z_1, z_2,..., z_{n-1}) \in C^{n-1}; |z_j| \leqslant 1, j = 1, 2,..., n-1\}$, respectively. Let Φ_1, Φ_2 and Φ_3 be the continuous functionals defined on S^*, K, and T_R, respectively, by*

$$\Phi_1(f) = \operatorname{Re} G(a_2,..., a_n), \qquad f \in S^*, \qquad f(z) = z + a_2 z + \cdots + a_n z^n + \cdots,$$
$$\Phi_2(g) = \operatorname{Re} H(b_2,..., b_n), \qquad g \in K, \qquad g(z) = z + b_2 z^2 + \cdots + b_n z^n + \cdots,$$
$$\Phi_3(h) = \operatorname{Re} G(d_2,..., d_n), \qquad h \in T_R, \qquad h(z) = z + d_2 z + \cdots + d_n z^n + \cdots,$$

Then

(1) *The maximum of* Φ_1 *over* S^* *is attained on an element of* S^*_n *only.*

(2) *The maximum of* Φ_2 *over* K *is attained on an element of* K_n *only.*

(3) *The maximum of* Φ_3 *over* T_R *is attained on an element of* $T_{R,n}$ *only.*

Proof. First note that by the well-known coefficient estimates for the classes S^*, K, and T_R (see [6, Chap. 2]) the functionals Φ_1, Φ_2, and Φ_3 are well defined over their whole correspondent classes. Taking into account the remarks which preceded the statement of Theorem 3, and for the class T_R the remark after the proof of Theorem 1 also, one obtains the conclusion of the theorem by an obvious application of part (c) of Corollary 1.

Remarks. (1) The part of Theorem 3 for the class S^* was proved by Hummel [4], who used variational methods.

(2) Using parts (a) and (b) of Corollary 1, one can similarly apply Theorem 1 to the classes S^*, K, and T_R. In order to transfer conditions (a) and (b) of Corollary 2 into corresponding conditions for functionals on these classes, one has to consider the form of the polynomials which define the Taylor coefficients of the functions in these classes in terms of the coefficients of the corresponding functions in the class P.

We conclude with another application of Theorem 1. Consider the class $E = \{f \in H(U): f(z) \neq 0: |f(z)| \leqslant 1, z \in U\}$. It is well known that for a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in E we have $\sum_{k=0}^{\infty} |a_k|^2 \leqslant 1$; hence in particular, $\sum_{k=0}^{n} |a_k|^2 \leqslant 1$ for every $n \geqslant 0$.

Let n be a positive integer and let W be a nonconstant analytic function of $n+1$ complex variables on a neighborhood of the set $\{z = (z_0, z_1 \cdots z_n) \in C^{n+1};\ \sum_{j=0}^{n} |z_j|^2 \leqslant 1\}$. Consider the functional Φ on E defined by $\Phi(f) = \operatorname{Re} W(a_0, a_1, \ldots, a_n), f \in E, f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$. We then have:

THEOREM 4. *The maximum of* Φ *over* E *is attained by a function* f_0 *in* E *of the form* $f_0 = e^{-\lambda_0 g_0}$ *only, where* λ_0 *is a nonnegative constant and* g_0 *is in* P_n.

Proof. Let $\bar{E}$ be the closure of E in $H(U)$. Clearly $\bar{E}$ is a compact subset of $H(U)$, and by the Theorem of Hurwitz $\bar{E} = E \cup \{0\}$, where 0 denotes the zero function in $H(U)$. The functional Φ can be extended continuously to $\bar{E}$ by putting $\Phi(0) = \operatorname{Re} W(0, 0, \ldots, 0)$. Since $\bar{E}$ is compact Φ attains its maximum on this set at some function f_0. Since W is not constant, it follows from the maximum principle for harmonic functions that $f_0 \not\equiv 0$; hence f_0 is in E. Every function f in E can be written in the form $f = e^{-\lambda g}$ for some nonnegative number λ and some function g in P. Let $f_0 = e^{-\lambda_0 g_0}$, and consider the functional Ψ defined on P by $\Psi(g) = \Phi(e^{-\lambda_0 g}), g \in P$. It is clear that $\Psi(g_0) = \max\{\Psi(g), g \in P\}$. Therefore, since for every $j = 1, 2, \ldots$, the jth Taylor coefficient of the function $e^{-\lambda_0 g}$ is a polynomial in the first j-Taylor coefficients of g, an application of

Corollary 1, part (c), yields that g_0 is in P_n, and the proof of Theorem 4 is complete.

Remark. For the functional $\Phi(f) = \operatorname{Re} a_n$, $f \in E$, $f(z) = a_0 + a_1 z + \cdots + a_n z + \cdots$, Theorem 4 was proved by Kirwan [5] by means of the variational methods of Goluzin [3].

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