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Capacity of shrinking condensers in the plane

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Abstract

We show that the capacity of a class of plane condensers is comparable to the capacity of corresponding "dyadic condensers". As an application, we show that for plane condensers in that class the capacity blows up as the distance between the plates shrinks, but there can be no asymptotic estimate of the blow-up.

Keywords: Condenser capacity; Two-dimensional potential theory; Potential theory on trees

1. Introduction

Let Ω be an open region in the complex plane and let E and K be disjoint subsets of $\overline{\Omega}$, with F closed and K compact. The *capacity* of the *condenser* (F, K) in Ω is

$$\operatorname{Cap}_\Omega(F, K) = \inf\left\{\|\nabla u\|^2_{L^2(\Omega)}\colon\ u \geqslant 1 \text{ on } K,\ u \leqslant 0 \text{ on } F\right\}.$$

The sets F and K are the *plates* of the condenser. The infimum is taken over functions u which are C^1 in Ω and continuous on $\Omega \cup F \cup K$. The capacity of a condenser, a notion arising in electrostatics, became part of mainstream Potential Theory in 1945 with the foundational articles by Pólya and Szegö [11] and Szegö [12], where the case of $\mathbb{R}^n$, $n \geqslant 2$, is considered. Condenser capacity has since become an important and useful notion in mathematics *per se*; for instance, in the theory of conformal ($n = 2$) and quasiconformal ($n \geqslant 2$) mappings and, more generally, in Geometric Measure Theory on metric spaces. A class of problems in the field deals with estimates of capacity for condensers the plates of which undergo geometric transformations of some

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kind: rigid movement, for instance, or degeneration of one plate. Here, we will consider some condensers in which the space between the plates shrinks. Intuition suggests that the capacity of such condensers must blow up: we will see that this is true, but in a weak sense only.

Problems of this kind have been considered before in the literature. In [10], the plates of the condenser are identical discs getting closer. In [9], the case of concentric circular arcs, which are symmetric with respect to the real axis, is considered. Other articles deal with different families of condensers and give rather precise estimates of how their capacity blows up as the distance between the plates vanishes. In this article, we consider a condenser in which one plate is a disc of radius increasing to one, while the other is a compact subset of the unit circle, subject to a constraint on its capacity only.

Let Δ be the unit disc in the complex plane and let $\mathbb{T}$ be its boundary, $\overline{\Delta} = \Delta \cup \mathbb{T}$. We denote by $\Delta(z, r)$ the disc of radius r centered at z and by $\overline{\Delta}(z, r)$ its closure. Given $E, F \subset \overline{\Delta}$, closed and disjoint, the capacity of the condenser (E, F) in Δ is

$$\operatorname{Cap}(E, F) = \inf\bigl\{\|\nabla u\|^2_{L^2(\Delta)}\colon\ u \geqslant 1 \text{ on } E,\ u \leqslant 0 \text{ on } F\bigr\},$$

where, for points $\zeta \in E \cup F \setminus \Delta$, we ask for the existence of $\lim_{r\to 1} u(r\zeta) \in \mathbb{R} \cup \{+\infty\}$. In this article, we define the *capacity* of $E \subseteq \mathbb{T}$ to be

$$\operatorname{Cap}(E) := \operatorname{Cap}\bigl(E, \overline{\Delta}(0, 1/2)\bigr).$$

The quantity $\operatorname{Cap}(E)$ is comparable with the logarithmic capacity of E. We are here interested in the behavior of $\operatorname{Cap}(E, \overline{\Delta}(0, r))$ as $r \to 1$, when $E \subseteq \mathbb{T}$ is closed and has positive capacity (if $\operatorname{Cap}(E) = 0$, then $\operatorname{Cap}(E, \overline{\Delta}(0, r)) = 0$ for all positive $r < 1$).

Theorem 1. *Let $E \subseteq \mathbb{T}$ be closed and* $\operatorname{Cap}(E) > 0$. *Then,*

$$\lim_{r\to 1^-} \operatorname{Cap}\bigl(E, \overline{\Delta}(0, r)\bigr) = +\infty.$$

It is not possible to find asymptotic estimates for the rate of convergence.

Theorem 2. *Let $0 < \epsilon < \epsilon_0$, with ϵ_0 small enough. Then, there is $C(\epsilon) > 0$ such that for all $r \in (0, 1)$ there is a closed subset $E = E_{\epsilon,r}$ of $\mathbb{T}$ with* $\operatorname{Cap}(E) = \epsilon$, *yet* $\operatorname{Cap}(E, \overline{\Delta}(0, r)) \leqslant C(\epsilon)$.

Theorem 2 could also be deduced from Haliste's desymmetrization result in [8]. In fact, we deduce it from an elementary, discrete desymmetrization inequality. Set desymmetrization was introduced in [7], and it has proved a powerful tool in potential theory.

If E has full capacity, $\operatorname{Cap}(E) = \operatorname{Cap}(\mathbb{T})$, then $E = \mathbb{T}$ and the problem of the rate reduces to an elementary calculation:

$$\operatorname{Cap}\bigl(\mathbb{T}, \overline{\Delta}(o, r)\bigr) = \frac{1}{\log r^{-1}} \sim (1 - r)^{-1} \quad \text{as } r \to 1.$$

It would be interesting to have an extension of Theorem 1 to the case $\epsilon_0 = \operatorname{Cap}(\mathbb{T})$.

Conjecture 3.

$$\inf\{\mathrm{Cap}(E, \overline{\Delta}(0,r)) \colon \mathrm{Cap}(E) = \mathrm{Cap}(\mathbb{T}) - \delta\} \approx \frac{\mathrm{Cap}(\mathbb{T}) - \delta}{1 - r + \delta}.$$

The conjecture, that is, is that the right "scale" governing the asymptotics of capacity for the condensers considered in this article is not given by a small set capacity, but by the small amount by which the closed set E fails to have full capacity (hence, to be the full boundary $\mathbb{T}$). We offer below some evidence in favor of the conjecture. If the conjecture were true, Theorem 2 would also hold without the assumption that ϵ be "small enough".

