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CONTINUOUS FRAME WAVELETS*

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Abstract Let π be a unitary representation of a locally compact topological group G on a separable Hilbert space $\mathcal{H}$. A vector $\psi \in \mathcal{H}$ is called a continuous frame wavelet if there exist $A, B > 0$ such that

$$A\|\phi\|^2 \leq \int_G |\langle \pi(g)\psi, \phi\rangle|^2 \mathrm{d}g \leq B\|\phi\|^2 \qquad (\phi \in \mathcal{H}),$$

in which dg is the left Haar measure of G. Similar to the study of wavelets, an essential problem in the study of continuous frame wavelets is how to characterize them under the given unitary representation. Moreover, we investigate a relation between admissible vectors of π and its components.

Key words Locally compact group; unitary representation; continuous frame; wavelet transform; direct integral

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1 Introduction

Let $\mathcal{H}$ be a separable Hilbert space and X be a locally compact Hausdorff space endowed with a positive Radon measure μ with supp$\mu = X$. A family $\{\psi_x\}_{x\in X}$ of vectors in $\mathcal{H}$ is called a continuous frame if the mapping $x \longmapsto \langle \psi_x, \phi\rangle$ is measurable for all $\phi \in \mathcal{H}$, and there exist constants $0 < A, B < +\infty$ such that

$$A\|\phi\|^2 \leq \int_X |\langle \psi_x, \phi\rangle|^2 \mathrm{d}\mu(x) \leq B\|\phi\|^2 \qquad (\phi \in \mathcal{H}). \tag{1.1}$$

A frame is said to be tight when $A = B$. Note that, if X is a countable set and μ the counting measure, then we obtain the usual definition of a (discrete) frame.

The frame operator S associated to $\{\psi_x\}_{x\in X}$ is defined in weak sense by

$$S : \mathcal{H} \longrightarrow \mathcal{H}, \qquad S\phi = \int_X \langle \psi_x, \phi\rangle \psi_x \mathrm{d}\mu(x).$$

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By (1.1), it follows that $\{\psi_x\}_{x\in X}$ is total in $\mathcal{H}$ and S is a bounded, positive, and boundedly invertible operator. In particular

$$A\|\phi\|^2 \le \langle S\phi,\phi\rangle \le B\|\phi\|^2 \qquad (\phi\in\mathcal{H}).$$

Moreover, as S is invertible and self-adjoint for all $\phi\in\mathcal{H}$, we have

$$\phi = S^{-1}S\phi = \int_X \langle\psi_x,\phi\rangle S^{-1}\psi_x \ \mathrm{d}\mu(x)$$

and

$$\phi = SS^{-1}\phi = \int_X \langle S^{-1}\psi_x,\phi\rangle \psi_x \ \mathrm{d}\mu(x).$$

The operator S is a multiple of the identity if and only if $\{\psi_x\}_{x\in X}$ is a tight frame (cf. [3, Subsection 5.3]).

In this article, we are mainly interested in taking a locally compact group G instead of X. More precisely, let $(\pi,\mathcal{H})$ be a unitary representation of G. For a fixed $\psi\in\mathcal{H}$, we try to find a necessary and sufficient condition to make $\{\pi(g)\psi\}_{g\in G}$ to be a continuous frame for $\mathcal{H}$. Such a ψ is called a continuous frame wavelet. The characterization of discrete frame wavelets was studied and extended by many authors in several different directions [4–6, 9].

We prove that admissible vectors and continuous frame wavelets are the same when the associated representation is irreducible. Moreover, we aim to find a relation between the continuous frame wavelets associated to a given representation and its irreducible subrepresentations.

