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Journal of Functional Analysis 213 (2004) 241–269
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

The Cayley transform and uniformly bounded representations ☆

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Received 5 June 2003; accepted 3 December 2003

Communicated by Alain Connes

Abstract

Let G be a simple Lie group of real rank one, with Iwasawa decomposition $KA\bar{N}$ and Bruhat big cell $NMA\bar{N}$. Then the space $G/MA\bar{N}$ may be (almost) identified with N and with K/M, and these identifications induce the (generalised) Cayley transform $\mathscr{C} : N \to K/M$. We show that $\mathscr{C}$ is a conformal map of Carnot–Caratheodory manifolds, and that composition with the Cayley transform, combined with multiplication by appropriate powers of the Jacobian, induces isomorphisms of Sobolev spaces $\mathscr{H}^{\alpha}(N)$ and $\mathscr{H}^{\alpha}(K/M)$. We use this to construct uniformly bounded and slowly growing representations of G.

Keywords: Simple Lie groups of real rank one; Cayley transform; Uniformly bounded representations

1. Introduction, notation, and preliminaries

Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$, where $n \geqslant 2$, and o denote the point $(0, \dots, 0, 1)$. Stereographic projection is a conformal bijection from $\mathbb{R}^n \cup \{\infty\}$ to S^n

☆ The research for this paper was carried out at the University of New South Wales. The authors thank the Australian Research Council, the *Progetto Cofinanziato M.U.R.S.T.* "*Analisi Armonica*", and the School of Mathematics of the University of the New South Wales for their support.

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0022-1236/$ - see front matter
doi:10.1016/j.jfa.2003.12.009

which maps ∞ to o. The conformal group G of S^n (or of $\mathbb{R}^n \cup \{\infty\}$) is a simple Lie group. Given an Iwasawa decomposition KAN of G, we write M for the centraliser of A in K, and then MAN is a parabolic subgroup, which we will abbreviate to P. We choose the decomposition so that MA stabilises both o and $-o$, and so that P is the stabiliser of o; then we may write $MA\bar{N}$ for the stabiliser $\bar{P}$ of $-o$. Then stereographic projection has a group theoretic interpretation: S^n may be identified with K/M, and $\mathbb{R}^n$ with N, and then the stereographic projection is essentially the identification of the cosets $n\bar{P}$ (where $n \in N$) and $k\bar{P}$ (where $k \in K$).

This paper is about an extension of this, namely, the generalised Cayley transform associated to a simple Lie group G of real rank one. In this case, $\mathbb{R}^n$ is replaced by a nilpotent Lie group N of Heisenberg type, and K/M by a sphere with a privileged subbundle of the tangent bundle—both are Carnot–Caratheodory manifolds. In this paper, we examine the Cayley transform from several points of view. Our main geometric result is that the Cayley transform is a conformal map of Carnot–Caratheodory manifolds. Our main analytic result is that composition with the Cayley transform and multiplication by a suitable power of the Jacobian induces isomorphisms from Sobolev spaces $\mathscr{H}^\alpha(K/M)$ to $\mathscr{H}^\alpha(N)$. This is obvious locally, but the global result requires more work. Finally, we tie these results into representation theory, and use them to construct uniformly bounded representations π_α of G on the spaces $\mathscr{H}^\alpha(K/M)$ and $\mathscr{H}^\alpha(N)$, and estimate the growth of the representations π_α on the space $\mathscr{H}^\beta(K/M)$ when $\alpha \neq \beta$. Our estimates are those required by Julg [17] for his proof of the Baum–Connes conjecture with coefficients for the groups $\mathrm{Sp}(n,1)$ and $F_{4,-20}$.

In order to deal with all the real rank one simple Lie groups G simultaneously, we use the Clifford algebra formulation of [5,6].

Our paper is structured as follows. In the rest of this section we recall some properties of Lorentz spaces that we will use in the sequel. In Section 2 we describe the Carnot–Caratheodory structure of N and K/M, and we study the Cayley transform. Section 3 deals with Sobolev spaces $\mathscr{H}^\alpha(N)$ and $\mathscr{H}^\alpha(K/M)$. In Section 4 we prove that $\mathscr{H}^\alpha(K/M)$ and $\mathscr{H}^\alpha(N)$ can be identified via the Cayley transform. Finally, in Section 5 we use our results to construct uniformly bounded and slowly growing representations of G.

1.1. Lorentz spaces

Suppose that (Ω, μ) is a measure space and that $f : \Omega \to \mathbb{C}$ is measurable. The *nonincreasing rearrangement* of f is the function f^* on $[0, +\infty)$ defined by

$$f^*(t) = \inf\{s \in [0, +\infty) : \mu(\{x \in \Omega : |f(x)| > s\}) \leqslant t\}.$$

The function f^* is nonincreasing, nonnegative, equimeasurable with $|f|$, and right-continuous. Next we define

$$\|f\|_{L^{p,q}(\Omega)} = \left(\frac{q}{p}\int_0^\infty (s^{1/p} f^*(s))^q \frac{ds}{s}\right)^{1/q}$$

when $1<p<\infty$ and $1<q<\infty$, and, when $1<p<\infty$ and $q=\infty$,

$$||f||_{L^{p,\infty}(\Omega)} = \sup\{s^{1/p}f^*(s) : s\in[0,+\infty)\}.$$

Definition 1.1. When $1<p<\infty$ and $1\leqslant q\leqslant\infty$, the Lorentz space $L^{p,q}(\Omega)$ consists of those measurable functions $f:\Omega\to\mathbb{C}$ for which $||f||_{L^{p,q}(\Omega)}$ is finite, modulo identification of functions which are equal almost everywhere.

It is easy to check that $L^{p,p}(\Omega)$ coincides isometrically with the usual Lebesgue space $L^p(\Omega)$. Moreover, if $q_1<q_2$, then $L^{p,q_1}(\Omega)$ is contained in $L^{p,q_2}(\Omega)$ and if $1<q<\infty$, then the dual space of $L^{p,q}(\Omega)$ is $L^{p',q'}(\Omega)$, where $p'=p/(p-1)$ and $q'=q/(q-1)$. A good reference for Lorentz spaces is [14]. We recall a few facts from that paper (see pp. 271 and 273).

Lemma 1.2. *Suppose that $1<p,q,r<\infty$ and $1/q=1/r+1/p$. Then there exists a constant $C(p,q,r)$, depending only on p, q and r, such that for every f in $L^{p,2}(\Omega)$ and m in $L^{r,\infty}(\Omega)$,*

$$||mf||_{L^{q,2}(\Omega)}\leqslant C(p,q,r)||m||_{L^{r,\infty}(\Omega)}||f||_{L^{p,2}(\Omega)}.$$

Lemma 1.3. *Suppose that Ω is a unimodular locally compact topological group, $1<p<q<\infty$ and $1/p+1/r=1/q+1$. Then there exists a constant $C(p,q,r)$, depending only on p, q and r, such that for every f in $L^{p,2}(\Omega)$ and k in $L^{r,\infty}(\Omega)$,*

$$||f*k||_{L^{q,2}(\Omega)}\leqslant C(p,q,r)||k||_{L^{r,\infty}(\Omega)}||f||_{L^{p,2}(\Omega)}.$$

We use the "variable constant convention", according to which constants are denoted by C, C', etc., and these are not necessarily equal at different occurrences. All "constants" are positive.

2. Heisenberg type groups and the Cayley transform

2.1. Properties of Heisenberg type groups

Let $\mathfrak{n}$ be a real nilpotent Lie algebra, with an inner product $\langle\cdot,\cdot\rangle$, such that $\mathfrak{n}$ may be written as an orthogonal sum $\mathfrak{v}\oplus\mathfrak{z}$, where $[\mathfrak{v},\mathfrak{v}]\subset\mathfrak{z}$ and $[\mathfrak{z},\mathfrak{z}]=\{0\}$. For Z in $\mathfrak{z}$, define the map $J_Z:\mathfrak{v}\to\mathfrak{v}$ by the formula

$$\langle J_Z X, Y\rangle = \langle Z,[X,Y]\rangle \quad \forall X,Y\in\mathfrak{v}.$$

Following Kaplan [18], we say that the Lie algebra $\mathfrak{n}$ is H-type if

$$J_Z^2 = -|Z|^2 I_{\mathfrak{v}} \quad \forall Z \in \mathfrak{z}, \tag{1}$$

where $I_{\mathfrak{v}}$ is the identity operator on $\mathfrak{v}$. We denote the dimensions of $\mathfrak{v}$ and $\mathfrak{z}$ by $d_{\mathfrak{v}}$ and $d_{\mathfrak{z}}$. From property (1), it follows that $\mathfrak{z} = [\mathfrak{v}, \mathfrak{v}]$, and moreover, if $\mathfrak{z}$ is nontrivial, then the dimension $d_{\mathfrak{v}}$ of $\mathfrak{v}$ is even. A connected, simply connected Lie group N whose Lie algebra is an H-type algebra is said to be a Heisenberg type group, or more briefly, an H-type group. The Iwasawa N-groups associated to all real rank one simple groups are H-type. For more information about H-type groups and their connection with Iwasawa N-groups, see [5,6].

We shall use the following properties of the map J, proved in [5, Section 1]:

$$\begin{aligned} J_Z J_{Z'} + J_{Z'} J_Z &= -2\langle Z, Z' \rangle I_{\mathfrak{v}} \quad \forall Z, Z' \in \mathfrak{z}, \\ [X, J_Z X] &= |X|^2 Z \quad \forall X \in \mathfrak{v}\ \forall Z \in \mathfrak{z}, \\ J_{[X,X']} X &= |X|^2 P_{J_{\mathfrak{z}} X} X' \quad \forall X, X' \in \mathfrak{v}, \end{aligned} \tag{2}$$

where $P_{J_{\mathfrak{z}} X} X'$ denotes the projection of X' onto the space $\{J_Z X : Z \in \mathfrak{z}\}$.

Since N is a simply connected nilpotent Lie group, the exponential mapping is bijective. For X in $\mathfrak{v}$ and Z in $\mathfrak{z}$, we denote by (X, Z) the element $\exp(X + Z)$ of the group N. By the Baker–Campbell–Hausdorff formula, the group law is given by

$$(X, Z)(X', Z') = \left(X + X', Z + Z' + \tfrac{1}{2}[X, X']\right)$$

for all X and X' in $\mathfrak{v}$ and all Z and Z' in $\mathfrak{z}$. The group N is unimodular and an element dn of Haar measure on N is $2^{d_{\mathfrak{z}}} dX\, dZ$, where dX and dZ are elements of the Lebesgue measures on the real vector spaces $\mathfrak{v}$ and $\mathfrak{z}$.

When $t > 0$, we define the homogeneous dilation δ_t on N by

$$\delta_t(X, Z) = (tX, t^2 Z) \quad \forall (X, Z) \in N.$$

It is easy to check that δ_t is a group automorphism. We write Q for the homogeneous dimension of N, i.e., $d_{\mathfrak{v}} + 2d_{\mathfrak{z}}$. The function $\mathcal{N}$ on N given by

$$\mathcal{N}(X, Z) = \left(\frac{|X|^4}{16} + |Z|^2\right)^{1/4}$$

is a homogeneous gauge. Define $d_N : N \times N \to \mathbb{R}_0^+$ by $d_N(n_1, n_2) = \mathcal{N}(n_1^{-1} n_2)$; then d_N satisfies the triangle inequality (see [8]), and is a metric.