The method we employ in proving Theorems 1 and 2 seems to be new in the context of condensers, and it might be useful in tackling similar problems. We will consider first, in Section 2, a discrete, "toy" version of the original estimates on a dyadic tree. The discrete problem turns out to be much easier to solve. On trees scaling arguments are natural and lead to precise formulas; the boundaries of "connected regions" are rather trivial and condensers are much simpler objects; more important: there is a precise recursive algorithm to compute the capacity of a set. In the tree context, we will prove analogs of Theorem 1 and of a sharper version of Theorem 2. Then, we show that the relevant quantities (capacities of sets and condensers, distance between the plates) can be transfered from the discrete setting to the disc setting and back, with estimates from above and below. In Theorem 1, we use the fact that, essentially, a unique function which is harmonic on the tree encodes the extremals for all condensers obtained by shrinking the space between the plates. In Theorem 2, the advantage is that a recursive argument on the tree, which is wholly precise, gives a good estimate for a condenser capacity in the disc: the loss of precision happens just once, passing from the tree to the disc.

The idea of using potential theory on trees to solve problems in the continuous setting is not new. In [6], Benjamini and Peres showed that logarithmic and "tree" capacity of a subset on the real line are comparable, and used this fact to explain the transience–recurrence dichotomy for a random walk on a tree in terms of classic logarithmic capacity. In [5], a different proof of the same result is given, and it is applied to the proof of a Nehari-type theorem for bilinear forms on the holomorphic Dirichlet space. In [4] it is shown in some generality (*Ahlfors regular metric spaces*, non-linear potentials) that Bessel-type set capacities are equivalent to analogous set capacities on trees. The novelty here is that the equivalence between discrete and continuous setting is extended to the capacities of some *condensers*.

This work was born from a question of Carl Sundberg, who asked me if something was known on the rate of convergence to infinity of the condensers considered in this article. It is a pleasure to thank him for the stimulating question and the organizers of the RAFROT 2010 Conference in Portorico, where the question was posed to me and where I gave a first (wrong) answer. Thanks also go to Dimitri Betsakos, for the useful comments and suggestions on a first draft of the paper.

2. Capacities on trees

Trees. We start by recalling some basic facts about trees. Let T be a dyadic tree with root $o \in T$. Each vertex of T is linked by an edge to three other vertices, except for the root, who is linked to just two vertices. The eventual edge between x and y is denoted by $[x, y]$ ($[x, y] = [y, x]$). A path $\Gamma_{x,y}$ between points $x, y \in T$ is defined as a sequence of edges: $\Gamma_{x,y} = \{[x_{j-1}, x_j] \colon j = 1, \dots, n\}$ with $x_0 = x$ and $x_n = y$ ($\Gamma_{x,y} = \Gamma_{y,x}$: we do not consider oriented paths). *The* path $\Gamma_{x,y} =: [x, y]$ in which no edge appears more than once is the *geodesic* between x and y. With slight abuse, we can consider $[x, y]$ as a subset of T. We identify the edge $[x, y]$ with the geodesic $[x, y]$ between neighboring points and pose $[x, x] = \{x\}$. The *natural distance* between $x, y \in T$ is $d(x, y) = \sharp[x, y] - 1$. We write $d(x) = d(x, o) = \sharp[x, o] - 1$. Given $x, y \in T$, we introduce the partial order: $x \leqslant y$ if $x \in [o, y]$. For each $x \in T$ there are two neighbors x_+ and x_- which follow x in the partial order. We say that $x_\pm$ are the *children* of x and that $x =: x_\pm^{-1}$ is their *parent*.

We also consider *half-infinite geodesics* $\gamma \subset T$ starting at $x \in T$, which might be defined as unions of geodesics $[x, x_n] \subset [x, x_{n+1}]$, with $d(x, x_n) \to +\infty$: $\gamma = \bigcup_{n \geqslant 0}[x, x_n]$. The set of the half-infinite geodesics starting at o is the *boundary* of T, denoted by ∂T. To avoid confusion, we consider ∂T as a set of geodesics' labels: $\zeta \in \partial T$ labels the geodesic $P(\zeta)$. By extension, we write $P(x) = [o, x]$ when $x \in T$. If $\zeta \in \partial T$ and $x \in T$, we set $[x, \zeta) := P(\zeta) \setminus P(x^{-1})$: the geodesic joining $x \in T$ and the boundary point ζ. Let $\overline{T} = T \cup \partial T$. Given $x \in T$, $S(x) \subseteq \overline{T}$ is the *successor set* of x: $S(x) = \{\zeta \in \overline{T}\colon x \in P(\zeta)\}$. We also set $T_x = S(x) \cap T = \{y \in T\colon y \geqslant x\}$, the subtree of T having root x. Note that $\partial T_x = \partial T \cap S(x)$ is the boundary of the rooted tree T_x.

Given $\zeta \neq \xi \in \overline{T}$, let $\zeta \wedge \xi = \max(P(\zeta) \cap P(\xi))$, where the maximum is taken w.r.t. the partial order. Assign now to each edge $[x, x^{-1}]$ in the tree the weight $2^{-d(x)}$ and define the length of a path in T as the sum of its edges. The minimal length of a path joining α and β in T is a new distance ρ, which extends to a distance on $\overline{T}$:

$$\rho(\alpha, \beta) = 2^{-d(\alpha \wedge \beta)} - \frac{1}{2}\big(2^{-d(\alpha)} + 2^{-d(\beta)}\big).$$

The metric space $(\partial T, \rho)$ is totally disconnected, perfect, compact and for $\xi, \zeta \in \partial T$ we have

$$\rho(\xi, \zeta) = 2^{-d(\xi \wedge \zeta)},$$

which is also called the *ultrametric distance* on ∂T. The tree T is the set of the isolated points in $(\overline{T}, \rho)$ and ∂T is the metric boundary of T in $(\overline{T}, \rho)$. In fact, we can identify $(\overline{T}, \rho)$ with the metric completion of (T, ρ) and ∂T with the points which have been added to T in order to make it complete w.r.t. the metric ρ. The set $S(x)$ is then the closure of T_x in $\overline{T}$.