2 Main Results

Throughout this article, we assume that G is a locally compact group with the left Haar measure $\mathrm{d}g$. By a representation $(\pi,\mathcal{H})$ we mean a continuous unitary representation of G on a separable Hilbert space $\mathcal{H}$. The representation π is called irreducible if its only closed invariant subspaces are $\{0\}$ and $\mathcal{H}$. If there exists a nonzero vector $\psi\in\mathcal{H}$ satisfying the admissibility condition

$$\int_G |\langle\pi(g)\psi,\phi\rangle|^2 \mathrm{d}g < +\infty \qquad (\phi\in\mathcal{H}), \tag{2.1}$$

then π is said to be a square integrable representation. Such a ψ is called an admissible vector. It is shown that (2.1) is equivalent to

$$C_\psi^2 := \frac{1}{\|\psi\|^2}\int_G |\langle\pi(g)\psi,\psi\rangle|^2 \mathrm{d}g < +\infty,$$

where π is irreducible. In this case, the set of all admissible vectors is an invariant subspace in $\mathcal{H}$. Hence, it is $\{0\}$ or dense in $\mathcal{H}$ (cf. [1, Section 8]).

For any admissible vector ψ, the mapping

$$W_\psi : \mathcal{H} \longrightarrow L^2(G), \qquad (W_\psi\phi)(g) = C_\psi^{-1}\langle\pi(g)\psi,\phi\rangle \qquad (\phi\in\mathcal{H},\ g\in G),$$

which is a linear isometry onto a (closed) subspace $\mathcal{H}_\psi$ of $L^2(G)$, is called a continuous wavelet transform on G. It is an intertwining operator between π and the left regular representation on G and its adjoint is W_ψ^{-1} on $\mathcal{H}_\psi$. Hence, a vector $\phi\in\mathcal{H}$ can be reconstructed uniquely by

$$\phi = \frac{1}{C_\psi}\int_G (W_\psi\phi)(g)\pi(g)\psi \mathrm{d}g.$$

For more details, see [1].

Definition 2.1 Let $(\pi, \mathcal{H})$ be a representation on G. A vector $\psi \in \mathcal{H}$ is called a continuous frame wavelet if there exist $A, B > 0$ such that

$$A\|\phi\|^2 \leq \int_G |\langle \pi(g)\psi, \phi\rangle|^2 \mathrm{d}g \leq B\|\phi\|^2,$$

for all $\phi \in \mathcal{H}$. In other words, $\{\pi(g)\psi\}_{g\in G}$ is a continuous frame for $\mathcal{H}$ with the frame bounds A and B.

Lemma 2.2 Let ψ be a continuous frame wavelet for an irreducible representation $(\pi, \mathcal{H})$. Then, the frame operator S is a multiple of the identity.

Proof By the definition of S, for all $\phi_1, \phi_2 \in \mathcal{H}$, we have

$$\langle S\phi_1, \phi_2\rangle = \int_G \langle \phi_1, \pi(g)\psi\rangle\langle \pi(g)\psi, \phi_2\rangle \mathrm{d}g.$$

Hence,

$$\begin{aligned} \langle S\pi(x)\phi_1, \phi_2\rangle &= \int_G \langle \pi(x)\phi_1, \pi(g)\psi\rangle\langle \pi(g)\psi, \phi_2\rangle \mathrm{d}g \\ &= \int_G \langle \pi(x)\phi_1, \pi(xg)\psi\rangle\langle \pi(xg)\psi, \phi_2\rangle \mathrm{d}g \\ &= \int_G \langle \phi_1, \pi(g)\psi\rangle\langle \pi(g)\psi, \pi(x^{-1})\phi_2\rangle \mathrm{d}g \\ &= \langle \pi(x)S\phi_1, \phi_2\rangle \end{aligned}$$

for all $x \in G$; that is, S commutes with every $\pi(x)$, so there exists $\lambda_\psi > 0$ such that $S = \lambda_\psi I$ by Schur's lemma (cf. [7, Subsection 3.1]).

Obviously, every continuous frame wavelet associated to a representation $(\pi, \mathcal{H})$ of G is an admissible vector, but the converse is not true (see Example 2.4). However, the next theorem shows that the converse holds when π is irreducible.

Theorem 2.3 Let $(\pi, \mathcal{H})$ be an irreducible representation on a locally compact group G and $\psi \in \mathcal{H}$. Then, ψ is a continuous frame wavelet if and only if $C_\psi < +\infty$.

