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Metric normal and distance function in the Heisenberg group

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Abstract We introduce a notion which is equivalent in the Heisenberg group $\mathbb{H}$ to that of segment normal to a surface. Then, we study some regularity properties of the function measuring the Carnot-Carathéodory distance from an Euclidean surface S in $\mathbb{H}$ in terms of the regularity of S.

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Contents

1 Introduction

Let $\mathbb{H} = \mathbb{H}^1$ be the Heisenberg group, the set of the triples $(x, y, t) \in \mathbb{R}^3$ with the product

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' - 2(xy' - x'y)).$$

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Let d be the Carnot-Charathéodory distance on $\mathbb{H}$, see Sect. 2 for the definition. If S is a closed subset of $\mathbb{H}$, the distance from a point P to S is

$$d_S(P) = \inf_{Q \in S} d(P, Q).$$

Let $S = \partial\Omega$ be the boundary of an open set Ω in $\mathbb{H}$. We define the *signed distance* from S as follows:

$$\delta_S(P) = \begin{cases} -d_S(P) & \text{if } P \in \Omega \\ d_S(P) & \text{if } P \notin \Omega. \end{cases} \tag{1}$$

Before stating our main result, we recall some standard notation and terminology.

A left invariant vector field V on $\mathbb{H}$ is *horizontal* if it is the linear combination of X and Y,

$$X = \partial_x + 2y\partial_t, \; Y = \partial_y - 2x\partial_t.$$

As usual, we identify a vector V at P with the unique left invariant vector field having value V at P. The linear space of the horizontal vector fields is denoted by $\mathcal{H}$, while $\mathcal{H}_P$ is the linear space of the horizontal vectors in $T_P\mathbb{H}$, the space tangent to $\mathbb{H}$ at P. If S is a surface in $\mathbb{H}$ and it is C^1 in the Euclidean sense, a point C in S is *characteristic* if $T_CS = \mathcal{H}_C$. We denote by $Char(S)$ the set of all characteristic points of S. If f is a real valued function defined on an open subset of $\mathbb{H}$, its *horizontal gradient* is $\nabla_{\mathbb{H}} f = (Xf)X + (Yf)Y$.

The aim of this paper is proving the following regularity theorem for δ_S.

Theorem 1.1 *Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$.*

(i) *If S is $C^{1,1}$ in the Euclidean sense, then $\nabla_{\mathbb{H}}\delta_S$ is a continuous function in an open neighborhood of $S - Char(S)$ in $\mathbb{H}$.*
(ii) *If S is C^k in the Euclidean sense, $k \geq 2$, then $\nabla_{\mathbb{H}}\delta_S$ and δ_S are of class C^{k-1}, in the Euclidean sense, in an open neighborhood of $S - Char(S)$ in $\mathbb{H}$.*

Theorem 1.1 extends to the Heisenberg group results proved in Euclidean space by Federer [6], in the case of the nonsigned distance, and by Krantz and Parks [10], in the case of the signed distance. The advantage of the signed distance over the nonsigned one is that the former can be regular even on the surface S. See [5] for an updated survey of the theory in the Euclidean space.

In the Heisenberg context, Monti and Serra-Cassano [13] proved that, if S is a closed subset of $\mathbb{H}$, then d_S satisfies the Eikonal equation *a.e.* :

$$|\nabla_{\mathbb{H}} d_S| = 1.$$

The proofs of the regularity results in Euclidean space rely on the metric properties of the vectors normal to the given surface. If the surface S is $C^{1,1}$, then for each point Q in S there exists a straight line n passing through Q which is normal to S. Moreover, there is an open neighborhood A of Q such that the distance between a point P in $n \cap A$ and S is realized by the distance between P and Q. The direction of n is that of the vector normal to S at Q. These considerations extend to the case of a hypersurface in a Riemannian manifold M, since the map $\gamma \mapsto \dot{\gamma}(0)$ establishes a 1-to-1 correspondence between geodesics γ such that $\gamma(0) = Q$ is fixed in M and unit vectors tangent to M at Q.

This fact has no clear analogue in sub-Riemannian geometry. The Carnot-Charathéodory distance between two points in $\mathbb{H}$ is realized by the *length* of a *geodesic.* Contrary to the Riemannian case, however, given a *horizontal vector* V at $Q \in \mathbb{H}$, there are infinitely many geodesics leaving Q and having tangent vector V at Q. To overcome this difficulty, we define the notion of *metric normal* to a surface in $\mathbb{H}$.

Let Q be a point in S. The *metric normal* to S at Q, denoted by $\mathcal{N}_Q S$, is the set of the points P such that

$$d_S(P) = d(P, Q).$$

The theorem below gives a geometric description of $\mathcal{N}_Q S$.

Theorem 1.2 *Let S be a $C^{1,1}$ Euclidean surface. Let Q be a non-characteristic point on S and let $\Pi_Q S$ be the Euclidean plane in $\mathbb{H}$ which is tangent to S at Q, in the Euclidean sense. Then, $\mathcal{N}_Q S$ is a nontrivial arc of the unique geodesic γ passing through Q and such that*

(i) $\dot{\gamma}(0) = N_P S$ *is the horizontal vector normal to S at Q*;
(ii) *the maximal length $l > 0$ over which γ is length minimizing in $\mathbb{H}$ is $l = \pi \cdot d(P, C)$, where C is the characteristic point of $\Pi_Q S$.*

If $\Pi_Q S$ does not have a characteristic point, (ii) holds with $l = \infty$. See Sect. 2 for a description of the geodesics in $\mathbb{H}$ and of their properties.

In this paper we consider the Heisenberg group with the lowest dimension. This object is interesting in itself. For instance, it is related with the study of the isoperimetric inequality in the Euclidean plane [11]. In [2] we use the results of this paper to study the *horizontal Hessian* of the distance function from a surface and some properties of the *mean curvature*.

This is the way in which the paper is structured. In Sect. 2 we give some preliminaries on the Heisenberg group. Section 3 and 4 are devoted to the determination of the metric normal for a Euclidean plane in $\mathbb{H}$, hence for a smooth surface. In Sect. 5 we prove some properties of the sets of *positive reach* in $\mathbb{H}$. Section 6 contains statements and proofs of various regularity results for the distance function. Finally, some properties of the cutlocus of a surface in $\mathbb{H}$ are discussed in Sect. 7.

2 Notation and preliminaries

In this section, we collect some basic definitions and known facts about the structure and the geometry of $\mathbb{H}$. There is a vast literature on sub-Riemannian geometry and Carnot groups. Just to quote a few titles, we refer the reader to [4], [7], [8], [11], [14], [15], [16], [18].

Left translation by $P \in \mathbb{H}$ is denoted by L_P, $L_P Q = P \cdot Q$. The left invariant vector fields X, Y do not commute, $[X, Y] = -4\partial_t$. The span of the vector fields X and Y is called *horizontal distribution*, and it is denoted by $\mathcal{H}$. The *fiber* of $\mathcal{H}$ at a point P of $\mathbb{H}$ is $\mathcal{H}_P = \text{span}\{X_P, Y_P\}$. The *Carnot-Charathéodory metric* in $\mathbb{H}$ is the sub-Riemannian metric that makes the left invariant vector fields X and Y pointwise orthonormal. The inner product in $\mathcal{H}_P$ is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $| \cdot |$.

An important element of $\mathbb{H}$'s structure is the *dilation group at the origin* $\{\delta_\lambda : \lambda \neq 0\}$,

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 t), \quad z = (x, y).$$

By left translation, a dilation group is defined at each point P of $\mathbb{H}$.