Definition 2.1 (see Cowling et al. [5]). We say that $\mathfrak{n}$ satisfies the J^2-*condition* if, for any X in $\mathfrak{v}$ and Z and Z' in $\mathfrak{z}$ such that $\langle Z, Z' \rangle = 0$, there exists Z'' in $\mathfrak{z}$ such that

$$J_Z J_{Z'} X = J_{Z''} X.$$

It is known [5] that an H-type group N is the Iwasawa N-group of a real rank one simple Lie group if and only if its Lie algebra satisfies the J^2-condition. We shall henceforth assume that this condition holds.

2.2. The Cayley transform

We write $\mathbb{S}$ for the unit sphere in $\mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$, i.e.,

$$\mathbb{S} = \{(X', Z', t') \in \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R} : |X'|^2 + |Z'|^2 + t'^2 = 1\}.$$

The Cayley transform $\mathscr{C} : N \to \mathbb{S}$, introduced in [6], is given by

$$\mathscr{C}(X, Z) = \frac{1}{\mathscr{B}(X, Z)} \left(\bar{\mathscr{A}}(X, Z)X, 2Z, -1 + \frac{|X|^4}{16} + |Z|^2 \right),$$

where $\mathscr{A}(X, Z)$ and $\bar{\mathscr{A}}(X, Z)$ denote the linear maps $1 + |X|^2/4 + J_Z$ and $1 + |X|^2/4 - J_Z$ on $\mathfrak{v}$, and the real number $\mathscr{B}(X, Z)$ is defined by

$$\mathscr{B}(X, Z) = \left(1 + \frac{|X|^2}{4}\right)^2 + |Z|^2.$$

Its inverse $\mathscr{C}^{-1} : \mathbb{S} \backslash \{o\} \to N$ is given by

$$\mathscr{C}^{-1}(X', Z', t') = \frac{1}{(1 - t')^2 + |Z'|^2} (2(1 - t' + J_{Z'})X', 2Z').$$

Here o stands for $(0, 0, 1)$. Note that if $(X', Z', t') = \mathscr{C}(X, Z)$, then

$$\mathscr{B}(X, Z) = \frac{4}{(1 - t')^2 + |Z'|^2}.$$

When the dependence on (X, Z) in N is clear, we just write $\mathscr{A}$, $\bar{\mathscr{A}}$ and $\mathscr{B}$.

Because we assume the J^2-condition, there is a simple Lie group G of real rank one whose Iwasawa N-group is N. The subgroups K, A and M of G are defined in the standard way: KAN is an Iwasawa decomposition of G, and M is the centraliser of A in K. The sphere $\mathbb{S}$ may be identified with K/M, and the map $\mathscr{C}$ is essentially the map treated by Helgason [13] (see Corollary 1.9 on p. 407 and the following sections).

There is a natural action of K on $\mathbb{S}$ by orthogonal transformations, which can be realized so that M stabilizes o and $-o$. We will write $k \cdot \zeta$ to indicate the result of applying k in K to ζ in $\mathbb{S}$.

For functions f on $\mathbb{S}$ and g on K/M, we denote by $f^\sharp$ and $g^\flat$ the corresponding functions on K/M and on $\mathbb{S}$.

We define $\mathscr{D}$ to be the function $(\mathscr{B}^{-1/4} \circ \mathscr{C}^{-1})^\sharp$ on K/M. Since $\mathscr{D}$ is M-invariant, we may define a function $d_{\mathbb{S}}$ on $\mathbb{S} \times \mathbb{S}$ by

$$d_{\mathbb{S}}(k_1 \cdot o, k_2 \cdot o) = \mathscr{D}(k_1^{-1} k_2 M) \quad \forall k_1, k_2 \in K.$$

In [2], Banner proved that $d_{\mathbb{S}}$ is a distance on $\mathbb{S}$.

Lemma 2.2. *For all n_1, n_2 in N,*

$$d_{\mathbb{S}}(\mathscr{C}(n_1), \mathscr{C}(n_2)) = \mathscr{B}^{-1/4}(n_1) \mathscr{B}^{-1/4}(n_2) \mathscr{N}(n_1^{-1} n_2).$$

Proof. Banner [2] showed that $4 d_{\mathbb{S}}(\zeta_1, \zeta_2)^4$ is equal to

$$\begin{aligned} &\langle X_1', X_2' \rangle^2 + |[X_1', X_2']|^2 + \left(1 - |X_1'|^2\right)\left(1 - |X_2'|^2\right) + 1 \\ &\quad + 2\left\langle \left(t_1' - J_{Z_1'}\right) X_1', \left(t_2' - J_{Z_2'}\right) X_2' \right\rangle - 2(\langle X_1', X_2' \rangle + \langle Z_1', Z_2' \rangle + t_1' t_2'), \end{aligned} \tag{3}$$

where $\zeta_1 = (X_1', Z_1', t_1')$ and $\zeta_2 = (X_2', Z_2', t_2')$. When $h = 1$ or 2, write ζ_h for

$$\mathscr{C}(n_h) = \mathscr{C}(X_h, Z_h) = \left(\mathscr{B}_h^{-1} \bar{\mathscr{A}}_h X_h, 2\mathscr{B}_h^{-1} Z_h, 1 - 2\mathscr{B}_h^{-1}(1 + |X_h|^2/4)\right),$$

where $\bar{\mathscr{A}}_h = 1 + |X_h|^2/4 - J_{Z_h}$ and $\mathscr{B}_h = (1 + |X_h|^2/4)^2 + |Z_h|^2$. Note that $t_h' - J_{Z_h'} = 1 - 2\mathscr{B}_h^{-1} \mathscr{A}_h$, so $4 d_{\mathbb{S}}(\zeta_1, \zeta_2)^4$ is equal to

$$\begin{aligned} &\left\langle \mathscr{B}_1^{-1} \bar{\mathscr{A}}_1 X_1, \mathscr{B}_2^{-1} \bar{\mathscr{A}}_2 X_2 \right\rangle^2 + \left|\left[\mathscr{B}_1^{-1} \bar{\mathscr{A}}_1 X_1, \mathscr{B}_2^{-1} \bar{\mathscr{A}}_2 X_2\right]\right|^2 \\ &\quad + 2\left\langle (1 - 2\mathscr{B}_1^{-1} \mathscr{A}_1) \mathscr{B}_1^{-1} \bar{\mathscr{A}}_1 X_1, (1 - 2\mathscr{B}_2^{-1} \mathscr{A}_2) \mathscr{B}_2^{-1} \bar{\mathscr{A}}_2 X_2 \right\rangle \\ &\quad + 1 + \left(1 - \mathscr{B}_1^{-1} |X_1|^2\right)\left(1 - \mathscr{B}_2^{-1} |X_2|^2\right) - 2\left\langle \mathscr{B}_1^{-1} \bar{\mathscr{A}}_1 X_1, \mathscr{B}_2^{-1} \bar{\mathscr{A}}_2 X_2 \right\rangle \end{aligned}$$

$$
\begin{aligned}
&- 2\langle 2\mathscr{B}_1^{-1}Z_1, 2\mathscr{B}_2^{-1}Z_2\rangle - 2\left(1 - 2\mathscr{B}_1^{-1}\left(1+\frac{|X_1|^2}{4}\right)\right)\left(1 - 2\mathscr{B}_2^{-1}\left(1+\frac{|X_2|^2}{4}\right)\right)\\
&= \mathscr{B}_1^{-1}\mathscr{B}_2^{-1}\Big(\mathscr{B}_1^{-1}\mathscr{B}_2^{-1}\langle\bar{\mathscr{A}}_1X_1,\bar{\mathscr{A}}_2X_2\rangle^2 + \mathscr{B}_1^{-1}\mathscr{B}_2^{-1}|[\bar{\mathscr{A}}_1X_1,\bar{\mathscr{A}}_2X_2]|^2 + \mathscr{B}_1\mathscr{B}_2\\
&\quad + 2\langle\bar{\mathscr{A}}_1X_1 - 2X_1, \bar{\mathscr{A}}_2X_2 - 2X_2\rangle + (\mathscr{B}_1 - |X_1|^2)\left(\mathscr{B}_2 - |X_2|^2\right) - 8\langle Z_1, Z_2\rangle\\
&\quad - 2\langle\bar{\mathscr{A}}_1X_1,\bar{\mathscr{A}}_2X_2\rangle - 2\left(\mathscr{B}_1 - 2\left(1+\frac{|X_1|^2}{4}\right)\right)\left(\mathscr{B}_2 - 2\left(1+\frac{|X_2|^2}{4}\right)\right)\Big),
\end{aligned}
$$

by (3). The equality

$$
\langle\bar{\mathscr{A}}_1X_1,\bar{\mathscr{A}}_2X_2\rangle^2 + |[\bar{\mathscr{A}}_1X_1,\bar{\mathscr{A}}_2X_2]|^2 = \mathscr{B}_1\mathscr{B}_2(\langle X_1, X_2\rangle^2 + |[X_1, X_2]|^2)
$$

is equivalent to the J^2-condition (see [5, p. 25], where $\mathscr{A}$ is defined slightly differently). By writing $\mathscr{A}_h$ and $\mathscr{B}_h$ in terms of X_h and Z_h and simplifying, we see that $4\mathscr{B}_1\mathscr{B}_2 d_{\mathbb{S}}(\zeta_1,\zeta_2)^4$ is equal to

$$
\begin{aligned}
&\frac{1}{4}\left(|X_1|^4 + 2|X_1|^2|X_2|^2 + |X_2|^4\right) + \langle X_1, X_2\rangle^2 - \left(|X_1|^2 + |X_2|^2\right)\langle X_1, X_2\rangle\\
&\quad + 4\left(|Z_1|^2 + |Z_2|^2\right) + 4\langle Z_1 - Z_2, [X_1, X_2]\rangle + |[X_1, X_2]|^2 - 8\langle Z_1, Z_2\rangle\\
&\quad = 4\mathscr{N}\left(n_1^{-1}n_2\right)^4,
\end{aligned}
$$

as required.

This explicit calculation does not consider the ambient group G; the result can also be derived by considering the relation between the Knapp–Stein intertwining operators for G in the compact and noncompact pictures; indeed, the kernels of these operators for the class-one principal series for G are powers of $\mathscr{N}$ in the noncompact picture and powers of $d_{\mathbb{S}}$ in the compact picture.

The following result was proved by Banner [2].

Corollary 2.3. *The Cayley transform $\mathscr{C}$ is* 1-*quasiconformal.*

Proof. It suffices to show that

$$
\lim_{r\to 0}\frac{\sup\{d_{\mathbb{S}}(\mathscr{C}(n_1),\mathscr{C}(n_2)) : \mathscr{N}(n_1^{-1}n_2) = r\}}{\inf\{d_{\mathbb{S}}(\mathscr{C}(n_1),\mathscr{C}(n_2)) : \mathscr{N}(n_1^{-1}n_2) = r\}} = 1 \quad \forall n_1 \in N.
$$

But if $\mathcal{N}(n_1^{-1}n_2) = \mathcal{N}(n_1^{-1}n_3) = r$, then

$$\frac{d_{\mathbb{S}}(\mathscr{C}(n_1), \mathscr{C}(n_2))}{d_{\mathbb{S}}(\mathscr{C}(n_1), \mathscr{C}(n_3))} = \frac{\mathscr{B}^{1/4}(n_3)}{\mathscr{B}^{1/4}(n_2)},$$

which can be made arbitrarily close to 1 by taking r small.