We introduce a sum operator I applying functions $\varphi : T \to \mathbb{R}$ to functions $I\varphi : \overline{T} \to \mathbb{R}$,

$$I\varphi(\zeta) = \sum_{x \in P(\zeta)} \varphi(x), \quad \zeta \in \overline{T}.$$

(We will consider $\varphi \geqslant 0$, hence convergence of the series when $P(\zeta)$ is infinite causes no ambiguity.) Its formal adjoint I^* acts on Borel measures μ on $\overline{T}$,

$$I^*\mu(x) = \int_{S(x)} d\mu, \quad x \in T.$$

The "Hardy" operator I on trees was first introduced, in connection with problems of classical function theory, in [2].

Tree capacities. Let E be a closed subset of ∂T. Its capacity is

$$\mathrm{Cap}^T(E) = \inf\big\{\|\varphi\|_{\ell^2}^2\colon \varphi \geqslant 0,\ I\varphi \geqslant 1 \text{ on } E\big\}.$$

For $n \geqslant 0$, integer, we consider a condenser capacity

$$\mathrm{Cap}_n^T(E) = \inf\bigl\{\|\varphi\|_{\ell^2}^2\colon \varphi \geqslant 0,\ I\varphi \geqslant 1 \text{ on } E,\ I\varphi(x) = 0\ \forall x \text{ s.t. } d(x) = n-1\bigr\}.$$

We set $\mathrm{Cap}_0^T(E) = \mathrm{Cap}^T(E)$.

For each $x \in T$, the tree T_x has boundary $\partial T_x = \partial S(x)$ and we can compute the capacity of sets $F \subseteq \partial T_x$ w.r.t. the root x. If $E \subseteq \partial T$ and $E_x = E \cap \partial T_x$, it is clear from the definitions and the trivial topology of T that

$$\mathrm{Cap}_n^T(E) = \sum_{x\colon d(x)=n} \mathrm{Cap}^{T_x}(E_x). \tag{1}$$

Here, $\mathrm{Cap}^{T_x}(E_x)$ is the capacity of E_x in T_x w.r.t. the root x.

Before we proceed, we give some basic properties of tree capacities. Proofs are sparse in the literature, or they are special cases of general theorems about capacities in metric spaces. A good source for the general theory is [1]. All properties are given a precise reference or proved in §5 of [4] (for general trees and weighted potentials), and they are proved in [3] (in the dyadic case).

(a) There exists a unique extremal function $h = \varphi$ for the definition of $\mathrm{Cap}^T(E)$. It satisfies (i) $\|h\|_{\ell^2}^2 = \mathrm{Cap}^T(E)$; (ii) $H = Ih = 1$ on E, but for a set of null capacity.
(b) The function h satisfies the algebraic relation $h(x) = h(x_+) + h(x_-)$ and $h(x) > 0$ everywhere on T.
(c) There is a unique positive, Borel measure μ supported on E (the equilibrium measure) with the property that $h = I^*\mu$. Moreover, $\mathrm{Cap}^T(E) = \mu(E)$. As a consequence, $\mathrm{Cap}^T(E) = h(o)$.
(d) $\lim_{x\to\partial T} H(x) = 1$ μ-a.e.
(e) Capacities satisfy a recursive relation:

$$\mathrm{Cap}_x^T(E_x) = \frac{\mathrm{Cap}^{T_{x_+}}(E_{x_+}) + \mathrm{Cap}^{T_{x_-}}(E_{x_-})}{1 + \mathrm{Cap}^{T_{x_+}}(E_{x_+}) + \mathrm{Cap}^{T_{x_-}}(E_{x_-})}.$$

The capacity of the full boundary is $\mathrm{Cap}^T(\partial T) = 1/2$.

Theorem 2 holds on trees.

Theorem 4. *$\forall \epsilon \in (0, 1/2)\ \exists R > 0\ \forall n\ \exists E \subset \partial T\colon \mathrm{Cap}^T(E) \geqslant \epsilon$, but $\mathrm{Cap}_n^T(E) \leqslant R$.*

In fact, we can be more precise:

$$R = \frac{\epsilon}{1 - 2\epsilon}.$$

Proof. Consider a set E such that $\mathrm{Cap}^T(E) = \epsilon$, a positive integer n, and suppose that, at each step $j = 1, \dots, n$ the set splits in two copies having the same capacity. Namely, the set E splits into two copies $E_+ \subseteq \partial T_{o_+}$ and $E_- \subseteq \partial T_{o_-}$ having equal capacities, and so on, iterating. In the end we get, corresponding to the 2^n points $x_1^n, \dots, x_{2^n}^n$ s.t. $d(x_j^n) = n$, 2^n sets $E_1^n \subset \partial T_{x_1}, \dots, E_{2^n}^n \subset \partial T_{x_{2^n}}$ having equal capacity.