The Heisenberg group is also endowed with a *rotation group*, which is useful in simplifying some calculations. For $\theta \in \mathbb{R}$, let $\mathcal{R}_\theta$ be the anti-clockwise rotation by an angle of amplitude θ around the origin in $\mathbb{R}^2$. Let

$$R_\theta(z,t) = (\mathcal{R}_\theta z, t)$$

be the rotation by θ around the t-axis. Composing with left translation, one can define rotations around any vertical line $\{(x,y,t) : (x,y) = (a,b)\}$. The map R_θ is an isometry of $\mathbb{H}$ and its differential acts on the fiber $\mathcal{H}_O$ as a rotation by θ. Under the usual identification between the Riemannian tangent space of $\mathbb{H}$ at O, $T_O\mathbb{H}$, and the Lie algebra $\mathfrak{h}$ of $\mathbb{H}$, the differential of R_θ can be thought of as a rotation on span$\{X, Y\}$, the first stratum of $\mathfrak{h}$. With respect to the basis $\{X, Y\}$,

$$dR_\theta V = \mathcal{R}_\theta V$$

whenever $V \in \text{span}\{X, Y\}$. With some abuse of notation, we denote dR_θ by R_θ.

The distance between two points P and Q in $\mathbb{H}$ is defined as follows. Consider an absolutely continuous curve γ in $\mathbb{R}^3$ joining P and Q, which is *horizontal*. That is, $\dot\gamma(t) = a(t)X_{\gamma(t)} + b(t)Y_{\gamma(t)}$ lies in $\mathcal{H}_{\gamma(t)}$, *a.e. t*. Its Carnot-Charathéodory length is $l_{\mathbb{H}}(\gamma)$,

$$l_{\mathbb{H}}(\gamma) = \int (a(t)^2 + b(t)^2)^{1/2} \mathrm{d}t$$

The *Carnot-Charathéodory distance* between P and Q, $d(P,Q)$, is the infimum of the Carnot-Charathéodory lengths of such curves. The infimum is actually a minimum, the distance between P and Q being realized by the length of a geodesic. By translation invariance, all geodesics are left translations of geodesics passing through the origin. The unit-speed geodesics at the origin [12], [11] are

$$\gamma_{O,\phi,W}(\sigma) = \begin{cases} x(\sigma) = \sin(\alpha(W))\frac{1-\cos(\phi\sigma)}{\phi} + \cos(\alpha(W))\frac{\sin(\phi\sigma)}{\phi} \\ y(\sigma) = \sin(\alpha(W))\frac{\sin(\phi\sigma)}{\phi} - \cos(\alpha(W))\frac{1-\cos(\phi\sigma)}{\phi} \\ t(\sigma) = 2\frac{\phi\sigma - \sin(\phi\sigma)}{\phi^2}. \end{cases} \tag{2}$$

Here, W is a unitary vector in $\mathcal{H}_O$ and $\alpha(W) \in [0, 2\pi)$ is unique with the property $\dot\gamma_{O,\phi,W}(0) = W$. The parameter ϕ lies in $\mathbb{R}$ and the geodesic is length minimizing over any interval of length $2\pi/|\phi|$. In the case $\phi = 0$, the geodesic is a straight line in the plane $\{t = 0\}$,

$$x(\sigma) = \cos(\alpha(W))\sigma, \quad y(\sigma) = \sin(\alpha(W))\sigma,$$

and we say that the geodesic is *straight*.

From these equations we deduce the parametric equations of the boundary of the ball $B(0,r)$: $(z,t) \in \partial B(0,r)$ if and only if there is $\phi \in [-2\pi/r, 2\pi/r]$ so that

$$\begin{cases} |z| = 2\frac{\sin(\phi r/2)}{\phi} \\ t = 2\frac{\phi r - \sin(\phi r)}{\phi^2}. \end{cases} \tag{3}$$

If $P = (z,t)$ and $z \neq 0$, then there exists a unique length minimizing geodesic connecting P and O. If $P = (0,t)$, $t \neq 0$, (i.e., if P belongs to the center of $\mathbb{H}$) then there is a one parameter family of length minimizing geodesics joining P and O. They are obtained by rotation around the t-axis of a single geodesic.

Given points $P = (z, t)$ and $P' = (z', t')$, they are joined by a unique length minimizing geodesic, unless $z = z'$.

Let

$$\gamma_{P,\phi,\alpha} = L_P \gamma_{O,\phi,\alpha}$$

The parameter ϕ is geometric in the following sense: $2\pi/|\phi|$ is the length of the intervals over which $\gamma_{P,\phi,W}$ is length minimizing and $sgn(\phi)$ is positive if and only if the t-coordinate increases with σ. Recall that in $\mathbb{H}$ the orientation of the t-axis is an intrinsic notion, unlike in Euclidean space. If $\gamma_{P,\phi,W}$ and $\gamma_{P',\phi',W'}$ have an arc in common, then $\phi = \pm\phi'$. No such easy relation exists for the parameter W. To change the orientation of a geodesic, observe that

$$\gamma_{P,\phi,W}(\sigma) = \gamma_{P,-\phi,-W}(-\sigma).$$

Unlike the Euclidean case, a geodesic γ leaving O is not determined by its tangent vector at the origin, $\dot{\gamma}(0) = W = \cos(\alpha(W))X_O + \sin(\alpha(W))Y_O$. The extra parameter we need is ϕ. Notice that ϕ is related to the dilation group as follows: $\delta_\lambda\left(\gamma_{O,\phi,W}\right)$ is a reparametrization of the geodesic $\gamma_{O,\phi/\lambda,W}$. That is, all geodesics γ leaving O and having fixed initial velocity $\dot{\gamma}(0) = v$ in $\mathcal{H}_O$ are dilated of each other (with the exception corresponding to $\phi = 0$). Hence, contrary to the Euclidean case, a geodesic is not invariant under the group of dilations. The case of the straight geodesics is the limiting one, corresponding to $\lambda \to 0$. In a precise sense, then, the set of non-straight geodesics at O is parametrized by the unit circle in $\mathcal{H}_O$ and by the dilation group, a feature of $\mathbb{H}$ with no Euclidean counterpart.

Let S be a surface in $\mathbb{H}$ which is C^1 in the Euclidean sense and such that, for some open Ω in $\mathbb{H}$, $S = \partial\Omega = \partial(\mathbb{H} - \overline{\Omega})$. We need some differential geometric notions about S.

Definition 2.1 *Let S be a surface in $\mathbb{H}$ as above and let $P \in S$ be a non-characteristic point. The Euclidean tangent space to S at P is denoted by T_PS. The* **direction tangent to** S **at** P *is the 1-dimensional space $V_PS = T_PS \cap \mathcal{H}_P$. The* **plane tangent to** S **at** P*, Π_PS, is the Euclidean plane in $\mathbb{H}$, tangent to S at P in the Euclidean sense. The* **direction normal to** S *at P is $N_PS = \mathcal{H}_P \ominus V_PS$.*

Let ν be the Euclidean exterior normal to S at P and let $\langle\cdot,\cdot\rangle$ be the Euclidean inner product. The **Pansu exterior normal to** S *at P, denoted by N_P^+S, is the unique horizontal vector V in the direction N_PS such that $\langle V, \nu\rangle > 0$.*

The *group tangent to S at P* [15] is the 2-dimensional vector space $G_PS = \mathcal{V}_PS \oplus \mathcal{T}$, where $\mathcal{T} = \{(0,0,t) : t \in \mathbb{R}\}$ is the center of $\mathbb{H}$ and $\mathcal{V}_PS$ is the one parameter subgroup of $\mathbb{H}$ generated by V_PS. One point we want to make in the present paper is that G_PS does not capture the complexity of the geodesics' set, while T_PS does, in a sense that will be made precise in the following two sections.