2.3. Vector fields and the Jacobian

We fix orthonormal bases $\{E_j\}_{j=1}^{d_{\mathfrak{v}}}$ of $\mathfrak{v}$ and $\{U_k\}_{k=1}^{d_{\mathfrak{z}}}$ of $\mathfrak{z}$. Given V in $\mathfrak{n}$, we also write V for the associated left-invariant vector field, i.e.,

$$Vf(n) = \frac{d}{dt} f(n \exp tV)\bigg|_{t=0} \quad \forall n \in N \ \forall f \in C^\infty(N).$$

Then

$$\begin{aligned} E_j f(X,Z) &= \partial_{x_j} f(X,Z) + \frac{1}{2} \sum_{k=1}^{d_{\mathfrak{z}}} \langle J_{U_k} X, E_j \rangle \partial_{z_k} f(X,Z) \\ U_k f(X,Z) &= \partial_{z_k} f(X,Z) \end{aligned} \tag{4}$$

for all smooth functions f on N and (X,Z) in N, where $j = 1, \ldots, d_{\mathfrak{v}}$ and $k = 1, \ldots, d_{\mathfrak{z}}$. We write $T_n^{(1)}(N)$ and $T_n^{(2)}(N)$ for the spans of the tangent vectors at n in N associated to the E_j and to the U_k.

We endow the sphere $\mathbb{S}$ with the Euclidean metric, which is $\mathrm{SO}(d_{\mathfrak{v}} + d_{\mathfrak{z}} + 1)$-invariant. The tangent space $T_o(\mathbb{S})$ at the point o decomposes orthogonally as $\mathfrak{v} \oplus \mathfrak{z}$. Since K acts orthogonally [6, Theorem 6.1], and $\mathrm{Ad}(M)$ preserves $\mathfrak{v}$ and $\mathfrak{z}$, the tangent space $T_\zeta(\mathbb{S})$ at any point ζ decomposes orthogonally as $T_\zeta^{(1)}(\mathbb{S}) \oplus T_\zeta^{(2)}(\mathbb{S})$, where $T_\zeta^{(1)}(\mathbb{S}) = k_*\mathfrak{v}$ and $T_\zeta^{(2)}(\mathbb{S}) = k_*\mathfrak{z}$ when $\zeta = k \cdot o$.

The Carnot–Caratheodory structure of N and $\mathbb{S}$ is reflected in the following lemma, linking the metrics and the special subspaces of the tangent spaces.

Lemma 2.4. *Suppose that $V \in T_n(N)$ and that γ is a smooth curve in N for which $\gamma(0) = n$ and $\dot\gamma(0) = V$. Then $d_N(n, \gamma(s)) = O(s)$ as $s \to 0$ if and only if $V \in T_n^{(1)}(N)$. In this case, $d_N(n, \gamma(s)) = |s||V|/2 + o(s)$ as $s \to 0$.*

Similarly, suppose that $V \in T_\zeta(\mathbb{S})$ and that γ is a smooth curve in $\mathbb{S}$ for which $\gamma(0) = \zeta$ and $\dot\gamma(0) = V$. Then $d_{\mathbb{S}}(\zeta, \gamma(s)) = O(s)$ as $s \to 0$ if and only if $V \in T_\zeta^{(1)}(\mathbb{S})$. In this case, $d_{\mathbb{S}}(\zeta, \gamma(s)) = |s|\,|V|/2 + o(s)$ as $s \to 0$.

Proof. We prove only the second part of the lemma, as the proof of the first part is similar but easier. The metric $d_{\mathbb{S}}$ and the subspace $T_\zeta^{(1)}(\mathbb{S})$ are both K-invariant, so it suffices to consider the case where $\zeta = o$.

Suppose that $X' \in \mathfrak{v}$ and $Z' \in \mathfrak{z}$. Then from (3),

$$
\begin{aligned}
&4d_{\mathbb{S}}((0,0,1),(X',Z',(1-|X'|^2-|Z'|^2)^{1/2}))^4 \\
&\quad = 2-|X'|^2-2(1-|X'|^2-|Z'|^2)^{1/2} \\
&\quad = 2-|X'|^2-2+(|X'|^2+|Z'|^2)+\frac{1}{4}(|X'|^2+|Z'|^2)^2+O\Big((|X'|^2+|Z'|^2)^3\Big) \\
&\quad = |Z'|^2+\frac{1}{4}(|X'|^2+|Z'|^2)^2+O\Big((|X'|^2+|Z'|^2)^3\Big).
\end{aligned}
$$

It follows that $V \in T_o^{(1)}(\mathbb{S})$ if and only if $d_{\mathbb{S}}(o,\gamma(s)) = O(s)$ as $s \to 0$, and when this is satisfied, $\lim_{s\to 0} |s|^{-1} d_{\mathbb{S}}(o,\gamma(s)) = |V|/2$ as $s \to 0$.

Lemma 2.5. *Suppose that $n \in N$ and that $\zeta = \mathscr{C}(n)$. For every vector V in $T_n^{(1)}(N)$ the vector $\mathscr{C}_* V$ lies in $T_\zeta^{(1)}(\mathbb{S})$, and vice versa. Moreover*

$$
|\mathscr{C}_* V| = \mathscr{B}^{-1/2}(n)|V| \quad \forall V \in T_n^{(1)}(N).
$$

Proof. Take V in $T_n(N)$, and a curve $\gamma : I \to N$ such $\gamma(0) = n$ and $\dot{\gamma}(0) = V$. From Lemma 2.2, $d_{\mathbb{S}}(\mathscr{C}(n), \mathscr{C}(\gamma(s))) = O(s)$ as $s \to 0$ if and only if $d_N(n,\gamma(s)) = O(s)$ as $s \to 0$. From Lemma 2.4, we deduce that $V \in T_n^{(1)}(N)$ if and only if $\mathscr{C}_* V \in T_\zeta^{(1)}(\mathbb{S})$. Further, in the case when these conditions hold,

$$
|\mathscr{C}_* V| = 2 \lim_{s\to 0} \frac{d_{\mathbb{S}}(\mathscr{C}(n), \mathscr{C}(\gamma(s)))}{|s|} = 2\mathscr{B}(n)^{-1/2} \lim_{s\to 0} \frac{d_N(n,\gamma(s))}{|s|} = \mathscr{B}(n)^{-1/2}|V|,
$$

as required. □

The Jacobian determinant $J_{\mathscr{C}^{-1}}$ of the map $\mathscr{C}^{-1}$ is given by

$$
J_{\mathscr{C}^{-1}}(\zeta) = \frac{d(\mathscr{C}^{-1}\zeta)}{d\sigma(\zeta)} = \left(\frac{4}{(1-t')^2 + |Z'|^2} \right)^{Q/2}
$$

for all $\zeta = (X', Z', t') \in \mathbb{S} \backslash \{o\}$, where σ denotes the standard measure on the sphere (see [6, p. 234]). Clearly $J_{\mathscr{C}} = J_{\mathscr{C}^{-1}}^{-1} \circ \mathscr{C} = \mathscr{B}^{-Q/2}$. This result may also be deduced from the previous lemma and the fact that the Jacobian of a 1-quasiconformal map of homogeneous manifolds is a multiple of the Qth power of the dilation factor.

2.4. Sublaplacians

Define the sublaplacian Δ on N by

$$(\Delta f \mid f)_N = \sum_{j=1}^{d_{\mathfrak{v}}} \int_N |E_j f(n)|^2 \, dn \quad \forall f \in C_c^\infty(N), \tag{5}$$

where $(\cdot \mid \cdot)_N$ denotes the usual inner product on $L^2(N)$. Since the fields E_j are divergence free,

$$\Delta = -\sum_{j=1}^{d_{\mathfrak{v}}} E_j^2.$$

Similarly, we define the sublaplacian $\mathscr{L}$ on the sphere $\mathbb{S}$ by

$$(\mathscr{L} f \mid f)_{\mathbb{S}} = \sum_{j=1}^{d_{\mathfrak{v}}} \int_{\mathbb{S}} |W_j f|^2 \, d\sigma \quad \forall f \in C^\infty(\mathbb{S}), \tag{6}$$

where $\{W_j\}_{j=1}^{d_{\mathfrak{v}}}$ is any set of vector fields defined almost everywhere on $\mathbb{S}$ which form an orthonormal basis for $T_\zeta^{(1)}(\mathbb{S})$ almost everywhere, and $(\cdot \mid \cdot)_{\mathbb{S}}$ denotes the usual inner product on $L^2(\mathbb{S})$. In what follows, we will define W_j by

$$W_j = |\mathscr{C}_* E_j|^{-1} \mathscr{C}_* E_j = J_{\mathscr{C}^{-1}}^{1/Q} \mathscr{C}_* E_j \quad \forall j \in \{1, \dots, d_{\mathfrak{v}}\}.$$

Note that $\mathscr{L}$ is K-invariant, because the action of K preserves $T^{(1)}(\mathbb{S})$, while the vector fields W_j are not K-invariant, nor are they divergence free.

In the real case, when $d_{\mathfrak{z}} = 0$, the operator $\mathscr{L}$ is the Laplace–Beltrami operator on the Euclidean sphere. In the complex case, when $d_{\mathfrak{z}} = 1$, the operator $\mathscr{L}$ is the laplacian considered by Geller [12] on the complex sphere.

Lemma 2.6. *For every β in $\mathbb{C}$ and (X, Z) in N,*

$$\Delta J_{\mathscr{C}}^{\beta}(X, Z) = J_{\mathscr{C}}^{\beta + 2/Q}(X, Z) \left[\frac{d_{\mathfrak{v}} \beta Q}{2} + \frac{|X|^2 \beta Q^2}{4} \left(-\beta + \frac{1}{2} - \frac{1}{Q} \right) \right].$$

Proof. This is a routine calculation, based on (2) and the equalities

$$E_j(J_{\mathscr{C}}^{-2/Q})(X, Z) = E_j(\mathscr{B})(X, Z) = \langle \mathscr{A} X, E_j \rangle$$

$$E_j(\mathscr{A} X) = \mathscr{A} E_j + \frac{1}{2} \Big(\langle X, E_j \rangle + J_{[X, E_j]} \Big) X \tag{7}$$

for all (X, Z) in N. $\square$

Theorem 2.7. *Let $\mathscr{L}$ be the sublaplacian on the sphere and let Δ be the sublaplacian on N. Then*

$$(\Delta(J_{\mathscr{C}}^{1/2-1/Q}\psi\circ\mathscr{C}) \mid J_{\mathscr{C}}^{1/2-1/Q}\varphi\circ\mathscr{C})_N = ((\mathscr{L}+b)\psi \mid \varphi)_{\mathbb{S}}$$

for all ψ and φ in $C^\infty(\mathbb{S})$, where $b=(Q-2)d_{\mathfrak{v}}/4$.