Proof Assume that ψ is an admissible vector, then W_ψ, the continuous wavelet transform associated to ψ, is a linear isometry. Therefore

$$C_\psi^{-2} \int_G |\langle \pi(g)\psi, \phi\rangle|^2 \mathrm{d}g = \|W_\psi \phi\|_2^2 = \|\phi\|^2 \qquad (\phi \in \mathcal{H}).$$

that is, $\{\pi(g)\psi\}_{g\in G}$ is a tight frame.

Notice that the above theorem indicates that every continuous frame wavelet ψ associated to an irreducible representation generates a tight continuous frame with the frame bound C_ψ^2. In the sequel, we aim to find a relation between admissible and continuous frame wavelet vectors when the representation π is not necessarily irreducible. First, define the Fourier transform of $f \in L^1(\mathbb{R})$ by

$$\widehat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-\mathrm{i}x\xi} \mathrm{d}x.$$

As usual, we extend the Fourier transformation to an isometry from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

Example 2.4 Let $G = (0,+\infty) \times_\tau \mathbb{R}$ be the affine group on $\mathbb{R}$ with the operation $(h',x').(h,x) = (hh', x+hx')$, and consider $(U, L^2(\mathbb{R}))$ as the quasi-regular representation on G. In fact,

$$U(h,x) : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}); \qquad (U(h,x)f)(y) = h^{\frac{-1}{2}} f\left(\frac{y-x}{h}\right),$$

for all $(h,x) \in G$ and $y \in \mathbb{R}$. Take

$$\mathcal{H}_1 = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\widehat{f}) \subseteq [0,+\infty)\},$$

$$\mathcal{H}_2 = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\widehat{f}) \subseteq (-\infty,0]\}.$$

Both of them are closed U invariant subspaces of $L^2(\mathbb{R})$ and $\mathcal{H}_1 \oplus \mathcal{H}_2 = L^2(\mathbb{R})$. Thus, U has two irreducible subrepresentations and therefore it is reducible [2]. Now, Theorem 2.3 shows that every admissible vector $\psi_i \in \mathcal{H}_i$ is a continuous frame wavelet for $\mathcal{H}_i$ $(i = 1,2)$. In this case, ψ_i is also an admissible vector for $(U, L^2(\mathbb{R}))$ even though it is not a continuous frame wavelet for $L^2(\mathbb{R})$. Furthermore, a straightforward calculation shows that $\{U(g)(\psi_1+\psi_2)\}_{g\in G}$ is a continuous frame with the bounds $\min\{C^2_{\psi_1}, C^2_{\psi_2}\}$ and $2\max\{C^2_{\psi_1}, C^2_{\psi_2}\}$. In particular, $\psi_1+\psi_2$ is a continuous frame wavelet for $L^2(\mathbb{R})$.

The following theorem is a more general setting.

Theorem 2.5 Let $(\pi,\mathcal{H})$ be a representation on locally compact group G, and let $(\pi,\mathcal{H})$ be decomposed into disjoint square integrable components $(\pi_i,\mathcal{H}_i)$, $i \in \mathbb{N}$, that is, $(\pi,\mathcal{H}) = (\oplus\pi_i, \oplus\mathcal{H}_i)$. If $\psi := (\psi_i) \in \mathcal{H}$ such that ψ_i is an admissible vector of $(\pi_i,\mathcal{H}_i)$ for every $i \in \mathbb{N}$, then ψ is an admissible vector for $(\pi,\mathcal{H})$ if and only if the mapping $i \longmapsto C_{\psi_i}$ belongs to l^∞

Proof Associated to admissible vectors $\psi_i \in \mathcal{H}_i$ and $\psi_j \in \mathcal{H}_j$, define the operators T_{ij} in weak sense by

$$T_{ij} : \mathcal{H}_i \longrightarrow \mathcal{H}_j, \qquad T_{ij}\phi_i = \int_G \langle \pi_i(g)\psi_i, \phi_i\rangle \pi_j(g)\psi_j \mathrm{d}g.$$