The following facts are easily established by direct calculation.

Proposition 2.1 *Let S be a smooth surface in $\mathbb{H}$, implicitly defined by $g(x, y, t) = 0$. Let $P \in S$ be non-characteristic. Then,*

$$V_PS = \text{span}\{Yg \cdot X - Xg \cdot Y\},\ N_PS = \text{span}\{Xg \cdot X + Yg \cdot Y\} = \text{span}\{\nabla_{\mathbb{H}} g\} \quad (4)$$

3 The metric normal to a plane and a to sphere

In this section, we compute the metric normal to a plane and to a sphere.

Definition 3.1 *Let E be a closed subset of $\mathbb{H}$, $P \in E$. The* **metric normal** *to E at P is the set $\mathcal{N}_P S$ of the points $Q \in \mathbb{H}$ such that $d(Q,E) = d(Q,P)$.*

See also [3], where a pathological occurrence of the metric normal to the unit sphere was used to study bi-Lipschitz functions in $\mathbb{H}$.

Lemma 3.1 *Let E be a closed subset of $\mathbb{H}$, $P \in E$. Let Q in $\mathcal{N}_P S$ and $\gamma : I \to \mathbb{H}$ be a length minimizing geodesic from Q to P. Then*

$$\gamma(I) \subseteq \mathcal{N}_P S.$$

Proof Let A be any point in γ, then $d_S(A) \le d(A,P)$. If there were P' in S such that $d(A,P') < d(A,P)$, then by the triangle inequality

$$d(Q,P') \le d(Q,A) + d(P',A) < d(Q,A) + d(P,A) = d(Q,P),$$

contradicting $Q \in \mathcal{N}_P S$. □

Theorem 3.1 *Let $\mathcal{P}$ be a plane in $\mathbb{H}$ and let $P \in \mathcal{P}$ be non-characteristic. Suppose that $\mathcal{P}$ has a characteristic point C and let Ω be one of the half-spaces having $\mathcal{P}$ as boundary. Then,*

$$\mathcal{N}_P\mathcal{P} = \gamma_{P,2/d(P,C),N_P^+\mathcal{P}}\left(\left[-\frac{\pi}{2}d(P,C), \frac{\pi}{2}d(P,C)\right]\right)$$

If $\mathcal{P}$ is a vertical plane, then $\mathcal{N}_P\mathcal{P}$ is the straight geodesic through P, in the direction $N_P\mathcal{P}$.

This result can be stated in terms of the Euclidean orthogonal projection $(z,t) \mapsto z$ from $\mathbb{H}$ onto the plane $\{t = 0\}$. The projection onto the $t = 0$ plane of γ is the circle c having as diameter the line joining C_1, the projection of C, and P_1, the projection of P.

Proof of the Proposition 3.1 The case when $\mathcal{P}$ is a vertical plane is elementary. Consider the case when $\mathcal{P}$ has a characteristic point.

In the statement of the proposition, everything is invariant under isometries. We can then left-translate everything so that $\mathcal{P}$ is the plane $t = 0$ and $C = O$. Possibly after applying the symmetry $(z,t) \mapsto (\bar{z}, -t)$, we can assume that Ω is the unique half-space having boundary $\mathcal{P}$ and containing a half line on the negative t-axis. Here, we have considered $z = x + iy$ as a complex number. Let $P \ne O$ be a point in $\mathcal{P}$. There exists a point $(0,0,T)$ on the vertical axis, $T > 0$, so that a length-minimizing geodesic from $(0,0,-T)$ to $(0,0,T)$, say γ, intersects $\mathcal{P}$ at P. The existence of T with the stated properties is guaranteed by a simple continuity argument. By the equation of geodesics we deduce $T = \frac{\pi}{2}d(O,P)^2$. $\gamma = \gamma_{P,\phi,W}$ is uniquely determined. We need to show that the parameters of γ are those in the statement of the theorem, and that γ is a parametrization of the metric normal.

Step 1 γ has the right parameter ϕ. Note that γ points upward. Hence, by the geodesic equation, we have $\phi \ge 0$. By symmetry with respect to $t = 0$, $d(P,(0,0,T)) = \pi/\phi$, one half of γ's total length. Inserting this in the equation for γ, and assuming after a rotation that P lies on the positive x-axis, we obtain that $P = (2/\phi, 0, 0)$. This shows that $d(P,0) = 2/\phi$.

Step 2 γ lies in the metric normal. We show that $(0,0,-T)$ belongs to $\mathcal{N}_P S$. Consider the ball $B = B((0,0,-T), d(P,(0,0,-T)))$. By maximizing the t coordinate in (3), we see that $B \cap \{(z,t) : t > 0\}$ is empty and that $\overline{B} \cap \{(z,t) : t > 0\}$ is the circle $c = \mathcal{P} \cap B(O, d(O,P))$ on $\mathcal{P}$, having radius $d(P,O)$. If $Q \in c$, then $d(Q,(0,0,-T))$ by rotational symmetry. If $Q \in \mathcal{P} - c$, then $d(Q,(0,0,-T)) > d(P,(0,0,-T))$.

Since $(0,0,-T)$ belongs to $\mathcal{N}_P S$, by Lemma 3.1 the lower half of γ's trace belongs to $\mathcal{N}_P S$. By symmetry, the upper half does, too.

Step 3 γ has the right parameter W. Let $c = c(s)$ be the projection of γ on $\mathcal{P}$. Then, $c(0) = c(2\pi/\phi) = 0$, $c(\pi/\phi) = P$ and c is a circle having Euclidean diameter OP. As s increases, $c(s)$ runs the circle clockwise. As to the tangent vectors, $\dot{c}(s)$ is the projection on the plane $\mathcal{P} = \{t = 0\}$ of the horizontal vector $\dot{\gamma}(s)$. After a rotation, we can suppose that $P = (2/\phi, 0, 0)$ lies on the positive x-axis. Then, $\dot{c}(\pi/\phi) = (0,-1)$, hence $\dot{\gamma}(\pi/\phi) = -Y(P)$. We only have to show that $-Y(P) = N_P^+\mathcal{P}$. Now, the space tangent to $\mathcal{P}$ at P contains just one horizontal direction (P is non-characteristic): that of the x-axis, X. Then, $N_P\mathcal{P} = \operatorname{span}\{Y(P)\}$, hence, since $N_P^+\mathcal{P}$ has to point upward, $N_P^+\mathcal{P} = -Y(P) = -\partial_y + \frac{4}{\phi}\partial_t$, as wished.

Step 4 γ contains the metric normal. Suppose now that Q belongs to both $\mathcal{N}_P\mathcal{P}$ and $\mathcal{N}_{P'}\mathcal{P}$. We show that, then, Q lies on the axis $x = y = 0$ and that P and P' lie on a Euclidean circle centered at O in $\mathcal{P}$.

After a group translation, the plane $L_{Q^{-1}}\mathcal{P}$ touches a closed Carnot Charathéodory ball B centered at the origin in $Q^{-1}P$ and $Q^{-1}P'$. This may happen only if $Q^{-1}P$ and $Q^{-1}P'$ both lie on the circle c of the points in B having highest (or lowest) t-coordinate, and $c = B \cap L_{Q^{-1}}\mathcal{P}$. Hence, the characteristic point of $\mathcal{P}$ lies on the same vertical line as the center of B, and this property is invariant under left translations. As a consequence, Q lies on $x = y = 0$ and P, P' are taken one into the other by a rotation around $x = y = 0$.