Let e_n be the capacity of any of E_j^n w.r.t. the root x_j^n. Indeed, $e_0 = \epsilon$ and

$$e_n = \frac{e_{n-1}}{2 - 2e_{n-1}},$$

by (e). Iterating, we find

$$\mathrm{Cap}_n^T(E) = 2^n e_n = \frac{2^n \epsilon}{2^n - (2^{n+1} - 2)\epsilon} \nearrow \frac{\epsilon}{1 - 2\epsilon}.$$

To finish the proof, we must show that, for any given ϵ in $(0, 1/2)$, there is a set E having capacity $\mathrm{Cap}^T(E) = \epsilon$, which is the union of 2^n subsets having equal capacity, each lying in some $I(x_j^n)$, $1 \leqslant j \leqslant 2^n$. This can be done if and only if we can find a subset E_j^n of $I(x_j^n)$ such that $\mathrm{Cap}^{T_{x_j^n}}(E_j^n) = e_n$; which (by obvious rescaling) is the same as finding a closed subset F of ∂T such that $\mathrm{Cap}^T(F) = e_n$. By induction and the fact that $\psi(t) := t/(2 - 2t)$ is a diffeomorphism of $[0, 1/2]$ onto itself, we have that $0 < e_n < 1/2$. Finally, it is easy, for each such e_n, to produce a set F with the desired capacity (for completeness, details are presented in Lemma 11 below). □

One might think that the splitting process could be continued for an infinite time, producing a stronger result. This is not the case: if one does not stop the procedure, the set E "fades away" and it will have null capacity, as Theorem 6 below shows.

It is also possible to prove, using an easy convexity argument, a quantitative, positive result justifying Conjecture 3.

Theorem 5. *Given a set E with $\mathrm{Cap}^T(E) = \epsilon$, one has the estimate*:

$$\inf\{\mathrm{Cap}_n^T(E)\colon \mathrm{Cap}^T(E) = \epsilon\} = 2^n e_n = \frac{\epsilon}{1 - (2 - 2^{1-n})\epsilon}.$$

The theorem's statement is more expressive if we replace $\epsilon = \mathrm{Cap}^T(\partial T) - \delta = 1/2 - \delta$. The estimate becomes

$$\inf\{\mathrm{Cap}_n^T(E)\colon \mathrm{Cap}^T(E) = \epsilon\} = \frac{1/2 - \delta}{(2 - 2^{1-n})\delta + 2^{-n}}: \tag{2}$$

the lower bound roughly doubles each time n increases by one, until 2^{-n} (the "Euclidean distance" between the plates of the condenser) reaches the scale of δ; after that point, it stabilizes. The difficulty in transfering this result to the continuous case consists in the fact that the scale is the amount by which E fails to have full capacity. This quantity, to the best of my knowledge, has never been investigated in depth: most applications involve estimates for sets having "small enough" capacity.

Proof of Theorem 5. Let $\psi : [0, 1/2] \to [0, 1/2]$ be the function

$$\psi(t) = \frac{t}{2(1 - t)}.$$

The function ψ is a continuous, increasing, strictly convex diffeomorphism of $[0, 1/2]$ onto itself. Let E be a fixed, closed subset of ∂T, and, for x in T, let $c(x) := \mathrm{Cap}^{T_x}(E \cap \partial T_x)$ be the capacity in the tree T_x of the portion of E lying in ∂T_x. The recursion relation for capacities can be written in the form

$$\frac{c(x_+) + c(x_-)}{2} = \psi\big(c(x)\big).$$

Let $\psi^{\circ n} = \psi \circ \cdots \circ \psi$ be the composition of ψ with itself n times: $\psi^{\circ n}$ is a continuous, increasing, strictly convex diffeomorphism of $[0, 1/2]$ onto itself, too. We claim that, for a in T fixed and n positive integer

$$\frac{1}{2^n} \sum_{x \geqslant a,\ d(x,a)=n} c(x) \geqslant \psi^{\circ n}\big(c(a)\big). \tag{3}$$

We prove this by induction. For $n = 1$ (3) holds with equality by the recursion relation. Suppose (3) holds for $n - 1$. Then,

$$\begin{aligned}
\frac{1}{2^n} \sum_{x \geqslant a,\ d(x,a)=n} c(x) &= \frac{1}{2}\left(\frac{1}{2^{n-1}} \sum_{x \geqslant a_+,\ d(x,a_+)=n-1} c(x) + \frac{1}{2^{n-1}} \sum_{x \geqslant a_-,\ d(x,a_-)=n-1} c(x)\right) \\
&\geqslant \frac{1}{2}\big(\psi^{\circ(n-1)}\big(c(a_+)\big) + \psi^{\circ(n-1)}\big(c(a_-)\big)\big) \\
&\geqslant \psi^{\circ(n-1)}\left(\frac{c(a_+) + c(a_-)}{2}\right) \\
&= \psi^{\circ n}\big(c(a)\big) \\
&= \frac{c(a)}{2^n - (2^{n+1} - 2)c(a)}.
\end{aligned}$$

The explicit calculation of $\psi^{\circ n}$ can be checked by induction. Set $a = o$ to finish the proof. □

Proof of Theorem 1 on trees.

Theorem 6. *If* $\mathrm{Cap}^T(E) > 0$, *then*

$$\lim_{n \to \infty} \mathrm{Cap}_n^T(E) = +\infty.$$

Proof. Let h be the extremal function for the definition of $\mathrm{Cap}^T(E)$ and let $H = Ih$. By properties (b) and (c),

$$0 < \mathrm{Cap}^T(E) = \sum_{x:\ d(x)=n} h(x).$$

Let x^{-1} be the parent of the point $x \in T$. By (d) and Egoroff's Theorem, for all $\delta > 0$ there is a set E_δ s.t. $\mu(E_\delta) < \delta$ and $1 - H(x^{-1}) \to 0$ uniformly as $x \to \zeta \in \partial T \setminus E_\delta$. Here, Egoroff's

Theorem is applied to the sequence of functions $H_n : \partial T \to \mathbb{R}$, $H_n(\zeta) = H(x)$ if $d(x) = n$ and $\zeta \in \partial S(x)$. By regularity of the measure μ, doubling δ, we can assume that E_δ is open; i.e. it is union of "arcs" of the form $\partial S(y)$.