From the Hölder inequality, it follows that $\|T_{ij}\| \le C_{\psi_i}C_{\psi_j}$. Moreover,

$$\begin{aligned}\pi_j(x)T_{ij} &= \int_G \langle \pi_i(g)\psi_i, \phi_i\rangle \pi_j(x)\pi_j(g)\psi_j \mathrm{d}g\\ &= \int_G \langle \pi_i(g)\psi_i, \pi_i(x)\phi_i\rangle \pi_j(g)\psi_j \mathrm{d}g\\ &= T_{ij}\pi_i(x)\end{aligned}$$

for all $x \in G$. Using Schur's Lemma, we see that T_{ij} is equal to zero when $i \ne j$ and T_{ii} is a multiple of the identity operator in $\mathcal{H}_i$. In fact, using Theorem 2.3, for every ϕ_i, we have

$$\langle \phi_i, T_{ii}\phi_i\rangle = \int_G |\langle \pi_i(g)\psi_i, \phi_i\rangle|^2 \mathrm{d}g = C^2_{\psi_i}\|\phi_i\|^2.$$

Thus,

$$\begin{aligned}\int_G |\langle \pi(x)\psi, \phi\rangle|^2 d(x) &= \sum_{i,j}\int_G \langle \pi_i(x)\psi_i, \phi_i\rangle\langle \phi_j, \pi_j(x)\psi_j\rangle d(x)\\ &= \sum_{i,j}\langle \phi_j, T_{ij}\phi_i\rangle\\ &= \sum_i C^2_{\psi_i}\|\phi_i\|^2.\end{aligned}$$

that is, $\psi \in \mathcal{H}$ is admissible if the mapping $i \longmapsto C_{\psi_i}$ belongs to l^∞.

Conversely, choose $u_i \in \mathcal{H}_i$ such that $\|u_i\| = 1$, then, $\phi = (\phi_i) = (|a_i|^{\frac{1}{2}} u_i) \in \mathcal{H}$ for every $(a_i) \in l^1$. Now, if $\psi \in \mathcal{H}$ is admissible, then,

$$\sum C_{\psi_i}^2 |a_i| = \int_G |\langle \pi(g)\psi, \phi\rangle|^2 \mathrm{d}g < \infty.$$

This completes the proof.

Corollary 2.6 With the assumptions as in Theorem 2.5, the followings hold:

(i) There exist $A, B > 0$ such that $A \le C_{\psi_i}^2 \le B$, for all i, if and only if ψ is a continuous frame wavelet for $\mathcal{H}$.

(ii) If $\psi \in \mathcal{H}$ is admissible, then all $\psi_i \in \mathcal{H}_i$ are admissible.

The above theorem leads us to this question: what happens if there exists a family of irreducible arbitrary subrepresentations? To solve the problem, we need the notion of direct integrals of unitary representations, including direct sums as special cases, that allows one to build a unitary representation out of irreducible ones.

Definition 2.7 Suppose that $\{(\pi_\gamma, \mathcal{H}_\gamma)\}_{\gamma\in\Gamma}$ is a family of representations on an arbitrary topological group G, and μ is a fixed measure on the parameter space Γ. Roughly speaking, the direct integral of the spaces $\mathcal{H}_\gamma$ with respect to μ, denoted by

$$\int^{\oplus} \mathcal{H}_\gamma \mathrm{d}\mu(\gamma),$$

is the Hilbert space of functions ϕ on Γ such that $\phi(\gamma) \in \mathcal{H}_\gamma$ and $\int \|\phi(\gamma)\|^2 \mathrm{d}\mu(\gamma) < +\infty$ with inner product

$$\langle \phi, \psi\rangle = \int_\Gamma \langle \phi(\gamma), \psi(\gamma)\rangle_\gamma \mathrm{d}\mu(\gamma).$$

Now, $\pi(g) := \int^{\oplus} \pi_\gamma(g)\mathrm{d}\mu(\gamma)$, the direct integral of the representations π_γ on Hilbert space $\int^{\oplus} \mathcal{H}_\gamma \mathrm{d}\mu(\gamma)$, is defined by $[\pi(g)\phi](\gamma) = \pi_\gamma(g)\phi(\gamma)$; for more details see [7, Subsection 7.4].