For each point P in $\mathcal{P} - \{O\}$, consider the geodesic arc

$$\beta_P = \gamma_{P,2/d(P,C),N_P^+\mathcal{P}}\left(\left[-\frac{\pi}{2}d(P,C), \frac{\pi}{2}d(P,C)\right]\right).$$

By steps 1-3, β_P lies in $\mathcal{N}_P\mathcal{P}$. As P varies, the union of the sets β_P fills up $\mathbb{H} - \{O\}$.

Suppose now that Q belongs to $\mathcal{N}_P\mathcal{P} - \beta_P$. Clearly, $Q \neq O$, and Q belongs to some $\beta_{P'} \subset \mathcal{N}_{P'}\mathcal{P}$. By the considerations above, then, Q lies on $x = y = 0$ and a rotation around $x = y = 0$ takes P' into P. The same rotation brings $\beta_{P'}$ onto β_P and fixes Q, hence $Q \in \beta_P$, a contradiction. □

By Theorem 3.1 and (2) we have an explicit expression for the metric normal to a plane.

Proposition 3.1 *The metric normal to $\Pi = \{t = 0\}$ at $P = (z,t)$, $z = (x,y)$, is the support of the geodesic arc*

$$\gamma_P(\sigma) = (u(\sigma),\ v(\sigma),\ s(\sigma)) = \begin{cases} \frac{x}{2}\left(1+\cos\left(\frac{2\sigma}{|z|}\right)\right) + \frac{y}{2}\sin\left(\frac{2\sigma}{|z|}\right) \\ \frac{y}{2}\left(1+\cos\left(\frac{2\sigma}{|z|}\right)\right) - \frac{x}{2}\sin\left(\frac{2\sigma}{|z|}\right) \\ \frac{|z|^2}{2}\left(\frac{2\sigma}{|z|} + \sin\left(\frac{2\sigma}{|z|}\right)\right) \end{cases}, \ |\sigma| \le \frac{\pi}{2}|z| \quad (5)$$

Let $w(\sigma) = (u(\sigma), v(\sigma))$. *Then, by (5),*

$$\begin{cases} |w| = |z| \cos\left(\frac{\sigma}{|z|}\right) \\ s = |z|^2 \left(\frac{\sigma}{|z|} + \frac{1}{2}\sin\left(\frac{2\sigma}{|z|}\right)\right). \end{cases} \tag{6}$$

Notice that for σ fixed, (6) gives a parametrization of the set of points having distance σ from Π.

The next lemma helps with calculations.

Remark 3.1 If p is the plane $t = ax + by + c$, its characteristic point is

$$C = (-b/2, a/2, c)$$

Proof The tangent space of p at (x, y, t) is spanned by $(1, 0, a)$ and $(0, 1, b)$. They are both horizontal iff $a = 2y$ and $b = -2x$. □

We also need this fact.

Lemma 3.2 *Let* $B = B(A, r)$ *be a ball in* $\mathbb{H}$, $Q \in \partial B(A, r)$ *and* γ *be a geodesic from* A *to* Q. *Then, if* A' *is a point of* γ, $B(A', r - d(A, A'))$ *is contained in* $B(A, r)$ *and* Q *belongs to* $\partial B(A', r - d(A, A'))$.

Moreover, if $A' \neq A$, *then* $\partial B(A, r) \cap \partial B(A', r - d(A, A')) = \{Q\}$.

Proof Let R be a point in $B(A', r - d(A, A'))$. Then,

$$d(R, A) \leq d(R, A') + d(A', A) = r.$$

Suppose, now that $A \neq A'$ and that $Q \neq Q' \in \partial B(A', r - d(A, A'))$. Since $Q' \notin \gamma$, we have $d(A, Q') < d(A, A') + d(A', Q') = d(A, Q)$. This contradicts $Q' \in \partial B(A, r)$, proving the last statement of the lemma. □

Lemma 3.3 *Let* $r > 0$, $\phi \in \mathbb{R}$, $|\ \phi r\ | < \pi$ *and* $Q \in \mathbb{H}$. *Suppose* γ *is a geodesic of parameter* ϕ *and suppose it has endpoints* Q *and* P. *If* $P \in \partial B(Q, r)$, *and* $B = B(Q, r)$, *then*

$$\Pi_P(\partial B) \cap \bar{B} = \{P\}.$$

Proof Consider the extension of γ from Q to Q', which we still call γ, where $d(P, Q') = \frac{\pi}{|\phi|} \geq r$. We can assume that $Q' = O$ is the origin and that the geodesic γ points upward while moving from $Q' = O$ to P. Consider the ball $B' = B(Q', P)$. Then, P is a point of maximal t-coordinate on B'. Hence, $\Pi_P B'$ is a plane parallel to $t = 0$. More precisely, $\Pi_P B'$ is the plane having equation $t = \frac{2R^2}{\pi}$. The plane $\Pi_P B'$ intersects $\partial B'$ in a Euclidean circle. By Lemma 3.2, $B \subset B'$ and P lies on the boundary of both B and B'. Hence, $\Pi_P B = \Pi_P B'$ and we obtain the thesis, again by Lemma 3.2. □

Proposition 3.2 *Let* $B = B(Q, r)$ *be a ball and let* P *be a point where its boundary is smooth. Let* $\gamma_P : I \to \mathbb{H}$ *be the maximal length minimizing geodesic starting at* Q *and passing through* P. *Then*

(i) $\mathcal{N}_P(\partial B) = \gamma_P(I)$;
(ii) *an open arc of* γ_P *containing* P *is contained in* $\mathcal{N}_P(\partial B) \cap \mathcal{N}_P(\Pi_P(\partial B))$.

Proof We prove (i) first. Let Q' be the other endpoint of γ_P. By the triangle inequality, $d(Q',P) = d_{\partial B}(Q')$. Hence, by Lemma 3.1, the arc of γ_P between P and Q' is contained in $\mathcal{N}_P(\partial B)$. Clearly, the arc of γ_P between P and Q is contained in $\mathcal{N}_P(\partial B)$, too. Hence $\gamma_P \subseteq \mathcal{N}_P(\partial B)$.

Suppose now that $A \in \mathcal{N}_P\partial B$ and $A \notin \gamma_P$. We assume, first, that $A \notin B$. Let η be a length minimizing geodesic between A and Q and let P' be its intersection with ∂B. Thus $d(A,P') \geq d_{\partial B}(A) = d(A,P)$ and $d(Q,P') = r = d(Q,P')$. Hence, $d(A,Q) \leq d(Q,P) + d(P,A) \leq d(Q,P') + d(P',A) = d(A,Q)$. This inequality is possible only if P belongs to a length minimizing geodesic between A and Q which extends the arc of γ_P between Q and P. This means that A belongs to a length minimizing extension of the geodesic between Q and P, but the maximal such extension is γ_P.

If $A \in B$, a similar argument shows that $A \in \gamma_P$. Details are left to the reader.

We now show (ii). By lemma 3.3, there exists a point Q' on γ_P such that the closure of $B' = B(Q', d(Q',P))$ meets $\Pi_P(\partial B') = \Pi_P(\partial B)$ in P only. Thus, by definition of metric normal, the arc of γ_P between P and Q' lies in $\mathcal{N}_P(\Pi_P(\partial B))$. By (i), on the other hand, γ_P is contained in $\mathcal{N}_P(\partial B)$.