Proof. From (6),

$$\begin{aligned}(\mathscr{L}\psi \mid \varphi)_{\mathbb{S}} &= \sum_{j=1}^{d_{\mathfrak{v}}} \left(J_{\mathscr{C}^{-1}}^{1/Q}\mathscr{C}_* E_j\psi \mid J_{\mathscr{C}^{-1}}^{1/Q}\mathscr{C}_* E_j\varphi\right)_{\mathbb{S}}\\ &= \sum_{j=1}^{d_{\mathfrak{v}}} \left(J_{\mathscr{C}}^{1/2-1/Q}E_j(\psi\circ\mathscr{C}) \mid J_{\mathscr{C}}^{1/2-1/Q}E_j(\varphi\circ\mathscr{C})\right)_N,\end{aligned}$$

and so from (5), the product rule, the definition of the inner product on $L^2(N)$, and integration by parts,

$$\begin{aligned}&\left(\Delta(J_{\mathscr{C}}^{1/2-1/Q}\psi\circ\mathscr{C}) \mid (J_{\mathscr{C}}^{1/2-1/Q}\varphi\circ\mathscr{C})\right)_N - (\mathscr{L}\psi \mid \varphi)_{\mathbb{S}}\\
&= \sum_{j=1}^{d_{\mathfrak{v}}}\Big[\left((E_jJ_{\mathscr{C}}^{1/2-1/Q})\psi\circ\mathscr{C} \mid (E_jJ_{\mathscr{C}}^{1/2-1/Q})\varphi\circ\mathscr{C}\right)_N\\
&\quad + \left((E_jJ_{\mathscr{C}}^{1/2-1/Q})\psi\circ\mathscr{C} \mid J_{\mathscr{C}}^{1/2-1/Q}E_j(\varphi\circ\mathscr{C})\right)_N\\
&\quad + \left(J_{\mathscr{C}}^{1/2-1/Q}E_j(\psi\circ\mathscr{C}) \mid (E_jJ_{\mathscr{C}}^{1/2-1/Q})\varphi\circ\mathscr{C}\right)_N\Big]\\
&= \sum_{j=1}^{d_{\mathfrak{v}}}\Big[\left((E_jJ_{\mathscr{C}}^{1/2-1/Q})^2\psi\circ\mathscr{C} \mid \varphi\circ\mathscr{C}\right)_N\\
&\quad + \left(J_{\mathscr{C}}^{1/2-1/Q}(E_jJ_{\mathscr{C}}^{1/2-1/Q}) \mid E_j(\bar{\psi}\circ\mathscr{C}\varphi\circ\mathscr{C})\right)_N\Big]\\
&= \sum_{j=1}^{d_{\mathfrak{v}}}\Big[\left((E_jJ_{\mathscr{C}}^{1/2-1/Q})^2\psi\circ\mathscr{C} \mid \varphi\circ\mathscr{C}\right)_N\\
&\quad - \left(E_j(J_{\mathscr{C}}^{1/2-1/Q}(E_jJ_{\mathscr{C}}^{1/2-1/Q}))\psi\circ\mathscr{C} \mid \varphi\circ\mathscr{C}\right)_N\Big]\\
&= -\sum_{j=1}^{d_{\mathfrak{v}}}\Big[\left(J_{\mathscr{C}}^{1/2-1/Q}E_j^2(J_{\mathscr{C}}^{1/2-1/Q})\psi\circ\mathscr{C} \mid \varphi\circ\mathscr{C}\right)_N\Big].\end{aligned}$$

The result now follows by Lemma 2.6, with β taken to be $1/2-1/Q$, and a change of variables. □

We now determine a fundamental solution of the "conformal sublaplacian" $(\mathscr{L}+b)$, i.e., a distribution u on the sphere such that $(\mathscr{L}+b)u$ is the Dirac delta at the point o.

Theorem 2.8. *The fundamental solution of* $\mathscr{L}+b$ *is a positive multiple of* $J_{\mathscr{C}^{-1}}^{1/2-1/Q}$.

Proof. First, for every β in $\mathbb{C}$ and $\zeta=(X',Z',t')$ in $\mathbb{S}\backslash\{o\}$,

$$(\mathscr{L}+b)J_{\mathscr{C}^{-1}}^{\beta}(\zeta)=-\left(\beta-\frac{1}{2}+\frac{1}{Q}\right)\left[\frac{d_{\mathfrak{v}}Q}{2}J_{\mathscr{C}^{-1}}^{\beta}(\zeta)+\frac{\beta Q^2}{4}|X'|^2J_{\mathscr{C}^{-1}}^{\beta+2/Q}(\zeta)\right]. \tag{8}$$

This formula is a simple consequence of the intertwining property of Theorem 2.7 applied to the function $J_{\mathscr{C}^{-1}}^{\beta}$, and Lemma 2.6.

Note that the function $J_{\mathscr{C}^{-1}}^{\beta}$ is regular away from the pole o. When (X',Z',t') is close to o on the sphere, $t'=(1-|X'|^2-|Z'|^2)^{1/2}$, and by arguing as in Lemma 2.4, we see that

$$\begin{aligned} J_{\mathscr{C}^{-1}}(X',Z',t') &= d_{\mathbb{S}}^{-2Q}(o,(X',Z',(1-|X'|^2-|Z'|^2)^{1/2})) \\ &= \left((|Z'|^2+\frac{|X'|^4}{4}+o(|X'|^4+|Z'|^2)\right)^{-Q/2} \\ &\asymp \mathscr{N}^{-2Q}(X',Z'). \end{aligned}$$

Therefore $J_{\mathscr{C}^{-1}}^{\beta}$ is integrable on the sphere if $\mathrm{Re}(\beta)<1/2$. Similarly, it follows that $(X',Z',t')\mapsto|X'|^2J_{\mathscr{C}^{-1}}^{\beta+2/Q}(X',Z',t')$ is integrable if $\mathrm{Re}(\beta)<1/2-1/Q$.

Following [4, Lemma 1.1], we deduce that the distribution-valued functions $\beta\mapsto J_{\mathscr{C}^{-1}}^{\beta}$ and $\beta\mapsto|X'|^2J_{\mathscr{C}^{-1}}^{\beta+2/Q}$ are holomorphic when $\mathrm{Re}(\beta)<1/2$ and $\mathrm{Re}(\beta)<1/2-1/Q$, and that they extend meromorphically to $\mathbb{C}$ with simple poles in $\{1/2+k/Q : k\in\mathbb{N}\}$ and $\{1/2+(k-1)/Q : k\in\mathbb{N}\}$.

When $\mathrm{Re}(\beta)<1/2-1/Q$, we can evaluate the sublaplacian of the distribution $J_{\mathscr{C}^{-1}}^{\beta}$ by ordinary differentiation. The conclusion follows by evaluating the limit as β tends to $1/2-1/Q$ of both sides in formula (8) and checking that the residue of the function $\beta\mapsto|X'|^2J_{\mathscr{C}^{-1}}^{\beta+2/Q}$ at $1/2-1/Q$ is a positive multiple of the Dirac delta at the point o. This can be done as in [4, Remark 1.10]. $\square$

3. Sobolev spaces on N and on K/M

3.1. Sobolev spaces on N

In this subsection, we recall some properties of Sobolev spaces on N. For further details the reader can refer to [4,11]. In [4], the proofs are given for only some

Iwasawa N groups, but these extend to all Heisenberg type groups with minor modifications (see [1]).

A function f on N is said to be *homogeneous* of degree γ (where $\gamma \in \mathbb{C}$) if

$$f \circ \delta_t = t^\gamma f \quad \forall t \in \mathbb{R}^+.$$

A distribution K on N is said to be a kernel of type γ if it coincides with a homogeneous function f_K of degree $\gamma - Q$ on $N \setminus \{(0,0)\}$. Below, we shall use the same notation K for the kernel K and the associated function f_K.

The sublaplacian Δ is a densely defined, essentially self-adjoint, positive operator on $L^2(N)$. Hence it has a spectral resolution given by

$$\Delta = \int_0^\infty \lambda \, dA_\lambda.$$

Folland [11, Proposition 3.9] proved that 0 is not an eigenvalue of Δ, and so we may define the operators Δ^α, for every α in $\mathbb{C}$, by the formula

$$\Delta^\alpha = \int_0^\infty \lambda^\alpha \, dA_\lambda.$$

He also proved the following results [11, Theorem 3.15, Proposition 3.17, 3.18].

Proposition 3.1. *The operators Δ^α have the following properties*:

(i) *Δ^α is closed on $L^2(N)$ for every α in $\mathbb{C}$;*
(ii) *if f is in $\mathrm{Dom}(\Delta^\alpha) \cap \mathrm{Dom}(\Delta^{\alpha+\beta})$, then $\Delta^\alpha f$ is in $\mathrm{Dom}(\Delta^\beta)$ and further $\Delta^\beta \Delta^\alpha f = \Delta^{\alpha+\beta} f$. In particular, $\Delta^{-\alpha} = (\Delta^\alpha)^{-1}$;*
(iii) *if $0 < \mathrm{Re}(\alpha) < Q$, then there exists a kernel $\mathcal{R}_\alpha$ of type α, smooth in $N \setminus \{(0,0)\}$, such that $\Delta^{-\alpha/2} f = f * \mathcal{R}_\alpha$ for all f in $\mathrm{Dom}(\Delta^{-\alpha/2})$.*

Definition 3.2. For α in $(-Q/2, Q/2)$, we define the homogeneous Sobolev space $\mathcal{H}^\alpha(N)$ to be the completion of the space of compactly supported smooth functions on N with respect to the norm

$$||f||_{\mathcal{H}^\alpha(N)} = (\Delta^\alpha f \,|\, f)_N^{1/2} = ||\, \Delta^{\alpha/2} f||_{L^2(N)}.$$

The spaces $\mathcal{H}^\alpha(N)$ and $\mathcal{H}^{-\alpha}(N)$ are dual relative to the usual L^2 inner product. For f in $\mathcal{H}^\alpha(N)$, we write $\partial_{x_j} f$ and $\partial_{z_h} f$ for its distributional derivatives.

We need a characterization of Sobolev spaces, due to Folland [11, Theorem 4.10] for the nonhomogeneous Sobolev spaces $\mathcal{H}^\alpha(N) \cap L^2(N)$ for positive α; his proof can be adapted without substantial changes to the case of homogeneous Sobolev spaces.

Theorem 3.3. *Suppose that* α *is in* $[1, Q/2)$. *Then* f *is in* $\mathscr{H}^\alpha(N)$ *if and only if* $E_j f$ *is in* $\mathscr{H}^{\alpha-1}(N)$ *when* $j = 1, \dots, d_\mathfrak{v}$; *moreover, the norms* $\|f\|_{\mathscr{H}^\alpha(N)}$ *and* $\sum_{j=1}^{d_\mathfrak{v}} \|E_j f\|_{\mathscr{H}^{\alpha-1}(N)}$ *are equivalent.*

The following Sobolev immersion properties hold.

Proposition 3.4. *If* f *is a continuous and compactly supported function on* N, *then* f *is in the Sobolev space* $\mathscr{H}^\alpha(N)$ *for every* α *in* $(-Q/2, 0]$, *and moreover*

$$\|f\|_{\mathscr{H}^\alpha(N)} \leqslant C_\alpha \|f\|_{L^{p,2}(N)},$$

where $1/p = 1/2 - \alpha/Q$.

Proof. The proposition is true if $\alpha = 0$. If α is in $(-Q/2, 0)$, then by item (iii) of Proposition 3.1,

$$\Delta^{\alpha/2} f = f * \mathscr{R}_{-\alpha},$$

where $\mathscr{R}_{-\alpha} = \Omega \mathscr{N}^{-(\alpha+Q)}$ and Ω is homogeneous of degree 0 and smooth away from the identity. Since $|\mathscr{R}_{-\alpha}| \leqslant C \mathscr{N}^{-(\alpha+Q)}$, and $(\mathscr{N}^{-(\alpha+Q)})^*(t) = C t^{-(1+\alpha/Q)}$ when $t > 0$, it follows that $\mathscr{R}_{-\alpha}$ is in $L^{r,\infty}(N)$ when $1/r = 1 + \alpha/Q$.

If $1/p = 1/2 - \alpha/Q$, then $1 < p < 2$ and $1/p + 1/r = 1/2 + 1$. Therefore, by Lemma 1.3 and the unimodularity of N,

$$\left\| \Delta^{\alpha/2} f \right\|_{L^2(N)} = \|f * \mathscr{R}_{-\alpha}\|_{L^2(N)} \leqslant C_\alpha \|\mathscr{R}_{-\alpha}\|_{L^{r,\infty}(N)} \|f\|_{L^{p,2}(N)},$$

as required. $\square$

By duality, we obtain the following result.

Corollary 3.5. *If* $0 < \alpha < Q/2$ *and* $1/p = 1/2 - \alpha/Q$, *then* $\mathscr{H}^\alpha(N)$ *is contained in* $L^{p,2}(N)$, *and moreover*

$$\|f\|_{L^{p,2}(N)} \leqslant C_\alpha \|f\|_{\mathscr{H}^\alpha(N)} \quad \forall f \in \mathscr{H}^\alpha(N).$$

The rest of this subsection deals with multiplier theorems on Sobolev spaces on N. Theorem 3.6 below was first proved in [4] for the Iwasawa N groups corresponding to the real, complex and quaternionic hyperbolic spaces. The methods arise in works of Strichartz [21] and Lohoué [19].