By rescaling, it is easy to see that

$$h(x) = \big(1 - H\big(x^{-1}\big)\big) \cdot \mathrm{Cap}^{T_x}(E_x). \tag{4}$$

Here is the proof of (4). Let h^x be the extremal function for E_x in T_x. Then, (i) h^x satisfies the additivity relation (b) in T_x and (ii) $\sum_{y \in P(\zeta) \setminus P(x^{-1})} h^x(y) \geqslant 1$ for nearly all ζ in E_x. An obvious choice of a function satisfying (i) and (ii) is $k^x = (1 - H(x^{-1}))^{-1} \cdot h$. It is easy to see that such guess has the minimizing property of the desired extremal function, hence that $k^x = h^x$. In fact, if k^x had not minimal ℓ^2 norm with properties (i) and (ii), we could modify h on T_x by setting $h|_{T_x} = (1 - H(x^{-1})) \cdot h^x$. This way, we would reduce the ℓ^2 norm of h on T, while retaining the property that $Ih \geqslant 1$ on E. By (c),

$$\mathrm{Cap}^{T_x}(E_x) = h^x(x) = \big(1 - H\big(x^{-1}\big)\big)^{-1} \cdot h(x),$$

proving (4).

Since H_n converges uniformly on $\partial T \setminus E_\delta$, there is $n(\delta)$ s.t., for $n \geqslant n(\delta)$ we have $1 - H(x^{-1}) = 1 - H_n(\zeta) \leqslant \delta$ if $d(x^{-1}) = n$ and $\zeta \in \partial S(x) \cap \partial T \setminus E_\delta$.

Putting all this together, with $n \geqslant n(\delta)$,

$$\begin{aligned} 0 < \mathrm{Cap}^T(E) &= \sum_{x:\, d(x)=n} h(x) \\ &= \sum_{d(x)=n,\ \partial S(x) \cap (\partial T \setminus E_\delta) \neq \emptyset} \big(1 - H\big(x^{-1}\big)\big) \cdot \mathrm{Cap}^{T_x}(E_x) + \sum_{d(x)=n,\ \partial S(x) \subset E_\delta} I^*\mu(x) \\ &\leqslant \sum_{d(x)=n,\ \partial S(x) \cap (\partial T \setminus E_\delta) \neq \emptyset} \big(1 - H\big(x^{-1}\big)\big) \cdot \mathrm{Cap}^{T_x}(E_x) + \mu(E_\delta) \\ &\leqslant \delta \sum_{d(x)=n,\ \partial S(x) \cap (\partial T \setminus E_\delta) \neq \emptyset} \mathrm{Cap}^{T_x}(E_x) + \delta. \end{aligned}$$

Thus,

$$0 < \mathrm{Cap}^T(E) \leqslant \delta \sum_{d(x)=n} \mathrm{Cap}^{T_x}(E_x) + \delta = \delta\big(\mathrm{Cap}_n^T(E) + 1\big),$$

and the result follows letting $\delta \to 0$. □

3. Continuous capacities vs. discrete capacities

The usual dyadic decomposition of the unit disc can be thought of as a tree structure T (as it is explained below). The boundary of the unit disc can be thought of as the boundary ∂T of the tree (this involves some technicalities, which are especially easy in our case, since the unit circle is topologically one-dimensional).

The following theorem is proved in [6]. A proof which applies to a more general case is in [4].

Theorem 7. *Let E be a closed subset of ∂T, identified with a closed subset of $\mathbb{T}$. Then,*

$$\mathrm{Cap}(E) \approx \mathrm{Cap}^T(E).$$

In this section, we prove a similar result for condenser capacities. For $r = 1 - 2^{-n}$, let

$$\mathrm{Cap}_n := \mathrm{Cap}\big(E, \overline{\Delta}(0, r)\big).$$

Theorem 8. *If E is a closed subset of $\mathbb{T}$, identified with a closed subset of ∂T, then*

$$\mathrm{Cap}_n(E) \approx \mathrm{Cap}_n^T(E).$$

The dyadic decomposition of the disc. For integers $n \geqslant 0$ and $1 \leqslant j \leqslant 2^n$, consider the *Bergman box*

$$Q(n, j) = \left\{z = re^{i\theta} \in \Delta\colon \frac{1}{2^{n+1}} < 1 - r \leqslant \frac{1}{2^n},\ \frac{j-1}{2^n} \leqslant \frac{\theta}{2\pi} < \frac{j}{2^n}\right\}, \tag{5}$$

and let $T = \{(n, j)\colon n \geqslant 0,\ 1 \leqslant j \leqslant 2^n\}$ be the set of such boxes. We associate to each $Q = Q(n, j)$ in T: a distinguished point $z(Q)$ in Q,

$$z(Q) = \big(1 - 2^{-n-1/2}\big)e^{i\frac{j-1/2}{2^n}};$$

a *Carleson box*

$$S(Q) = \left\{z = re^{i\theta} \in \Delta\colon 0 < 1 - r \leqslant \frac{1}{2^n},\ \frac{j-1}{2^n} \leqslant \frac{\theta}{2\pi} < \frac{j}{2^n}\right\};$$

and a distinguished boundary arc $I(Q)$ in $\mathbb{T}$,

$$I(Q) = \left\{e^{i\theta} \in \Delta\colon \frac{j-1}{2^n} \leqslant \frac{\theta}{2\pi} < \frac{j}{2^n}\right\}.$$

We will freely use obvious variations on the notation just introduced. For instance, we write $I(n, j) = I(Q)$ when $Q = Q(n, j)$. Also, we might write $Q = Q(I)$ if $I = I(Q)$. Etcetera.

The tree structure. The set T is given a *tree structure*, which will be denoted by the same letter T. The points of T are the *vertices*. There is an *edge* of the tree between (n, j) and (m, i) if $n = m + 1$ and $I(n, j) \subseteq I(m, i)$ ($I(n, j)$ is one of the two halves of the arc $I(m, i)$) or, vice versa, if $m = n + 1$ and $I(m, i) \subseteq I(n, j)$. The *level* of the box $Q = Q(n, j)$ is $d_T(Q) := n$; so that $|I(Q)| = 2^{-d_T(Q)}$. Note that there is just one vertex $o := (0, 1)$ having level $d_T(o) = 0$: it is the *root* of the tree T. Boxes and labels for boxes are sometimes identified: $Q(n, j) \equiv (n, j)$.