A similar argument shows that there exists an arc of γ_P external to B, which lies in $\mathcal{N}_P(\partial B) \cap \mathcal{N}_P(\Pi_P(\partial B))$. □

Remark 3.2 Lemma 3.3 implies that part of ∂B is convex in the Euclidean sense. In other words that both principal curvatures are non-negative.

4 The metric normal to a smooth surface

In this section, we discuss the metric normal to smooth surfaces. We consider surfaces satisfying an inner-and-outer ball condition. Many of the results extend, with obvious modifications, to surfaces just satisfying an inner ball condition. When no ambiguity is possible, we shall identify a curve $\gamma : I \to \mathbb{H}$ with its trace $\gamma(I)$.

Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$. We say that S satisfies condition (TB) at P if there are open balls $B_1 \subseteq \Omega$ and $B_2 \subseteq \mathbb{H} - \overline{\Omega}$ containing P on their boundary.

Theorem 4.1 *Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$.*

(i) *If P belongs to S and S has tangent plane $\Pi_P S$ at P, then $\mathcal{N}_P S$ is an arc (eventually degenerate) of a geodesic. Moreover, if $\mathcal{N}_P S$ is nontrivial, then its intersection with $\mathcal{N}_P \Pi_P S$ is a nontrivial geodesic arc containing P.*
(ii) *Suppose, further, that S satisfies (TB) at P. Then S has a tangent plane at P and $\mathcal{N}_P S$ is a nontrivial geodesic arc having endpoints in Ω and $\mathbb{H} - \overline{\Omega}$, respectively.*

An arc on a curve is degenerate if it reduces to a point. We say that S has tangent plane $\Pi_P S$ at P if $d_{Euc}(Q, \Pi_P S) = o(|Q - P|)$ as $Q \to P$ in S. Observe that, by (i), if $\mathcal{N}_P S$ is nontrivial, then it can be parametrized by any equation parametrizing $\mathcal{N}_P \Pi_P S$.

Proof (i) If $\mathcal{N}_P S$ reduces to P alone, there is nothing to prove. Otherwise, let $P \neq Q \in \mathcal{N}_P S$, $Q \in \Omega$ and let γ be any geodesic from P to Q (actually, there is only one such geodesic).

Then, $B(Q, d(P,Q)) \subseteq \Omega$ and $P \in \partial B(Q, d(P,Q))$. Now, either $\partial B(Q, d(P,Q))$ is smooth at P or P is one of the "poles" of $\partial B(Q, d(P,Q))$. In the first case γ

lies in $\mathcal{N}_P\partial B(Q,d(P,Q))$. Hence, by Proposition 3.2, a non trivial arc of γ lies in $\mathcal{N}_P\Pi_P(\partial B(Q,d(P,Q))) = \mathcal{N}_P\Pi_P S$, since $\Pi_P(\partial B(Q,d(P,Q))) = \Pi_P S$. The curve γ has then the same equation of $\mathcal{N}_P\Pi_P$.

If P is, say, the North Pole of $B(Q,d(P,Q))$, then S, being external to $B(Q,d(P,Q))$, can not have tangent plane at P. The reason is that there is a proper Euclidean cone external to $B(Q,d(P,Q))$, which is tangent to the boundary of $B(Q,d(P,Q))$ at P.

Up to this point, we have proved that $\gamma_1 = \mathcal{N}_P S \cap \overline{\Omega}$ and $\gamma_2 = \mathcal{N}_P S \cap (\mathbb{H} - \Omega)$ are geodesic arcs and that any of them, if nontrivial, shares a nontrivial arc with $\mathcal{N}_P\Pi_P S$. We now show that $\mathcal{N}_P S = \gamma_1 \cup \gamma_2$ is a length minimizing geodesic. Let Q_j the endpoint of γ_j other than P and let l_j be the (intrinsic) length of γ_j. Then, the balls $B_j = B(Q_j, l_j)$ do not intersect and P belongs to the boundary of both. Let η be any horizontal curve between Q_1 and Q_2. Since η has to meet the boundaries of B_1 and B_2, we have that $l_{\mathbb{H}}(\eta) \geq l_{\mathbb{H}}(\gamma_1) + l_{\mathbb{H}}(\gamma_2)$. Hence, $d(Q_1,Q_2) = l_{\mathbb{H}}(\gamma_1) + l_{\mathbb{H}}(\gamma_2)$. This shoes that $\gamma_1 \cup \gamma_2$ is a (length minimizing) geodesic arc.

(ii) Since B_1 and B_2 have empty intersection, the point P can not be a pole of either B_1 or B_2. Being ∂B_1 and ∂B_2 smooth at P in the Euclidean sense, ∂B_1, ∂B_2 and S have the same tangent plane $\Pi = \Pi_P S$ at P. The centers Q_1 and Q_2 of B_1 and B_2, respectively, belong to $\mathcal{N}_P S$, by definition. Hence, by Proposition 3.1, the geodesic γ_j between Q_j and P is contained in $\mathcal{N}_P S$, $j = 1,2$. Thus $\mathcal{N}_P S$, which is a geodesic arc by (i), is non degenerate and it does not have P as endpoint. □

Corollary 4.1 *Let S be a surface in $\mathbb{H}$, which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$. Assume that C is a characteristic point of S. If S has tangent plane at C, then S can not satisfy (TB) at C.*

Proof If (TB) held at C, then $\mathcal{N}_C S$ would be nontrivial, hence $\mathcal{N}_C\Pi_C S$ would nontrivial. But C is characteristic for $\Pi_C S$, hence $\mathcal{N}_C\Pi_C S$ is degenerate, by Theorem 3.1. □

Definition 4.1 *Let S be a C^1 surface in the Euclidean sense in $\mathbb{H}$, which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$, and let $P \in S$. The* **oriented metric normal to** S **at** P, *$\mathcal{N}_P^+ S$, is the unique parametrization of $\mathcal{N}_P S$ such that $\delta_S(\mathcal{N}_P^+ S(\sigma), P) = \sigma$.*

This means that $\mathcal{N}_P^+ S(\sigma) \in \Omega$ for $\sigma < 0$ and $\mathcal{N}_P^+ S(\sigma) \in \mathbb{H} - \overline{\Omega}$ for $\sigma > 0$. In particular, if $\mathcal{N}_P^+ S$ is nontrivial, then

$$\dot{\mathcal{N}}_P^+ S(0) = N_P^+ S.$$

Let S be a differentiable surface (in the Euclidean sense) in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$. Suppose that, locally, S has equation $g = 0$, where $g : \mathbb{H} \to \mathbb{R}$ is differentiable in the Euclidean sense and $\nabla g \neq 0$ pointwise on S. Let P be a point in S. Then P is characteristic for S if and only if $\nabla_{\mathbb{H}} g(P) = 0$. Let P be noncharacteristic for S and let C be the characteristic point of $\Pi_P S$. (If $\Pi_P S$ has no characteristic point, i.e., if it has equation $ax + by + c = 0$, we say that $\Pi_P S$ *has characteristic point at infinity*). The equation of $\Pi_P S$ is

$$\Pi_P S = \{(x,y,t) : \partial_x g(P)(x - x(P)) + \partial_y g(P)(y - y(P)) + \partial_t g(P)(t - t(P)) = 0\}.$$

The coordinates of C are:

$$C = \left(\frac{g_y(P)}{2g_t(P)}, -\frac{g_x(P)}{2g_t(P)}, t(P) + x(P)\frac{g_y(P)}{g_t(P)} + y(P)\frac{g_x(P)}{g_t(P)} \right). \tag{7}$$