For k in $\mathbb{Z}^+$, let $\mathfrak{D}^k$ denote the set of all differential operators of the form

$$E_{j_1} E_{j_2} \cdots E_{j_k},$$

where $1 \leqslant j_1, j_2, \dots, j_k \leqslant d_{\mathfrak{v}}$; $\mathfrak{D}^0$ denotes the set containing the identity operator.

For a real number γ and a nonnegative integer d, denote by $\mathscr{M}_{\gamma,d}(N)$ the space of functions m in $C^d(N\backslash\{(0,0)\})$ such that

$$|D^k m| \leqslant C_k \mathcal{N}^{-\gamma-k} \quad \forall D^k \in \mathfrak{D}^k \ \forall k \in \{0,1,2,\dots,d\}.$$

For a real number a, define $\lceil a \rceil$ to be the integer such that $a - \lceil a \rceil$ is in $(-1,0]$, and, for α and β real, define the nonnegative integer $d_{(\alpha,\beta)}$ by

$$d_{(\alpha,\beta)} = \begin{cases} \max(0, \lceil \alpha \rceil) & \text{if } \beta \geqslant 0, \\ \max(0, \lceil -\beta \rceil) & \text{if } \beta < 0. \end{cases}$$

Theorem 3.6. *Suppose that $-Q/2 < \alpha \leqslant \beta < Q/2$, $\gamma = \beta - \alpha$ and $d = d_{(\alpha,\beta)}$, and that m is in $\mathscr{M}_{\gamma,d}(N)$. Then pointwise multiplication by the function m defines a bounded operator from $\mathscr{H}^\beta(N)$ to $\mathscr{H}^\alpha(N)$.*

Proof. For m in $\mathscr{M}_{\gamma,d}(N)$, define the operator $\Lambda(m)$ by $\Lambda(m)f = mf$, for every measurable function f on N. The proof may be divided into five steps.

First, if $\alpha \leqslant 0$, $\beta \geqslant 0$ and $\gamma = \beta - \alpha$, then $\Lambda(m)$ is bounded from $\mathscr{H}^\beta(N)$ to $\mathscr{H}^\alpha(N)$ for all m in $\mathscr{M}_{\gamma,d}(N)$. This is an immediate consequence of Lemma 1.2, Proposition 3.4 and Corollary 3.5.

Next, if $\alpha = \beta = 1$, then $\Lambda(m)$ is bounded on $\mathscr{H}^1(N)$ for all m in $\mathscr{M}_{0,1}(N)$. This follows from Theorem 3.3, the previous step, and the equality

$$E_j(mf) = (E_j m)f + m(E_j f) \quad \forall j \in \{1,\dots,d_{\mathfrak{v}}\} \ \forall f \in C_c^\infty(N).$$

Third, if $-1 \leqslant \alpha \leqslant 1$, then $\Lambda(m)$ is bounded on $\mathscr{H}^\alpha(N)$ for all m in $\mathscr{M}_{0,1}(N)$, by complex interpolation and duality.

Fourth, if $0 < \alpha < Q/2$ and $d = \lceil \alpha \rceil$, then $\Lambda(m)$ is bounded on $\mathscr{H}^\alpha(N)$ for all m in $\mathscr{M}_{0,d}(N)$. Indeed, if $D^d \in \mathfrak{D}^d$ and f is a smooth function on N, then $D^d(mf)$ is a finite sum of terms of the form $D^s m D^{d-s} f$, where $D^s \in \mathfrak{D}^s$ and $D^{d-s} \in \mathfrak{D}^{d-s}$; the result now follows from Theorem 3.3, Proposition 3.4, Corollary 3.5, and the preceding steps.

Finally, the case where $-Q/2 < \alpha \leqslant \beta < Q/2$ follows by duality and complex interpolation. □

Further details can be found in [1].

Corollary 3.7. *Suppose that $-Q/2 < \alpha \leqslant \beta < Q/2$, and γ is in $\mathbb{C}$ and $\mathrm{Re}(\gamma) = \beta - \alpha$. If m in $C^\infty(N\backslash\{(0,0)\})$ is homogeneous of degree $-\gamma$, then pointwise multiplication by m defines a bounded operator from $\mathscr{H}^\beta(N)$ to $\mathscr{H}^\alpha(N)$.*

Proof. The hypotheses imply that m is in $\mathscr{M}_{\mathrm{Re}(\gamma),d}(N)$ for every nonnegative integer d. The result follows by Theorem 3.6. □

We shall be interested in the case where the multiplier function is a power of the Jacobian of the Cayley transform multiplied by a polynomial.

The degree of a polynomial in the variables $x_1, x_2, \dots, x_{d_\mathfrak{v}}, z_1, z_2, \dots, z_{d_\mathfrak{z}}$ is defined by saying that the monomial $x_j^r z_i^s$ has degree $r + 2s$.

Lemma 3.8. *Suppose that* $\gamma \in \mathbb{C}$ *and that* $D^k \in \mathfrak{D}^k$ *where* $k \in \mathbb{N}$. *Then*

(i) $D^k(J_\mathscr{C}^{\gamma/2Q}) = \sum_{0 \leqslant r \leqslant 3k} J_\mathscr{C}^{(\gamma+k+r)/2Q} P_r$, *where* P_r *is a polynomial of degree* r;
(ii) $|D^k(J_\mathscr{C}^{\gamma/2Q})| \leqslant C_{k,\gamma} J_\mathscr{C}^{(\mathrm{Re}(\gamma)+k)/2Q}$;
(iii) $|D^k(J_\mathscr{C}^{\gamma/2Q})| \leqslant C_{k,\gamma} \mathscr{N}^{-\mathrm{Re}(\gamma)-k}$ *on* $N \backslash \{(0,0)\}$, *when* $\mathrm{Re}(\gamma) \geqslant -k$.

Proof. This is a routine calculation, based on formulae (7). $\square$

Corollary 3.9. *Suppose that* $m_{z,t} = (J_\mathscr{C}^{z/Q} \circ \delta_{e^t}) J_\mathscr{C}^{-z/Q}$, *where* $z \in \mathbb{C}$ *and* $t \in \mathbb{R}$, *and that* $-Q/2 < \alpha < Q/2$. *Then there exists a constant* C, *independent of* t, *such that*

$$\|m_{z,t} f\|_{\mathscr{H}^\alpha(N)} \leqslant C(1 + e^{-2t\,\mathrm{Re}(z)}) \|f\|_{\mathscr{H}^\alpha(N)} \quad \forall f \in \mathscr{H}^\alpha(N).$$

Proof. Recall that $J_\mathscr{C} = \mathscr{B}^{-Q/2}$. First we prove that $m_{z,t}$ is in $\mathscr{M}_{0,d}(N)$ for every nonnegative integer d. For every (X, Z) in N,

$$\begin{aligned} |m_{z,t}(X,Z)| &= \left| \left(J_\mathscr{C}^{z/Q} \circ \delta_{e^t} \right) J_\mathscr{C}^{-z/Q}(X,Z) \right| \\ &= \left(\frac{(1 + |X|^2/4)^2 + |Z|^2}{(1 + e^{2t}|X|^2/4)^2 + e^{4t}|Z|^2} \right)^{\mathrm{Re}(z)/2} \\ &= e^{-2\,\mathrm{Re}(z)t} \left(\frac{(1 + |X|^2/4)^2 + |Z|^2}{(e^{-2t} + |X|^2/4)^2 + |Z|^2} \right)^{\mathrm{Re}(z)/2}. \end{aligned}$$

Considering separately the four cases for the signs of t and $\mathrm{Re}(z)$, we see that $|m_{z,t}| \leqslant e^{-2\,\mathrm{Re}(z)t}$ if $\mathrm{Re}(z)t < 0$, and $|m_{z,t}| \leqslant 1$ if $\mathrm{Re}(z)t \geqslant 0$.

For every D^d in $\mathfrak{D}^d$, $D^d(m_{z,t})$ is a finite sum of terms of the form

$$D^s \left(J_\mathscr{C}^{z/Q} \circ \delta_{e^t} \right) D^{d-s} \left(J_\mathscr{C}^{-z/Q} \right),$$

where $D^s \in \mathfrak{D}^s$ and $D^{d-s} \in \mathfrak{D}^{d-s}$, and by Lemma 3.8, each of these terms can be estimated as follows:

$$\begin{aligned}\left|D^s\left(J_{\mathscr{C}}^{z/Q}\circ\delta_{e^t}\right)D^{d-s}\left(J_{\mathscr{C}}^{-z/Q}\right)\right| &= \left|e^{ts}\left(D^sJ_{\mathscr{C}}^{z/Q}\right)\circ\delta_{e^t}D^{d-s}\left(J_{\mathscr{C}}^{-z/Q}\right)\right|\\ &\leqslant C|m_{z,t}|e^{ts}\left(J_{\mathscr{C}}^{s/2Q}\circ\delta_{e^t}\right)J_{\mathscr{C}}^{(d-s)/2Q}\\ &\leqslant C\left(1+e^{-2t\,\mathrm{Re}(z)}\right)\mathscr{N}^{-s}J_{\mathscr{C}}^{(d-s)/2Q}\\ &\leqslant C\left(1+e^{-2t\,\mathrm{Re}(z)}\right)\mathscr{N}^{-d}\end{aligned}$$

on $N\setminus\{(0,0)\}$. Arguing as in the third step of the proof of Theorem 3.6, it is easy to see that, when $0<\alpha<Q/2$ and $d=\lceil\alpha\rceil$,

$$\begin{aligned}&||m_{z,t}f||_{\mathscr{H}^\alpha(N)}\\ &\leqslant C\left(\sum_{s=1}^{d}\sum_{D^s\in\mathfrak{D}^s}||D^s(m_{z,t})||_{L^{Q/s,\infty}(N)}+||m_{z,t}||_{L^\infty(N)}\right)||f||_{\mathscr{H}^\alpha(N)}\\ &\leqslant C\left(\left(1+e^{-2t\,\mathrm{Re}(z)}\right)\sum_{s=1}^{d}\left|\left|\mathscr{N}^{-s}\right|\right|_{L^{Q/s,\infty}(N)}+||m_{z,t}||_{L^\infty(N)}\right)||f||_{\mathscr{H}^\alpha(N)}\\ &\leqslant C\left(1+e^{-2t\,\mathrm{Re}(z)}\right)||f||_{\mathscr{H}^\alpha(N)}.\end{aligned}$$

The case where $-Q/2<\alpha<0$ can be treated by duality. □

3.2. The Fourier transform on K/M

We refer to [10,15,16] for further information about this subsection.

If f and g are integrable functions on K/M, and g is M-invariant, then we may define their convolution $f\bigstar g$ by

$$f\bigstar g(k_1M)=\int_K f(kM)g(k^{-1}k_1M)\,dk,$$

where dk is an element of the normalized Haar measure on K (so $\int_K dk=1$). We recall that the pair (K,M) is a Gelfand pair, i.e., the convolution algebra of M-bi-invariant integrable functions on K is commutative.

We will be interested in the Fourier transform of the powers of the function $\mathscr{D}$, which are M-invariant functions on K/M.