We begin with the easy inequality in Theorem 8.

Lemma 9.

$$\mathrm{Cap}_n^T(E) \lesssim \mathrm{Cap}_n(E).$$

Proof. Consider the subtrees T_x of T, $d(x) = n$, viewed as trees of Bergman boxes, as above. For each α in T_x, let $z(\alpha)$ be the center of the box $Q(\alpha)$ in Δ. Let φ be the extremal function for the definition of $\mathrm{Cap}_n(E)$ and define a function $h : T \to \mathbb{R}$ by

$$h(\alpha) := \varphi\big(z(\alpha)\big) - \varphi\big(z\big(\alpha^{-1}\big)\big).$$

It is clear that $h(\beta) = 0$ for $d(\beta) \leqslant n - 1$ and that

$$\sum_{\gamma=x}^{\alpha} h(\gamma) = \varphi\big(z(\alpha)\big) \quad \forall \alpha \in T_x.$$

Estimating differences $h(\alpha) := \varphi(z(\alpha)) - \varphi(z(\alpha^{-1}))$ and integrating, we see that

$$\|h\|^2_{\ell^2(T)} \lesssim \|\nabla\varphi\|^2_{L^2(\Delta)}. \tag{6}$$

In fact, φ is harmonic in the annulus $\{re^{i\theta}\colon 0 < 1 - r \leqslant 2^{-n}\}$, hence

$$\begin{aligned}
&\big|\varphi\big(z(\alpha)\big) - \varphi\big(z\big(\alpha^{-1}\big)\big)\big| \\
&\quad = \int_{z(\alpha^{-1})}^{z(\alpha)} \nabla\varphi(w)\cdot dw \\
&\quad \lesssim \big(1 - \big|z(\alpha)\big|\big)\big|\nabla\varphi\big(w(\alpha)\big)\big| \\
&\qquad \text{for some } w(\alpha) \text{ in the closure of } Q(\alpha) \cup Q\big(\alpha^{-1}\big) \\
&\quad = \big(1 - \big|z(\alpha)\big|\big)\left|\frac{1}{|B_\alpha|}\int_{B_\alpha} \nabla\varphi(w)\, dA(w)\right| \\
&\qquad \text{by the Mean Value Property,} \\
&\qquad \text{where } dA \text{ is area measure and } B_\alpha \text{ is a small disc centered at } w(\alpha) \\
&\qquad \text{having radius and distance from } \mathbb{T} \text{ comparable to } \big(1 - \big|z(\alpha)\big|\big) \\
&\quad \lesssim \left(\int_{B_\alpha} \big|\nabla\varphi(w)\big|^2\, dA(w)\right)^{1/2} \\
&\qquad \text{by Jensen's inequality.}
\end{aligned}$$

Estimate (6) follows, since the discs B_α have bounded overlapping.

On the other hand, as $\alpha \to \zeta \in \partial T$ in T, $z(\alpha) \to \Lambda(\zeta)$, the image of ζ in $\mathbb{T}$, nontangentially. In turn, this implies that

$$Ih(\alpha) = \varphi\big(z(\alpha)\big) \to 1,$$

but for a set of null capacity in ∂T (actually, but for the preimage through Λ of a set of null capacity in $\mathbb{T}$; but by Theorem 7 this is the same as null capacity in ∂T).

Then, h is admissible for the definition of tree capacity E; hence (6) implies the lemma. □

We now come to the more difficult inequality in Theorem 8,

$$\mathrm{Cap}_n(E) \lesssim \mathrm{Cap}_n^T(E). \tag{7}$$

We start with a localization lemma for the condenser capacity.

Fix integer $n \geqslant 1$, large enough, and let $E_j = E \cap I_{n,j}$, where $I_{n,j}$ $(1 \leqslant j \leqslant 2^n)$ is the dyadic arc on $\mathbb{T}$ defined before. Let $A_n = \Delta \setminus \overline{\Delta(0, 1-2^{-n})}$ be the annulus and let $R \subset A_n$ be the curvilinear rectangle

$$R = \left\{re^{it} \in A_n\colon\ \frac{-2}{2^n} \leqslant \frac{t}{2\pi} \leqslant \frac{3}{2^n}\right\},$$

and let $I'_R = \partial R \cap \partial\Delta(0, 1-2^{-n})$ be the side of R which is closest to the center of Δ. We also need I_R, the union of I'_R and of the parts of ∂R lying on the radii $\frac{t}{2\pi} = \frac{-2}{2^n}$ and $\frac{t}{2\pi} = \frac{3}{2^n}$. Define

$$\mathrm{Cap}_R\big(I'_R, E_0\big) = \inf\big\{\|\nabla\varphi\|^2_{L^2(R)}\colon\ \varphi|_{I_R} = 0,\ \varphi|_{E_0} \geqslant 1\big\}$$

to be the capacity of the condenser (I'_R, E_0) in R.

Lemma 10.

$$\mathrm{Cap}_R(I_R, E_0) \lesssim \mathrm{Cap}_R\big(I'_R, E_0\big).$$

By trivial comparison, the opposite inequality $\mathrm{Cap}_R(I_R, E_0) \geqslant \mathrm{Cap}_R(I'_R, E_0)$ holds.