The Heisenberg distance between P and C has a simple expression:

$$d(P,C) = 2\frac{|\nabla_{\mathbb{H}} g|}{|[X,Y]g|} = \frac{|\nabla_{\mathbb{H}} g|}{2|\partial_t g|}. \tag{8}$$

If $\Omega = \{g < 0\}$ in a neighborhood of P, then the unit normal vector in the Pansu sense which points outside S, has equation

$$N_P^+ S = \frac{\nabla_{\mathbb{H}} g(P)}{|\nabla_{\mathbb{H}} g(P)|}.$$

By Theorems 4.1 and 3.1, the equation of $\mathcal{N}_P^+ S$ can be written in terms of g's partial derivatives: $\mathcal{N}_P^+ S = P \cdot \eta$ (left translation by P), where $\eta = (u, v, s)$ and

$$\eta(\sigma) = \begin{cases} u(\sigma) = \frac{1}{4\partial_t g}\left\{Yg(P)\left(1 - \cos\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right)\right) + Xg(P)\sin\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right)\right\} \\ v(\sigma) = \frac{1}{4\partial_t g}\left\{-Xg(P)\left(1 - \cos\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right)\right) + Yg(P)\sin\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right)\right\} \\ s(\sigma) = \frac{|\nabla_{\mathbb{H}} g(P)|^2}{8(\partial_t g(P))^2}\left\{\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|} - \sin\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right)\right\} \end{cases} \tag{9}$$

Observe that $\mathcal{N}_P^+ S$ points upwards if $\partial_t g(P) > 0$ and it points downwards if $\partial_t g(P) < 0$. If $\partial_t g(P) = 0$, then $\Pi_P S$ has characteristic point C at infinity. Hence, $d(P,C) = \infty$ and (9) becomes

$$\eta(\sigma) = \left(\frac{Xg(P)}{|\nabla_{\mathbb{H}} g(P)|}\sigma, \frac{Yg(P)}{|\nabla_{\mathbb{H}} g(P)|}\sigma, 0\right) \tag{10}$$

Equations (9) and (10) show that, for some smooth function Φ,

$$\mathcal{N}_P^+ S(\sigma) = \Phi(\sigma, P, \nabla_{\mathbb{H}} g(P), [X,Y]g(P)).$$

In order to write $\mathcal{N}_P^+ S$, the knowledge of $\nabla_{\mathbb{H}} g$ alone is not sufficient. On the other hand, we do not need *all* of the horizontal second order derivatives, but just $[X,Y]$. The continuity of $\nabla_{\mathbb{H}} g$ and $[X,Y]g$ is equivalent to the requirement that g is C^1 in the Euclidean sense.

We think that this is a sufficient motivation for our choice of considering surfaces which are C^1 in the Euclidean sense, at least outside their characteristic set. C^1 regularity should be henceforth considered as an intrinsic requirement which is intermediate between $C^1_{\mathbb{H}}$ and $C^2_{\mathbb{H}}$.

5 Sets of positive reach

In this section we investigate the notion of *set of positive reach* in the Heisenberg group. This notion was used by Federer [6] to prove several regularity results for the distance function in $\mathbb{R}^n$.

Definition 5.1 *Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$ and let S_+ an open subset of S. We say that S satisfies condition (UTB) (locally uniformly tangent ball) on S_+ if, for each point $P_0 \in S_+$, we can find $r > 0$ and $h > 0$ such that, for all P in $B(P_0, r)$, condition (TB) holds at P with balls B_1 and B_2 of radius h.*

Theorem 5.1 *Let S be a surface in* $\mathbb{H}$, $S = \partial\Omega$ *and* $S = \partial(\mathbb{H} - \overline{\Omega})$, *where* Ω *is open in* $\mathbb{H}$. *If S is* $C^{1,1}$ *in the Euclidean sense and if Char(S) is the set of S's characteristic points, then S satisfies (UTB) on* $S - Char(S)$.

Proof Federer showed that a $C^{1,1}$ Euclidean surface S satisfies (UTB) on the whole of S with Euclidean balls instead of Heisenberg balls.

Let P be a non characteristic point on S. Recall that $\Pi_P S$ is the Euclidean plane tangent to S at P and let γ be the metric normal to $\Pi_P S$ at P pointing inside Ω. Let $s \geq 0$ be a number smaller than the length of γ and consider the point C_s on γ having distance s from P. Suppose τ_s is the left translation taking C_s to O and let S', P' γ', Ω' be the images under τ_s of S, P and γ, Ω, respectively. By continuity and by the assumption that S is C^2, there is $r > 0$ such that, for each $s < \delta(r)$, there exists a Euclidean ball $B^s_{Euc}(D_s, r)$ of radius r contained in Ω' and having P' on its boundary.

We will show that there exists a Heisenberg ball $B(R,\eta)$ such that: (a) $B(R,\eta)$ is centered at a point R of γ'; (b) $B(R,\eta)$ is contained in $B^s_{Euc}(D_s, r)$; (c) $B(R,\eta)$ has P' on its boundary. The ball $\tau_s^{-1}B(R,\eta) = B(\tau_s^{-1}R,\eta)$ is contained in Ω and has P on its boundary. Exchanging Ω with $\mathbb{H}-\overline{\Omega}$, we see that (TB) holds at P. Since our procedure is stable as P varies in $S - \mathrm{Char}(S)$, (UTB) holds on $S - \mathrm{Char}(S)$.

Since P' is noncharacteristic, a subarc of γ' starting at P' is contained in $B^s_{Euc}(D_s, r)$, but for the point P'. It suffices to prove the following claim.

Claim $B^s_{Euc}(D_s, r)$ contains a Heisenberg ball having P' on its boundary. We need an estimate for the Euclidean curvatures of the Heisenberg ball with center at the origin.

Lemma 5.1 *Let* $B(0,s)$ *be the Heisenberg ball centered in* 0 *with radius s. For every point* $P \in \partial B(0,s)$ *such that* $P = P(x,y,t)$ *with* $(x,y) \neq (0,0)$, *there exists a positive number* $s_0(P)$ *such that the principal Euclidean curvatures in P as* $s \to 0$ *are, respectively*:

$$k_1^E = \frac{1}{s\sqrt{1+\frac{1}{16}\phi^2}}(1+o(1))$$

and

$$k_2^E = \frac{3}{4s^3}(1+o(1)).$$

where $o(1) \to 0$ *as* $s \to 0$, *uniformly w.r.t.* $|\phi| \leq C$, *for any fixed* $C > 0$.

Proof From (5) and $\alpha = \frac{\phi s}{2}$ we get the following parametrization of the Carnot ball of radius s:

$$\begin{cases} x = s\cos\theta\,\frac{\sin\alpha}{\alpha} \\ y = s\sin\theta\,\frac{\sin\alpha}{\alpha} \\ t = \frac{s^2}{2}\,\frac{2\alpha - \sin(2\alpha)}{\alpha^2}. \end{cases}$$

We can calculate the Euclidean curvatures k_1^E, k_2^E of the Carnot ball:

$$k_1^E = \frac{g'(\alpha)}{f(\alpha)\sqrt{f'(\alpha)^2 + (\frac{s}{2}g'(\alpha))^2}},$$

$$k_2^E = \frac{1}{f'(\alpha)}\left(\frac{g'(\alpha)}{\sqrt{f'(\alpha)^2 + (\frac{s}{2}g'(\alpha))^2}}\right)',$$

where $f(\alpha) = s\frac{\sin\alpha}{\alpha}$ and $g(\alpha) = s^2\frac{2\alpha-\sin(2\alpha)}{2\alpha^2}$.