The following theorem shows that how we can construct an admissible vector for π.

Theorem 2.8 Let $(\pi, \mathcal{H})$ be a direct integral of irreducible representations $\{(\pi_\gamma, \mathcal{H}_\gamma)\}_{\gamma\in\Gamma}$ on G. Also, let $\psi = (\psi(\gamma)) \in \mathcal{H}$. Then,

(i) If $\psi \in \mathcal{H}$ is admissible, then every $\psi(\gamma) \in \mathcal{H}_\gamma$ is admissible provided that $\mu(\{\gamma\}) \ne 0$.

(ii) If the mapping $\gamma \longmapsto C_{\psi(\gamma)}$ belongs to $L^2(\mu)$ and almost all $\psi(\gamma) \in \mathcal{H}_\gamma$ is admissible, then, $\psi \in \mathcal{H}$ are admissible with respect to $(\pi, \mathcal{H})$.

Proof Suppose that $\psi \in \mathcal{H}$ is admissible. Take

$$\phi(\gamma) = \begin{cases} 0 & \text{for } \gamma \ne \beta\ , \\ \psi(\beta) & \text{for } \gamma = \beta. \end{cases}$$

Clearly, $\phi = (\phi(\gamma)) \in \mathcal{H}$ and

$$\begin{aligned}\int_G |\langle \pi(g)\psi, \phi\rangle|^2 \mathrm{d}g &= \int_G \left| \int_\Gamma \langle \pi_\gamma(g)\psi(\gamma), \phi(\gamma)\rangle \mathrm{d}\mu(\gamma)\right|^2 \mathrm{d}g \\ &= \mu(\{\beta\}) \int_G |\langle \pi_\beta(g)\psi(\beta), \psi(\beta)\rangle|^2 \mathrm{d}g.\end{aligned}$$

This proves (i). To show (ii) assume that $\psi(\gamma) \in \mathcal{H}_\gamma$ is an admissible vector of $(\pi_\gamma, \mathcal{H}_\gamma)$ then using Hölder inequality and Theorem 2.3, we have

$$\begin{aligned}
&\int_G |\langle \pi(g)\psi, \phi\rangle|^2 \mathrm{d}g \\
=& \int_G \langle \pi(g)\psi, \phi\rangle \langle \phi, \pi(g)\psi\rangle \mathrm{d}g \\
=& \int_\Gamma \int_\Gamma \int_G \langle \pi_\gamma(g)\psi(\gamma), \phi(\gamma)\rangle \langle \phi(\beta), \pi_\beta(g)\psi(\beta)\rangle \mathrm{d}g\mathrm{d}\mu(\gamma)\mathrm{d}\mu(\beta) \\
\leq& \int_\Gamma \int_\Gamma \left(\int_G |\langle \pi_\gamma(g)\psi(\gamma), \phi(\gamma)\rangle|^2 \mathrm{d}g \right)^{\frac{1}{2}} \left(\int_G |\langle \phi(\beta), \pi_\beta(g)\psi(\beta)\rangle|^2 \mathrm{d}g \right)^{\frac{1}{2}} \mathrm{d}\mu(\gamma)\mathrm{d}\mu(\beta) \\
=& \int_\Gamma \int_\Gamma C_{\psi(\gamma)} \|\phi(\gamma)\| C_{\psi(\beta)} \|\phi(\beta)\| \mathrm{d}\mu(\gamma)\mathrm{d}\mu(\beta) \\
=& \left(\int_\Gamma C_{\psi(\gamma)} \|\phi(\gamma)\| \mathrm{d}\mu(\gamma) \right)^2 < \infty.
\end{aligned}$$

Remark 2.9 By Theorem 7.38 of [7], every unitary representation on G is a direct integral of its irreducible representations. Hence, Theorem 2.8 holds for all unitary representation of G in which the components are also square integrable.

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