Proof. To prove the lemma, we use a cut-off argument. Let χ be a smooth cutoff function on A_n:

$$\chi\big(re^{it}\big) = \begin{cases} 1 & \text{if } \frac{-1}{2^n} \leqslant \frac{t}{2\pi} \leqslant \frac{2}{2^n}; \\ 0 & \text{if } \frac{t}{2\pi} \leqslant \frac{-2}{2^n} \text{ or } \frac{t}{2\pi} \geqslant \frac{3}{2^n}. \end{cases}$$

We can choose χ in such a way that $0 \leqslant \chi \leqslant 1$ on A_n and that

$$\|\nabla\chi\|^2_{L^2(A_n)} \approx 1.$$

Let φ be the extremal function for $\mathrm{Cap}_R(I'_R, E_0)$. Then, $\varphi \cdot \chi$ is an admissible function for $\mathrm{Cap}_R(I'_R, E_0)$. It suffices, then, to prove that

Claim. $\|\nabla(\varphi\cdot\chi)\|^2_{L^2(A_n)} \lesssim \mathrm{Cap}_R(I'_R, E_0)$.

We have $\|\nabla(\varphi\cdot\chi)\|^2_{L^2(A_n)} \lesssim \|\chi\nabla\varphi\|^2_{L^2(A_n)} + \|\varphi\nabla\chi\|^2_{L^2(A_n)} = I + II$. The first summand is o.k.: $I \leqslant \|\nabla\varphi\|^2_{L^2(R)} = \mathrm{Cap}_R(I'_R, E_0)$. About the second, the integrand is supported in

$$Q = \left\{re^{it}\in A_n\colon \frac{-2}{2^n}\leqslant\frac{t}{2\pi}\leqslant\frac{-1}{2^n}\right\}\cup\left\{re^{it}\in A_n\colon \frac{2}{2^n}\leqslant\frac{t}{2\pi}\leqslant\frac{3}{2^n}\right\}$$

and we are done if we show that

$$M^2 := \sup_{z\in Q}\left|\varphi(z)\right|^2 \lesssim \mathrm{Cap}_R(I_R, E_0).$$

Let $K := \{z\in R\colon \varphi(z)\geqslant M/2\} = \bigsqcup_j K_j$, where each K_j is a connected component of K: K_j is closed in R and its closure in the plane meets the boundary of R, by the maximum principle (φ, being extremal, is harmonic in R). Let K'_j be a component of K having a point in Q and having nonempty interior (there must be one, by definition of M and by continuity of φ). If the closure of K'_j does not meet $I_{n,0}$, the arc containing E_0, we can replace φ by $M/2$ on K'_j, strictly reducing the Dirichlet integral of φ on R, which contradicts the extremality of φ. Then, there is a continuum K'_j joining a point z_0 in Q and a point z' in $Q_0 = \{re^{it}\in A_n\colon 0\leqslant\frac{t}{2\pi}\leqslant\frac{1}{2^n}\}$ on which $\varphi\geqslant M/2$. Let

$$Q' = \left\{re^{it}\in A_n\colon \frac{-1}{2^n}\leqslant\frac{t}{2\pi}\leqslant 0\right\}\cup\left\{re^{it}\in A_n\colon \frac{1}{2^n}\leqslant\frac{t}{2\pi}\leqslant\frac{2}{2^n}\right\}$$

and let $I'_1 = \partial Q'\cap\partial\Delta(0, 1-2^{-n})$, $I'_2 = \partial Q'\cap\partial\Delta(0,1)$. Obvious comparison shows that

$$\begin{aligned}
1 &\approx \mathrm{Cap}_{Q'}\left(I'_1, I'_2\right)\\
&\leqslant \left\|\nabla\left(\frac{\varphi}{M/2}\right)\right\|^2_{L^2(Q')}\\
&\quad \text{because the function } \frac{\varphi}{M/2} \text{ is admissible for the condenser capacity}\\
&\leqslant \frac{4}{M^2}\|\nabla\varphi\|^2_{L^2(R)}\\
&= \frac{4}{M^2}\mathrm{Cap}_R\left(I'_R, E_0\right).
\end{aligned}\tag{8}$$

i.e., $M^2\lesssim\mathrm{Cap}_R(I'_R, E_0)$, as wished. □

We now come to the proof of (7).

Let R_j be a rectangle as R, but built starting from the set E_j. Let

$$E^{(k)} = \bigsqcup_{j=5n+k} E_j,\quad j = 0,1,2,3,4.$$

Since the sum of the extremal functions for the five pieces of E is admissible for E,

$$\mathrm{Cap}_n(E) \leqslant 5\sum_{k=0}^{4} \mathrm{Cap}_n\big(E^{(k)}\big). \tag{9}$$

Also, by comparison:

$$\mathrm{Cap}_n\big(E^{(k)}\big) \leqslant \sum_m \mathrm{Cap}_R(I_{5m+k}, E_{5m+k}). \tag{10}$$

In fact, if φ_m are extremal functions for $\mathrm{Cap}_{R_{5m+k}}(I_{R_{5m+k}}, E_{5m+k})$, extended to be zero in $A_n \setminus R_{5m+k}$, then

$$\varphi = \sum_m \varphi_m$$

is admissible for $\mathrm{Cap}_n(E^{(k)})$ and $\|\nabla\varphi\|^2_{L^2} = \sum_m \|\varphi_m\|^2_{L^2(R_{5m+k})}$. The inequality follows by definition of capacity.

By (9), (10) and Lemma 10, then:

$$\mathrm{Cap}_n(E) \lesssim \sum_{k=0}^{4}\sum_m \mathrm{Cap}_{R_{5m+k}}\big(I'_{R_{5m+k}}, E_{5m+k}\big) = \sum_l \mathrm{Cap}_{R_l}\big(I'_{R_l}, E_l\big). \tag{11}$$

The quantity $\mathrm{Cap}_{R_l}(I'_{R_l}, E_l)$ verifies the condition under which capacity can be discretized as in [6] or [4]. In fact, the proof of Theorem 7 can be adapted without changes to show that

$$\mathrm{Cap}_{R_l}\big(I'_{R_l}, E_l\big) \approx \mathrm{Cap}^{T_{x_l}}(E_j).$$

Summing over l and using *additivity* of these special capacities in the tree T,

$$\begin{aligned} \mathrm{Cap}_n(E) &\lesssim \sum_l \mathrm{Cap}^{T_{x_l}}(E_j) \\ &= \mathrm{Cap}_n^T(E), \end{aligned}$$

as wished. The proof of Theorem 8 is finished.