As a consequence we obtain

$$k_1^E = \frac{G'(\alpha)}{F(\alpha)\sqrt{F'(\alpha)^2 + (sG'(\alpha))^2}}$$

and

$$k_2^E = \frac{1}{F'(\alpha)}\left(\frac{G'(\alpha)}{\sqrt{F'(\alpha)^2 + (sG'(\alpha))^2}}\right)',$$

where $F(\alpha) = \frac{\sin\alpha}{\alpha}$ and $G(\alpha) = \frac{2\alpha-\sin(2\alpha)}{2\alpha^2}$. On the other hand, for fixed ϕ and as $\alpha \to 0$,

$$F'(\alpha) = -\frac{\alpha}{3} + \frac{1}{30}\alpha^3 + o(\alpha^4),$$

$$G'(\alpha) = \frac{2}{3} - \frac{2}{5}\alpha^2 + o(\alpha^3),$$

and

$$F''(\alpha) = -\frac{1}{3} + \frac{1}{10}\alpha^2 + o(\alpha^3),$$

$$G''(\alpha) = -\frac{4}{5}\alpha + o(\alpha^2).$$

Hence,

$$k_1^E = \frac{1}{s\sqrt{1+\frac{1}{16}\phi^2}}(1 + o(1))$$

and

$$k_2^E = \frac{3}{4s^3}(1 + o(1)).$$

□

We now complete the proof of Theorem 5.1. If s is small enough, by Lemma 5.1 there exists an open neighborhood $\mathcal{V}$ of P' such that $B(0,s) \cap \mathcal{V}$ is contained in $B_{Euc}(D_s, r)$. Consider now all Heisenberg balls $B(R,\eta)$ with $R \in \gamma'$. By Lemma 3.2, $B(R,\eta) \subseteq B(0,s)$. As $\eta \to 0$, the ball $B(R,\eta)$ shrinks to P'. Hence, $B(R,\eta)$ is contained in $B(0,s) \cap \mathcal{V} \subseteq B_{Euc}(D_s, r)$, if η is small enough. Clearly, $B(R,\eta)$ contains P' on its boundary. This proves the claim, hence Theorem 5.1. □

Following Federer, [6] we introduce the notion of reach.

Definition 5.2 *Let Ω be an open subset of $\mathbb{H}$ with boundary S. We denote by $Unp(S)$ the set of the points P in $\mathbb{H}$ such that there exists a unique point Q in S which is nearest to P,*

$$d(P,S) = d(P,Q).$$

We say that $Q = \xi(P) = \xi_S(P)$ is the **projection** *of P onto S.*

Let U be an open subset of S. S has **locally positive reach** *on U if for all Q in U an open neighborhood of Q in $\mathbb{H}$ is contained in $Unp(S)$.*

The notion of positive reach is related to (UTB) via an exponential-like map.

Definition 5.3 *Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$ and suppose that S is C^1 in the Euclidean sense. Let*

$$\mathcal{C} = \{(P,s):\ P \in S,\ s \in dom(\mathcal{N}_P^+ S)\} \subseteq S \times \mathbb{R}$$

where $dom(\mathcal{N}_P S)$ is the domain of $\mathcal{N}_P S$. The **exponential map** *associated with S is the map*

$$\exp_S : \mathcal{C} \to \mathbb{H},\ \exp_S(P,s) = \mathcal{N}_P^+ S(s).$$

We call this map *exponential* because it extends the notion of exponential map in Riemannian manifolds. The points $P \in S$ are in $1-1$ correspondence with the oriented metric normals to S via the map $P \mapsto \mathcal{N}_P^+ S$. Also, each oriented metric normal is a geodesic segment. The map $\exp_S$ parametrizes a point $Q = \exp_S(P,s)$ of $\mathbb{H}$ according to the geodesic γ, normal to S, to which Q belongs and to the position of Q on γ. Note that $\exp_S$ is not necessarily $1-1$.

Theorem 5.2 *Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$ and suppose that S is C^1 in the Euclidean sense.*

Then, $\exp_S$ is a homeomorphism of $int(\mathcal{C})$ onto an open subset of $int(Unp(S))$. Moreover, $S \cap int(Unp(S)) = S \cap \exp_S(int(\mathcal{C}))$.

Proof Let $\mathcal{U} \subseteq S \times \mathbb{R}$ be the interior of $\mathcal{C}$.

The map $\exp_S$ is $1-1$ on $\mathcal{U}$. Suppose $\exp_S(P,s) = \exp_S(P',s') = Q$. By definition of metric normal, $s = \delta_S(Q) = s'$. Suppose $s > 0$. The case $s < 0$ is similar. Since $\mathcal{U}$ is open, $\mathcal{N}_P S$ can be extended to an interval $[0, s+\epsilon]$ for some positive ϵ. Let $Q' = \mathcal{N}_P S(s+\epsilon)$. Thus, if $P \neq P'$,

$$d(Q',S) \leq d(Q',P') < d(Q',Q) + d(Q,P') = d(Q',P) = d(Q',S),$$

contradiction. The strict inequality depends on the fact that Q' can not lie on the extension of a geodesic between P' and Q. As a consequence, we have that $\exp_S(\mathcal{U}) \subseteq Unp(S)$.

The map $\exp_S$ is continuous on $\mathcal{C}$, since S is C^1 in the Euclidean sense and, by (9), η continuously depends on $s, P, \nabla_{\mathbb{H}} g(P)$ and $[X,Y]g(P)$.

The map $\exp_S$ maps $\mathcal{C}$ *onto* $\mathbb{H}$. Let $Q \in \mathbb{H}$, let P be a point on S such that $d(Q,P) = d(Q,S)$ and let $\delta_S(Q) = s$. Then $(P,s) \in \mathcal{C}$ and $\exp_S(P,s) = Q$.

Consider now

$$G : Unp(S) \to S \times \mathbb{R},\ G(Q) = (\xi_S(Q), \delta_S(Q)).$$

By Lemma 6.3 below, G is a continuous function. Then $\exp_S|_{\mathcal{U}}$ is a homeomorphism, because $G \circ \exp_S|_{\mathcal{U}} = Id$. Hence, $\mathcal{U}$ and $\exp_S(\mathcal{U})$ are homeomorphic, $\mathcal{U}$ is locally identifiable with an open subset of $\mathbb{R}^3$, hence, by Brouwer's theorem on the invariance of domain (see [17]), $\exp_S(\mathcal{U}) \subseteq Unp(S)$ is an open subset of $\mathbb{H}$, contained in $Unp(S)$. This shows that $\exp_S$ is a homeomorphism of $\mathcal{U}$ onto an open subset of $int(Unp(S))$.