For an irreducible representation τ of K, we denote by V^τ the vector space of the representation and by V^τ the subspace of M-invariant vectors, i.e.,

$$V^\tau_M = \{v \in V^\tau : \tau(m)v = v \quad \forall m \in M\}.$$

Then $V^\tau_M = \{0\}$ or V^τ_M is one-dimensional; in the latter case, we say that τ is class one, we write $\tau \in T$, and we define the spherical function ϕ_τ by

$$\phi_\tau(k) = \langle v_0 \mid \tau(k)v_0 \rangle \quad \forall k \in K,$$

where v_0 is a unit vector in V^τ_M. The Fourier transform of an M-invariant function f on K/M is given by

$$\widehat{f}(\tau) = \int_K f(kM)\overline{\phi_\tau(k)}\, dk \quad \forall \tau \in T.$$

Let τ be a class one representation and v be in the representation space V^τ; we define the function f_v on K by

$$f_v(k) = \langle v \mid \tau(k)v_0 \rangle \quad \forall k \in K.$$

The function f_v is continuous and M-right-invariant, so it may be identified with a continuous function on K/M. This gives a correspondence between vectors in V^τ and continuous functions on K/M, and henceforth we will think of V^τ as a space of functions on K/M. It is known that

$$L^2(K/M) = \bigoplus_{\tau \in T} V^\tau. \tag{9}$$

Then any f in $L^2(K/M)$ may be written as $\sum_{\tau \in T} a_\tau Y_\tau$, where the a_τ are complex numbers and the Y_τ are unit vectors in V^τ, and $\sum_{\tau \in T} |a_\tau|^2 = ||f||^2_{L^2(\mathbb{S})}$.

In the degenerate case where $d_\mathfrak{z} = 0$, provided that $d_\mathfrak{v} \geqslant 2$, the subspaces of spherical harmonics of degree d are K-invariant and irreducible. For every d in $\mathbb{N}$, we denote by τ_d the corresponding class one representation.

In the other cases, the space of spherical harmonics of a given degree are not always irreducible; one has to restrict attention to ''bigraded'' spherical harmonics. Roughly speaking, to any class one representation there corresponds a pair of integers (d,h). When $d_\mathfrak{z} = 1$ (the complex case) we consider (d,h) in $\mathbb{N}^2$. When $d_\mathfrak{z} = 3$ (the quaternionic case) or $d_\mathfrak{z} = 7$ (the octonionic case), we consider only pairs of integers (d,h) where $d \geqslant h \geqslant 0$.

Johnson and Wallach [15,16] proved the next result.

For every α in $\mathbb{C}\backslash\{0,-1,-2,\dots\}$, define

$$A_\alpha = 2^{-\alpha+Q/2} \frac{\Gamma((d_\mathfrak{v} + d_\mathfrak{z} + 1)/2)\Gamma(\alpha)}{\Gamma((Q+2\alpha)/4)\Gamma((d_\mathfrak{v}+2+2\alpha)/4)}.$$

In the real case, the Fourier transforms of the powers of $\mathscr{D}$ can be thought of as functions of d in $\mathbb{N}$, and $(\mathscr{D}^{2\alpha-Q})^\wedge(\tau_d)$ is equal to

$$A_\alpha \prod_{j=1}^{d}\left(\frac{Q/2-\alpha+j-1}{Q/2+\alpha+j-1}\right) = A_\alpha \frac{\Gamma(Q/2-\alpha+d)}{\Gamma(Q/2-\alpha)}\frac{\Gamma(Q/2+\alpha)}{\Gamma(Q/2+\alpha+d)}. \tag{10}$$

In the other cases, $(\mathscr{D}^{2\alpha-Q})^\wedge(\tau_{d,h})$ is equal to

$$\begin{aligned} A_\alpha \prod_{j=1}^{d}\left(\frac{Q/2-\alpha+2j-2}{Q/2+\alpha+2j-2}\right)\prod_{j=1}^{h}\left(\frac{d_\mathfrak{v}-2\alpha+4j-2}{d_\mathfrak{v}+2\alpha+4j-2}\right) \\ = A_\alpha \frac{\Gamma(Q/4-\alpha/2+d)}{\Gamma(Q/4+\alpha/2+d)}\frac{\Gamma(Q/4+\alpha/2)}{\Gamma(Q/4-\alpha/2)} \\ \times \frac{\Gamma((d_\mathfrak{v}+2-2\alpha)/4+h)}{\Gamma((d_\mathfrak{v}+2+2\alpha)/4+h)}\frac{\Gamma((d_\mathfrak{v}+2+2\alpha)/4)}{\Gamma((d_\mathfrak{v}+2-2\alpha)/4)} \end{aligned} \tag{11}$$

for (d,h) in $\mathbb{N}^2$ or (d,h) such that $d \geqslant h \geqslant 0$.

The point of this result in the work of Johnson and Wallach is to compute the Fourier transform of the Knapp–Stein intertwining operator (in the compact picture); the positivity of this operator implies the existence of complementary series.

3.3. Sobolev spaces on K/M

Define the sublaplacian $\mathscr{L}^\sharp$ on K/M by

$$\mathscr{L}^\sharp f^\sharp = (\mathscr{L}f)^\sharp \quad f \in C^\infty(\mathbb{S}).$$

By Theorem 2.8,

$$(\mathscr{L}^\sharp + b)^\wedge = C((\mathscr{D}^{2-Q})^\wedge)^{-1},$$

where the constant C depends on $d_\mathfrak{v}$ and $d_\mathfrak{z}$ and can be computed by evaluating the Fourier transform of $(\mathscr{L}^\sharp + b)f$ when $f = 1$ on K. Using formulae (10) and (11), we see that, in the real case,

$$(\mathscr{L}^\sharp + b)^\wedge(\tau_d) = \left(\frac{Q}{2} - 1 + d\right)\left(\frac{Q}{2} + d\right); \tag{12}$$

while in the other cases,

$$(\mathscr{L}^\sharp + b)^\wedge(\tau_{d,h}) = \left(\frac{Q}{2} - 1 + 2d\right)\left(\frac{d_\mathfrak{v}}{2} + 2h\right). \tag{13}$$

Since $\mathscr{L}^\sharp$ is M-invariant, it acts on V^τ by scalar multiples, say

$$\mathscr{L}^\sharp Y_\tau = \lambda_\tau Y_\tau \quad \forall \tau \in T \ \forall Y_\tau \in V^\tau;$$

then

$$\lambda_\tau = (\mathscr{L}^\sharp + b)^\wedge(\tau) - b.$$

Suppose that α is real. We can define real powers of the conformal sublaplacian by requiring that

$$(\mathscr{L}^\sharp + b)^\alpha Y_\tau = (\lambda_\tau + b)^\alpha Y_\tau \quad \forall Y_\tau \in V^\tau.$$

Take f in $L^2(K/M)$. According to (9), we write f as $\sum_{\tau \in T} a_\tau Y_\tau$, where the Y_τ are unit vectors in V^τ and a_τ lie in $\mathbb{C}$. Suppose that

$$\sum_{\tau \in T} |a_\tau|^2 (\lambda_\tau + b)^{2\alpha} < \infty.$$

Then

$$(\mathscr{L}^\sharp + b)^\alpha f = \sum_{\tau \in T} (\lambda_\tau + b)^\alpha a_\tau Y_\tau.$$

Definition 3.10. For real α, we define the Sobolev space $\mathscr{H}^\alpha(K/M)$ to be the completion of the space of smooth functions on K/M, with respect to the norm

$$||f||_{\mathscr{H}^\alpha(K/M)} = \left|\left|(\mathscr{L}^\sharp + b)^{\alpha/2} f\right|\right|_{L^2(K/M)} \quad \forall f \in C^\infty(K/M).$$

We note that if $\alpha < \beta$, then $\mathscr{H}^\beta(K/M)$ is continuously embedded in $\mathscr{H}^\alpha(K/M)$, for the eigenvalues λ_τ are nonnegative. The spaces $\mathscr{H}^\alpha(K/M)$ and $\mathscr{H}^{-\alpha}(K/M)$ are dual with respect to the L^2 inner product.

Proposition 3.11. *Suppose that $\alpha \geqslant 2$. Then f is in $\mathscr{H}^\alpha(K/M)$ if and only if f is in $L^2(K/M)$ and $\mathscr{L}^\sharp f$ is in $\mathscr{H}^{\alpha-2}(K/M)$; moreover the expressions $||f||_{\mathscr{H}^\alpha(K/M)}$ and $||f||_{L^2(K/M)} + ||\mathscr{L}^\sharp f||_{\mathscr{H}^{\alpha-2}(K/M)}$ are equivalent.*

Proof. This is routine. $\square$

Proposition 3.12. *Suppose that α is in $(-Q/2, Q/2)\backslash\mathbb{Z}$. Then the operator*

$$f \mapsto f \bigstar \mathscr{D}^{2\alpha - Q}$$

is bounded with a bounded inverse from $\mathscr{H}^{-\alpha}(K/M)$ *to* $\mathscr{H}^{\alpha}(K/M)$. *Its norm is bounded by a constant multiple of* $|c(\alpha)|^{-1}$, *and its inverse is a constant multiple of*

$$f \mapsto c(\alpha)c(-\alpha) f \bigstar \mathscr{D}^{-2\alpha-Q},$$

where

$$c(\alpha) = \frac{\Gamma((Q-2\alpha)/4)\Gamma((d_{\mathfrak{v}}+2-2\alpha)/4)}{\Gamma(\alpha)2^{\alpha}}.$$

Proof. The Fourier transforms of $\mathscr{D}^{2\alpha-Q}$ and $(\mathscr{L}^{\sharp}+b)^{\alpha}$ can be evaluated using formulae (10)–(13). It is easy to check that

$$|(\mathscr{D}^{2\alpha-Q})^{\wedge}(\tau)((\mathscr{L}^{\sharp}+b)^{\alpha})^{\wedge}(\tau)| \leqslant C|c(\alpha)|^{-1} \quad \forall \tau \in T,$$

where C depends on the dimensions $d_{\mathfrak{v}}$ and $d_{\mathfrak{z}}$. Suppose that f is in $C^{\infty}(K/M)$ and $f = \sum_{\tau \in T} a_{\tau} Y_{\tau}$. Then

$$f \bigstar \mathscr{D}^{2\alpha-Q} = \sum_{\tau \in T} (\mathscr{D}^{2\alpha-Q})^{\wedge}(\tau) a_{\tau} Y_{\tau}.$$

Moreover,

$$\begin{aligned} ||f \bigstar \mathscr{D}^{2\alpha-Q}||^2_{\mathscr{H}^{\alpha}(K/M)} &= \sum_{\tau \in T} |((\mathscr{L}^{\sharp}+b)^{\alpha/2})^{\wedge}(\tau)|^2 |(\mathscr{D}^{2\alpha-Q})^{\wedge}(\tau)|^2 |a_{\tau}|^2 \\ &= \sum_{\tau \in T} |\lambda_{\tau}+b|^{2\alpha} |(\mathscr{D}^{2\alpha-Q})^{\wedge}(\tau)|^2 |\lambda_{\tau}+b|^{-\alpha} |a_{\tau}|^2 \\ &\leqslant C|c(\alpha)|^{-2} \sum_{\tau \in T} |\lambda_{\tau}+b|^{-\alpha} |a_{\tau}|^2 \\ &= C|c(\alpha)|^{-2} ||(\mathscr{L}^{\sharp}+b)^{-\alpha/2} f||^2_{L^2(K/M)} \\ &= C|c(\alpha)|^{-2} ||f||^2_{\mathscr{H}^{-\alpha}(K/M)}. \end{aligned}$$

The inverse may be evaluated by observing that the product of the Fourier transforms of $\mathscr{D}^{2\alpha-Q}$ and $\mathscr{D}^{-2\alpha-Q}$ is $(c(\alpha)\,c(-\alpha))^{-1} 2^{Q}\,\Gamma((d_{\mathfrak{v}}+d_{\mathfrak{z}}+1)/2)^2$, i.e.,

$$\mathscr{D}^{2\alpha-Q} \bigstar \mathscr{D}^{-2\alpha-Q} = 2^{Q} \frac{\Gamma((d_{\mathfrak{v}}+d_{\mathfrak{z}}+1)/2)^2}{c(\alpha)c(-\alpha)} \delta,$$

where δ denotes the Dirac delta at the identity. □

Analogous results can be proved for the group N. The Fourier transforms of powers of the homogeneous norm and of the sublaplacian Δ on N are well known [4,7]. Similar arguments to those we used for K/M yield the following proposition.