Proofs of the main theorems.

Proof of Theorem 1. Since $r \mapsto \mathrm{Cap}(E, \overline{\Delta}(0, r))$ is increasing, it suffices to test the conclusion of the theorem on $r = 1 - 2^{-n}$, for integer n. By Theorem 8,

$$\mathrm{Cap}\big(E, \overline{\Delta}\big(0, 1 - 2^{-n}\big)\big) \gtrsim \mathrm{Cap}_n^T(E) \to \infty$$

as $n \to \infty$, by Theorem 6. □

Proof of Theorem 2. If $\epsilon > 0$ is small enough, then, by Theorem 8 (rather, by the special case proved in [6] and [4]), if $\mathrm{Cap}(E) \leqslant \epsilon$, then $0 < \mathrm{Cap}^T(E) \leqslant \epsilon' < \mathrm{Cap}_n^T(\partial T) = 1/2$. By Theorem 4, there is $R(\epsilon)$ s.t. for all n there is E with $\mathrm{Cap}^T(E) \leqslant \epsilon'$ and $\mathrm{Cap}_n(E) \leqslant R(\epsilon)$. By Theorem 8, this implies Theorem 2. □

We finish with the proof of a lemma used in the proof of Theorem 2.

Lemma 11. *For each $0 \leqslant e \leqslant 1/2$ there is a closed subset E of ∂T such that $\mathrm{Cap}^T(E) = e$.*

Proof. Let $\Lambda : \partial T \to [0,1]$ be the map associating to a geodesic ζ in ∂T, $P(\zeta) = \{\zeta_k = (k, j_k)\colon k \geqslant 0\}$ being an enumeration of its vertices where $d(\zeta_n) = n$, the point t in $[0,1]$ such that

$$e^{2\pi i t} = \bigcap_{k \geqslant 0} \overline{I\big(Q(k, j_k)\big)}.$$

We assume that the geodesic "to the extreme left" maps to 0, while that to the "extreme right" maps to 1.

It is easy to prove that the map Λ is continuous (in fact, Lipschitz) w.r.t. the metrics ρ on ∂T and Euclidean on $[0,1]$. Define a function $f : [0,1] \to [0,1/2]$ by

$$f(t) = \mathrm{Cap}^T\big(\Lambda^{-1}\big([0,t]\big)\big).$$

Clearly $f(0) = 0$, $f(1) = 1/2$ and f increases. It suffices to prove that f is continuous.

We have the inequalities (for $h > 0$):

$$\begin{aligned} f(t) &\leqslant f(t+h) \\ &= \mathrm{Cap}^T\big(\Lambda^{-1}\big([0,t+h]\big)\big) \leqslant \mathrm{Cap}^T\big(\Lambda^{-1}\big([0,t]\big)\big) + \mathrm{Cap}^T\big(\Lambda^{-1}\big([t,t+h]\big)\big) \\ &\quad \text{by subadditivity of capacity} \\ &= f(t) + o_{h\to 0}(1), \end{aligned}$$

by regularity of capacity: $\lim_{h\to 0} \mathrm{Cap}^T(\Lambda^{-1}([t,t+h])) = \mathrm{Cap}^T(\Lambda^{-1}([t,t])) = 0$. Hence, f is right continuous.

Similarly, one shows that $f(t-h) \leqslant f(t) \leqslant f(t-h) + o_{h\to 0}(1)$, deducing that f is left continuous. □

References

[1] D.R. Adams, L.I. Hedberg, Function Spaces and Potential Theory, Grundlehren Math. Wiss., vol. 314, Springer-Verlag, Berlin, 1996, xii+366 pp.

[2] Nicola Arcozzi, Richard Rochberg, Eric Sawyer, Carleson measures for analytic Besov spaces, Rev. Mat. Iberoam. 18 (2) (2002) 443–510.

[3] N. Arcozzi, R. Rochberg, E. Sawyer, Capacity, Carleson measures, boundary convergence, and exceptional sets, in: Perspectives in Partial Differential Equations, Harmonic Analysis and Applications, in: Proc. Sympos. Pure Math., vol. 79, Amer. Math. Soc., Providence, RI, 2008, pp. 1–20.

[4] Nicola Arcozzi, Richard Rochberg, Eric Sawyer, Brett Wick, Nonlinear potential theory on trees, graphs and Ahlfors regular metric spaces, preprint.

[5] Nicola Arcozzi, Richard Rochberg, Eric Sawyer, Brett D. Wick, Bilinear forms on the Dirichlet space, Anal. PDE 3 (1) (2010) 21–47.
[6] Itai Benjamini, Yuval Peres, Random walks on a tree and capacity in the interval, Ann. Inst. Henri Poincaré Probab. Stat. 28 (4) (1992) 557–592.
[7] V.N. Dubinin, Change of harmonic measure in symmetrization, Mat. Sb. (N.S.) 124(166) (2) (1984) 272–279 (in Russian).
[8] K. Haliste, On an extremal configuration for capacity, Ark. Mat. 27 (1) (1989) 97–104.
[9] D. Karp, Capacity of a condenser whose plates are circular arcs, Complex Var. Theory Appl. 50 (2) (2005) 103–122.
[10] Frank Leppington, Harold Levine, On the capacity of the circular disc condenser at small separation, Proc. Cambridge Philos. Soc. 68 (1970) 235–254.
[11] G. Pólya, G. Szegö, Inequalities for the capacity of a condenser, Amer. J. Math. 67 (1945) 1–32.
[12] G. Szegö, On the capacity of a condenser, Bull. Amer. Math. Soc. 51 (1945) 325–350.