Proposition 3.13. *Suppose that α is in $(-Q/2, Q/2)\backslash\mathbb{Z}$. The map*

$$F \mapsto F * \mathcal{N}^{2\alpha-Q}$$

is an invertible bounded operator from $\mathscr{H}^{-\alpha}(N)$ to $\mathscr{H}^{\alpha}(N)$. Its norm is bounded by a constant multiple of $|c(\alpha)|^{-1}$, and its inverse is a constant multiple of

$$F \mapsto c(\alpha)c(-\alpha)F * \mathcal{N}^{-2\alpha-Q},$$

where

$$c(\alpha) = \frac{\Gamma((Q-2\alpha)/4)\Gamma((d_{\mathfrak{v}}+2-2\alpha)/4)}{\Gamma(\alpha)2^{\alpha}}.$$

Theorem 3.14. *If f is a continuous function on K/M, then f is in the Sobolev space $\mathscr{H}^{\alpha}(K/M)$ for every α in $(-Q/2, 0]$ and moreover*

$$||f||_{\mathscr{H}^{\alpha}(K/M)} \leqslant C_{\alpha}||f||_{L^{p,2}(K/M)},$$

where $1/p = 1/2 - \alpha/Q$. If $0 \leqslant \alpha < Q/2$, then $\mathscr{H}^{\alpha}(K/M)$ is contained in $L^{p,2}(K/M)$, where $1/p = 1/2 - \alpha/Q$, and moreover

$$||f||_{L^{p,2}(K/M)} \leqslant C_{\alpha}||f||_{\mathscr{H}^{\alpha}(K/M)} \quad \forall f \in \mathscr{H}^{\alpha}(K/M).$$

Proof. The proofs of these results are variants of the proofs of Proposition 3.4. When $\alpha = 0$, the result is trivial. Arguing as in Proposition 3.12, one can prove that when α is in $(-Q/2, 0)\backslash\mathbb{Z}$, there exists a constant C such that

$$||f||_{\mathscr{H}^{\alpha}(K/M)} \leqslant C||f \bigstar \mathscr{D}^{-\alpha-Q}||_{L^2(K/M)}$$

for all f in $\mathscr{H}^{\alpha}(K/M)$. We only need to know that $\mathscr{D}^{-\alpha-Q}$ is in $L^{r,\infty}(\mathbb{S})$ when $1/r = 1 + \alpha/Q$. This can be proved as in Theorem 2.8. The result follows by Lemma 1.3, complex interpolation and duality. $\square$

4. The main result

The purpose of this section is to show that the operator T_{α}, defined by

$$T_{\alpha}F = \left(J_{\mathscr{C}^{-1}}^{1/2-\alpha/Q}F \circ \mathscr{C}^{-1}\right)^{\sharp} \quad \forall F \in C_c^{\infty}(N),$$

is bounded with a bounded inverse from $\mathscr{H}^{\alpha}(N)$ to $\mathscr{H}^{\alpha}(K/M)$, when $-Q/2 < \alpha < Q/2$. Formally,

$$T_{\alpha}^{-1}f = J_{\mathscr{C}}^{1/2-\alpha/Q}f^{\flat} \circ \mathscr{C} \quad \forall f \in C^{\infty}(K/M).$$

It is straightforward that T_0 is a multiple of an isometry. Moreover, by Theorem 2.7, the operator T_1 is a multiple of an isometry.

The preceding formulae make sense also for complex values of α, and sometimes we shall consider the operators T_α for complex α.

Lemma 4.1. *Suppose that α is in $\mathbb{C}$ and $\operatorname{Re}(\alpha)<0$. Then*

$$T_{-\alpha}(F * \mathcal{N}^{-2\alpha-Q}) = \sigma(\mathbb{S})(T_\alpha F) \bigstar \mathcal{D}^{-2\alpha-Q} \quad \forall F \in C_c^\infty(N).$$

Proof. Suppose that α is in $\mathbb{C}$ and $\operatorname{Re}(\alpha)<0$. By Lemma 2.2, for every F in $C_c^\infty(N)$,

$$\begin{aligned}
&(T_{-\alpha}(F * \mathcal{N}^{-2\alpha-Q}))^\flat(\zeta)\\
&\quad = J_{\mathscr{C}^{-1}}^{1/2+\alpha/Q}(\zeta) \int_N F(n_1)\mathcal{N}^{-2\alpha-Q}(n_1^{-1}\mathscr{C}^{-1}(\zeta))\, dn_1\\
&\quad = \int_N F(n_1) J_{\mathscr{C}}^{1/2+\alpha/Q}(n_1)\, d_{\mathbb{S}}^{-2\alpha-Q}(\mathscr{C}(n_1),\zeta)\, dn_1\\
&\quad = \int_{\mathbb{S}} J_{\mathscr{C}^{-1}}^{1/2-\alpha/Q}(\zeta_1)(F\circ\mathscr{C}^{-1})(\zeta_1)\, d_{\mathbb{S}}^{-2\alpha-Q}(\zeta_1,\zeta)\, d\sigma(\zeta_1).
\end{aligned}$$

Therefore

$$\begin{aligned}
T_{-\alpha}(F * \mathcal{N}^{-2\alpha-Q})(kM) &= \sigma(\mathbb{S}) \int_K T_\alpha F(k_1 M)\mathcal{D}^{-2\alpha-Q}(k_1^{-1}kM)\, dk_1\\
&= \sigma(\mathbb{S})(T_\alpha F) \bigstar \mathcal{D}^{-2\alpha-Q},
\end{aligned}$$

as required. $\square$

Our main result is the following relation between homogeneous Sobolev spaces on N and Sobolev spaces on K/M. The real case of this theorem may be found in Branson [3].

Theorem 4.2. *Suppose that $-Q/2<\alpha<Q/2$. Then the map*

$$T_\alpha : F \mapsto (J_{\mathscr{C}^{-1}}^{1/2-\alpha/Q} F\circ\mathscr{C}^{-1})^\sharp$$

is a bounded invertible operator from $\mathscr{H}^\alpha(N)$ to $\mathscr{H}^\alpha(K/M)$.

Proof. We suppose that $Q>4$. When $Q\leqslant 4$, matters can be simplified.

Let F be a smooth function with compact support on N. By Theorem 2.7 and Leibniz' rule,

$$\begin{aligned}[(\mathcal{L}^\sharp + b)(T_\alpha F)]^\flat \circ \mathscr{C} &= J_{\mathscr{C}}^{-1/2-1/Q} \Delta (J_{\mathscr{C}}^{1/2-1/Q} (T_\alpha F)^\flat \circ \mathscr{C}) \\ &= J_{\mathscr{C}}^{-1/2-1/Q} \Delta (J_{\mathscr{C}}^{(\alpha-1)/Q} F) \\ &= \left[T_{\alpha-2} \left(\Delta F + m_\alpha^{(2)} F + \sum_{j=1}^{d_{\mathfrak{v}}} m_{\alpha,j}^{(1)} E_j F \right) \right]^\flat \circ \mathscr{C}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} m_\alpha^{(2)}(X,Z) &= (\alpha - 1) J_{\mathscr{C}}^{2/Q}(X,Z) \left[\frac{d_{\mathfrak{v}}}{2} + \left(\frac{Q}{8} - \frac{\alpha}{4} \right) |X|^2 \right] \\ m_{\alpha,j}^{(1)}(X,Z) &= (1-\alpha) J_{\mathscr{C}}^{2/Q}(X,Z) \langle \mathscr{A} X, E_j \rangle, \end{aligned}$$

for all (X,Z) in N. Note that $m_\alpha^{(2)}$ and $m_{\alpha,j}^{(1)}$ are in $\mathscr{M}_{2,d}$ and $\mathscr{M}_{1,d}$, for every nonnegative integer d.

We extend the definition of T_α to obtain an analytic family of operators T_z for $\mathrm{Re}(z)$ in $[0, Q/2)$, by

$$T_z F = \left(J_{\mathscr{C}^{-1}}^{1/2-z/Q} F \circ \mathscr{C}^{-1} \right)^\sharp \quad \forall F \in C_c^\infty(N).$$

It is easy to prove that, when $\mathrm{Re}(z) = 0$, the operators T_z are multiples of isometries from $L^2(N)$ to $L^2(K/M)$.

Suppose now that $\mathrm{Re}(z) = 2$, i.e., $z = 2 + iy$ where y is in $\mathbb{R}$. Since T_0 is L^2 bounded, from Theorem 3.3, Theorem 3.6 and formula (14),

$$\begin{aligned} & \|T_{2+iy} F\|_{\mathscr{H}^2(K/M)} \\ &= \left\| (\mathcal{L}^\sharp + b)\left(T_2 (J_{\mathscr{C}}^{iy/Q} F) \right) \right\|_{\mathscr{H}^0(K/M)} \\ &= \left\| T_0 \left(\Delta \left(J_{\mathscr{C}}^{iy/Q} F \right) + m_2^{(2)} J_{\mathscr{C}}^{iy/Q} F + \sum_{j=0}^{d_{\mathfrak{v}}} m_{2,j}^{(1)} E_j \left(J_{\mathscr{C}}^{iy/Q} F \right) \right) \right\|_{L^2(K/M)} \\ &\leqslant C \left(\left\| \Delta (J_{\mathscr{C}}^{iy/Q} F) \right\|_{L^2(N)} + \left\| m_2^{(2)} J_{\mathscr{C}}^{iy/Q} F \right\|_{L^2(N)} + \sum_{j=0}^{d_{\mathfrak{v}}} \left\| m_{2,j}^{(1)} E_j (J_{\mathscr{C}}^{iy/Q} F) \right\|_{L^2(N)} \right) \\ &\leqslant C \left\| J_{\mathscr{C}}^{iy/Q} F \right\|_{\mathscr{H}^2(N)} \leqslant C \|F\|_{\mathscr{H}^2(N)}. \end{aligned}$$

By Stein's complex interpolation theorem [20], the operators T_α are bounded from $\mathscr{H}^\alpha(N)$ to $\mathscr{H}^\alpha(K/M)$ for every α in $[0, 2]$.

We write $[0, Q/2) = \bigcup_{h=0}^{Q/2-1} [h, h+1)$ and we proceed by induction on h. We have just proved that T_α is bounded when α is in $[0, 2]$. Now suppose that T_α is bounded when α is in $[0, h+1)$, and let α be in $[h+1, h+2)$. By (14), the inductive hypothesis, Theorems 3.3 and 3.6,

$$\begin{aligned}
&||T_\alpha F||_{\mathscr{H}^\alpha(K/M)} \\
&\leqslant C\left(||\Delta F||_{\mathscr{H}^{\alpha-2}(N)} + \left|\left|m_\alpha^{(2)} F\right|\right|_{\mathscr{H}^{\alpha-2}(N)} + \sum_{j=1}^{d_\mathfrak{v}} \left|\left|m_{\alpha,j}^{(1)} E_j F\right|\right|_{\mathscr{H}^{\alpha-2}(N)} \right) \\
&\leqslant C\left(||F||_{\mathscr{H}^\alpha(N)} + \sum_{j=1}^{d_\mathfrak{v}} ||E_j F||_{\mathscr{H}^{\alpha-1}(N)} \right) \\
&\leqslant C||F||_{\mathscr{H}^\alpha(N)}.
\end{aligned}$$

This proves that T_α is bounded when α is in $[0, Q/2)$.

Now take α in $(-Q/2, 0)\backslash\mathbb{Z}$ and F in $\mathscr{H}^\alpha(N)$. By Proposition 3.12 and Lemma 4.1,

$$(T_\alpha F) = Cc(\alpha)c(-\alpha)T_{-\alpha}(F * \mathscr{N}^{-2\alpha-Q}) \bigstar \mathscr{D}^{2\alpha-Q}.$$

From above, $T_{-\alpha}$ is bounded from $\mathscr{H}^{-\alpha}(N)$ to $\mathscr{H}^{-\alpha}(K/M)$, and it follows T_α is bounded from $\mathscr{H}^\alpha(N)$ to $\mathscr{H}^\alpha(K/M)$ when α is in $(-Q/2, 0)\backslash\mathbb{Z}$ by Propositions 3.12 and 3.13. More precisely,

$$\begin{aligned}
||T_\alpha F||_{\mathscr{H}^\alpha(K/M)} &= C|c(\alpha)\, c(-\alpha)| \left|\left|T_{-\alpha}(F * \mathscr{N}^{-2\alpha-Q}) \bigstar \mathscr{D}^{2\alpha-Q}\right|\right|_{\mathscr{H}^\alpha(K/M)} \\
&\leqslant C|c(-\alpha)| \left|\left|F * \mathscr{N}^{-2\alpha-Q}\right|\right|_{\mathscr{H}^{-\alpha}(N)} \\
&\leqslant C||F||_{\mathscr{H}^\alpha(N)}.
\end{aligned}$$

As before, we use complex interpolation to prove that T_α is bounded when α is an integer and $-Q/2 < \alpha < 0$.

Finally, denote by T_α^* the adjoint of T_α. Note that $T_\alpha^* = T_{-\alpha}^{-1}$. By duality, it follows that T_α^{-1} is bounded from $\mathscr{H}^\alpha(K/M)$ to $\mathscr{H}^\alpha(N)$ when α is in $(-Q/2, Q/2)$. $\square$

5. Representations

We now apply our work on the Cayley transformation to representation theory. Recall that the simple group G acts on the quotient space $G/\bar{P}$, where $\bar{P}$ is a parabolic subgroup of G, and that $G/\bar{P}$ may be identified with the one-point compactification $N \cup \{\infty\}$ of N or with K/M. This gives rise to (nonmeasure-preserving) actions of G on $N \cup \{\infty\}$ and on K/M (we will just write $g \cdot n$ and $g \cdot kM$ to indicate the result of applying g to n or to kM). In turn these actions give rise to representations of G on spaces of functions on $N \cup \{\infty\}$ and K/M.

For λ in $\mathbb{C}$, we define representations π_λ and $\tilde{\pi}_\lambda$ on functions on N and on K/M by the formulae

$$\pi_\lambda(g)f(n) = \left(\frac{d(g^{-1}\cdot n)}{dn}\right)^{1/2-\lambda/Q} f(g^{-1}\cdot n)$$

$$\tilde{\pi}_\lambda(g)f(kM) = \left(\frac{d(g^{-1}\cdot kM)}{dkM}\right)^{1/2-\lambda/Q} f(g^{-1}\cdot kM),$$

where the "derivatives" are to be interpreted in the sense of Radon–Nikodym. These are known as the noncompact and compact versions of the class 1 principal series representations of G.

Now the action of G on K/M induces an action on $\mathbb{S}$ (denoted similarly), and clearly

$$\frac{d\sigma(g^{-1}\cdot\zeta)}{d\sigma(\zeta)} = \frac{d(g^{-1}\cdot kM)}{d(kM)}.$$

Further, the Cayley transform $\mathscr{C}$ maps $N\cup\{\infty\}$ bijectively to $\mathbb{S}$, and if $\zeta = \mathscr{C}n$, then

$$\begin{aligned}\frac{d\sigma(g^{-1}\cdot\zeta)}{d\sigma(\zeta)} &= \frac{d\sigma(g^{-1}\cdot\mathscr{C}n)}{d\sigma(\mathscr{C}n)}\\ &= \frac{d\sigma(g^{-1}\cdot\mathscr{C}n)}{d(g^{-1}\cdot n)}\left(\frac{d\sigma(\mathscr{C}n)}{dn}\right)^{-1}\frac{d(g^{-1}\cdot n)}{dn}\\ &= \frac{d\sigma(\mathscr{C}g^{-1}\cdot n)}{d(g^{-1}\cdot n)}\left(\frac{d\sigma(\mathscr{C}n)}{dn}\right)^{-1}\frac{d(g^{-1}\cdot n)}{dn}\\ &= J_{\mathscr{C}}(g^{-1}\cdot n)J_{\mathscr{C}}(n)^{-1}\,\frac{d(g^{-1}\cdot n)}{dn}.\end{aligned}$$

Consequently, the representations π_λ and $\tilde{\pi}_\lambda$ are linked by the equality

$$\tilde{\pi}_\lambda(g)T_\lambda = T_\lambda\pi_\lambda(g) \quad \forall g\in G.$$

Consider the Cartan decomposition $G = KAK$ of the simple group G. For real t, denote by a_t the element in A which acts on N by the dilation

$$a_t\cdot(X,Z) = \delta_{e^t}(X,Z) = (e^tX, e^{2t}Z) \quad \forall (X,Z)\in N.$$

Theorem 5.1. *Suppose that $\alpha\in(-Q/2,Q/2)$. Then there exists a constant C such that, for any λ in $\mathbb{C}$,*

$$||\tilde{\pi}_\lambda(ka_tk')f||_{\mathscr{H}^\alpha(K/M)} \leqslant C\cosh(t(\mathrm{Re}(\lambda)-\alpha))||f||_{\mathscr{H}^\alpha(K/M)}$$

for all t in $\mathbb{R}$, all k and k' in K, and all f in $C^\infty(K/M)$.

Proof. It is clear that $||\tilde{\pi}_\lambda(k)f||_{\mathscr{H}^\alpha(K/M)} = ||f||_{\mathscr{H}^\alpha(K/M)}$ for all k in K, so it suffices to estimate $||\tilde{\pi}_\lambda(a_t)f||_{\mathscr{H}^\alpha(K/M)}$. It is easy to check that, for any a_t in A,

$$\pi_\lambda(a_t)F = e^{-Qt(1/2-\lambda/Q)} F \circ \delta_{e^{-t}} \quad \forall t \in \mathbb{R} \ \forall F \in C_c^\infty(N).$$

Since $\Delta(F \circ \delta_{e^{-t}}) = e^{-2t}(\Delta F) \circ \delta_{e^{-t}}$,

$$\begin{aligned} ||\pi_\lambda(a_t)F||_{\mathscr{H}^\alpha(N)} &= \left|\left|\Delta^{\alpha/2}(e^{-Qt(1/2-\lambda/Q)} F \circ \delta_{e^{-t}})\right|\right|_{L^2(N)} \\ &= e^{-Qt(1/2-\mathrm{Re}(\lambda)/Q)-\alpha t} \left|\left| \left(\Delta^{\alpha/2} F\right) \circ \delta_{e^{-t}} \right|\right|_{L^2(N)} \\ &= e^{t(\mathrm{Re}(\lambda)-\alpha)} ||F||_{\mathscr{H}^\alpha(N)}. \end{aligned}$$

Note that, for any f in $C^\infty(K/M)$,

$$\tilde{\pi}_\lambda(a_t)f = T_\alpha\big(\pi_\lambda(a_t)(m_{\lambda-\alpha,t} T_\alpha^{-1} f)\big),$$

where $m_{\lambda-\alpha,t} = \left(J_{\mathscr{C}}^{(\lambda-\alpha)/Q} \circ \delta_{e^t}\right) J_{\mathscr{C}}^{-(\lambda-\alpha)/Q}$. Therefore, by Theorem 4.2 and Corollary 3.9,

$$\begin{aligned} ||\tilde{\pi}_\lambda(a_t)f||_{\mathscr{H}^\alpha(K/M)} &\leqslant C e^{t(\mathrm{Re}(\lambda)-\alpha)} ||m_{\lambda-\alpha,t} T_\alpha^{-1} f||_{\mathscr{H}^\alpha(N)} \\ &\leqslant C(e^{t(\mathrm{Re}(\lambda)-\alpha)} + e^{-t(\mathrm{Re}(\lambda)-\alpha)}) ||T_\alpha^{-1} f||_{\mathscr{H}^\alpha(N)} \\ &\leqslant C \cosh(t(\mathrm{Re}(\lambda)-\alpha)) ||f||_{\mathscr{H}^\alpha(K/M)}, \end{aligned}$$

as required. □

Corollary 5.2. *Suppose that $\lambda \in \mathbb{C}$ and that $\mathrm{Re}(\lambda) = \alpha \in (-Q/2, Q/2)$. Then there exists a constant C such that*

$$||\tilde{\pi}_\lambda(k a_t k')f||_{\mathscr{H}^\alpha(K/M)} \leqslant C ||f||_{\mathscr{H}^\alpha(K/M)}$$

for all t in $\mathbb{R}$, all k and k' in K, and all f in $C^\infty(K/M)$.

The proof of this is immediate. Recall that a representation π of a group G on a Hilbert space is said to be uniformly bounded if there is a constant C such that $||\pi(g)|| \leqslant C$ for all g in G; then we have established that π_λ acts uniformly boundedly on $\mathscr{H}^\alpha(K/M)$ when $\alpha = \mathrm{Re}(\lambda)$. Note however that the bound which is uniform for g in G may not be uniform in λ, and may blow up as $\alpha \to \pm Q/2$.

A substitute for uniform boundedness, for which the constants remain bounded as $\mathrm{Re}(\lambda)$ tends to $\pm Q/2$ (but at the cost of losing uniform boundedness), is slow growth.

Fix a proper nonnegative real-valued function f on a locally compact group G; then a representation π of G as bounded operators on a Hilbert space is said to be

ε-slowly growing if $||\pi(g)|| \leqslant Ce^{\varepsilon f(g)}$ for all g in G. In the case of a semisimple Lie group, it is natural to define f by $f(kak') = ||\log a||$, where $a \in A^+$, and $k, k' \in K$. From our theorem, it follows that π_λ is an $\varepsilon Q/2$ slowly growing representation on $\mathscr{H}^\alpha(K/M)$ when $\alpha = (1-\varepsilon)\mathrm{Re}(\lambda)$, for all λ such that $|\mathrm{Re}(\lambda)| \leqslant Q/2$, and further, the constant C may be taken to be uniform for real λ. This is an important feature of our result for the applications envisaged by Julg [17].

Finally, it should be mentioned that the uniform boundedness of the representations π_λ (where $|\mathrm{Re}(\lambda)| < Q/2$) in the noncompact picture was established in [4]. Recently, Dooley [9] showed that, in the noncompact picture, the representations are uniformly bounded when $\lambda \in (-Q/2, Q/2)$, with a bound independent of λ. For Julg's application, a result for the compact picture is needed. The proof here enables us to transfer Dooley's result to the compact picture, but as our estimates for the map T_λ between Sobolev-type spaces on N and on K/M blow up as we approach the edge of the critical strip, the transferred estimates in the compact picture also blow up as we approach the edge of the strip. Our result gives results for both the compact and noncompact pictures; if it could be improved (for instance, by changing the Sobolev spaces on N and K/M) to give estimates for T_λ which stay bounded, then we could obtain uniform boundedness à la Dooley for both the compact and noncompact pictures.

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