We still have to show that $S \cap int(Unp(S)) = S \cap \exp_S(\mathcal{U})$. Let $\mathcal{U}_1 = \{P \in S :\ (P,0) \in \mathcal{U}\}$. $\mathcal{U}_1$ is open in S. Clearly, $\mathcal{U}_1 \subseteq S \cap int(Unp(S))$. We show that $S \cap int(Unp(S)) \subseteq \mathcal{U}_1$. Let $P_0 \in S \cap int(Unp(S))$ and let $\mathcal{V}$ be an open neighborhood of P_0 in $int(Unp(S))$. Then, $G(\mathcal{V}) \subseteq \mathcal{C}$, but G is the inverse function of $\exp_S$, hence $G(\mathcal{V}) = \exp_S^{-1}(\mathcal{V})$, which is open in $\mathcal{C}$. Hence, $G(\mathcal{V}) \subset \mathcal{U}$ and, since $G(P) = (P,0)$ when $P \in S$, this shows that the open subset $\mathcal{V} \cap S$ of S is contained in $\mathcal{U}_1$. □

Theorem 5.3 *Let S be a surface in $\mathbb{H}$, $S = \partial\Omega$ and $S = \partial(\mathbb{H} - \overline{\Omega})$, where Ω is open in $\mathbb{H}$. Let S_1 be an open subset of S. If S is C^1 in the Euclidean sense, then S satisfies (UTB) on S_1 if and only if it satisfies (PR) on S_1.*

Proof One easily verifies that S satisfies (PR) on S_1 if and only if $S_1 \subseteq int(Unp(S))$. On the other hand, S satisfies (UTB) on S_1 if and only if $S_1 \times \{0\} \subset \mathcal{U} = int(\mathcal{C})$. The theorem now follows from Theorem 5.2. □

From Theorems 5.1 and 5.3 we obtain the following.

Corollary 5.1 *Let S be a surface in $\mathbb{H}$, $S = \partial\Omega$ and $S = \partial(\mathbb{H} - \overline{\Omega})$, where Ω is open in $\mathbb{H}$ and suppose that S is $C^{1,1}$ in the Euclidean sense. Then, Ω satisfies (PR) on $S - \mathrm{Char}(S)$.*

6 Regularity of the distance function

The main result of this section is the following theorem.

Theorem 6.1 *Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$.*

(i) *If S is C^1, and S satisfies (UTB) on $S - \mathrm{Char}(S)$, then $\nabla_{\mathbb{H}}\delta_S$ is a continuous function in an open neighborhood of $S - \mathrm{Char}(S)$ in $\mathbb{H}$. In particular, the conclusion holds if S is $C^{1,1}$ in the Euclidean sense.*
(ii) *If S is C^k in the Euclidean sense, $k \geq 2$, then $\nabla_{\mathbb{H}}\delta_S$ and δ_S are of class C^{k-1}, in the Euclidean sense, in an open neighborhood of $S - \mathrm{Char}(S)$ in $\mathbb{H}$.*

Theorem 1.1 follows now from Theorems 5.1 and 6.1.

The proof of Theorem 6.1 will come after we prove a number of lemmata.

Lemma 6.1 *Let f be a Lipschitz function in the open set W, g a continuous function in W and V any linear combination of X and Y. If $Vf = g$ where $Vf = g$ exists. Then $Vf = g$ in W hence f is $C^1(W)$.*

Proof Following Federer, [6] Lemma 4.7, without loss of generality we can assume $V = X$. Let P be any point in W, and $r > 0$ so that $B(P, 2r) \subset W$. By Rademacher's theorem in [16] and Fubini's theorem, for a.e. $Q \in B(P, r)$ we have that, whenever $|\tau| < r$,

$$f(Q \cdot (\tau, 0, 0)) - f(Q) = \int_0^\tau Xf(Q \cdot (s, 0, 0))ds = \int_0^\tau g(Q \cdot (s, 0, 0))\mathrm{d}s.$$

By the continuity of f and g it follows that

$$f(P \cdot (\tau, 0, 0)) - f(P) = \int_0^\tau g(P \cdot (s, 0, 0))\mathrm{d}s.$$

Differentiating with respect to τ we get $Xf(P) = g(P)$. □

Lemma 6.2 *Let P in $Unp(S)\backslash S$, such that δ_S is differentiable at P. Then,*

$$\nabla_{\mathbb{H}}\delta_S(P) = \dot{\mathcal{N}}^{+}_{\xi(P)}(\delta_S(P)). \tag{11}$$

Proof Let $\tilde{\gamma} = \tilde{\gamma}_P$ be the geodesic connecting P with $\xi(P)$, parametrized to have $\tilde{\gamma}(0) = P$ and $\tilde{\gamma}(1) = \xi(P)$. Hence, $\tilde{\gamma}$ has speed $d(P, \xi(P)) = d_S(P)$. The geodesic $\tilde{\gamma}$ is a reparametrization of an arc of $\mathcal{N}^+_{\xi(P)}S$,

$$\mathcal{N}^+_{\xi(P)}S(s) = \tilde{\gamma}\left(1 - \frac{s}{\delta_S(P)}\right).$$

By Lemma 3.1,

$$d_S(\tilde{\gamma}(t)) = d(\tilde{\gamma}(t), \xi(P)) = (1-t)d_S(P).$$

Hence

$$-d_S(P) = \frac{d}{dt}|_{t=0} d_S(\tilde{\gamma}(t)) = \langle \dot{\tilde{\gamma}}(0), \nabla_{\mathbb{H}} d_S(P) \rangle.$$

On the other hand, by Cauchy-Schwarz and by the Eikonal equation (rather, by the easy inequality $|\nabla_{\mathbb{H}} d_S(P)| \leq 1$),

$$\langle \dot{\tilde{\gamma}}(0), \nabla_{\mathbb{H}} d_S(P) \rangle \geq -|\dot{\tilde{\gamma}}(0)| \cdot |\nabla_{\mathbb{H}} d_S(P)| \geq -d_S(P).$$

As a consequence inequalities are actual equalities and this implies

$$d_S(P) \nabla_{\mathbb{H}} d_S = -\dot{\tilde{\gamma}}(0),$$

which is equivalent to (11). □

Lemma 6.3 *Let E be a closed subset of $\mathbb{H}$. The map $P \mapsto \xi(P)$ is continuous on $Unp(E)$.*

The proof of Federer, ([6] Theorem 4.8 (4)) extends to the Heisenberg case without changes.

Lemma 6.4 *Let S be an Euclidean C^1 surface. The map $P \mapsto \dot{\mathcal{N}}^+_{\xi(P)}(\delta_S(P))$ is a continuous section of the horizontal fiber restricted to $Unp(S)$.*

Proof The key observation is that $\dot{\mathcal{N}}^+_{\xi(P)}(\delta_S(P))$ is obtained by $\dot{\mathcal{N}}^+_{\xi(P)}(0) = N^+_P S$ by means of a rotation. Let θ be the angle

$$\theta = -\frac{2\delta_S(P)}{d(\xi(P), C)},$$

where C is the characteristic point of $\Pi_{\xi(P)}S$ ($\theta = 0$ when $d(\xi(P), C) = 0$). Then,

$$\dot{\mathcal{N}}^+_{\xi(P)}(\delta_S(P)) = (R_\theta)_{*,P} N^+_{\xi(P)}(0). \quad (12)$$

Here, $(R_\theta)_*$ is rotation by θ in $\mathcal{H}$ and $(R_\theta)_{*,P}V$ is the evaluation at P of the horizontal vector field $(R_\theta)_*V$. In complex notation

$$(R_\theta)_*(aX + ibY) = e^{i\theta}(aX + ibY).$$

In the right hand side of (12), we denote by $N^+_{\xi(P)}(0)$ both a horizontal vector at $\xi(P)$ and its extension to a left invariant (horizontal) vector field. Equation (12) is an immediate consequence of the geodesic equation (2).

In view of Lemma 6.3 and the assumption that S is C^1 in the Euclidean sense, hence that $N^+_{\xi(P)}(0)$ and $d(\xi(P), C)$ are continuous functions of $\xi(P) \in S$, (12) implies that $\dot{\mathcal{N}}^+_{\xi(P)}(\delta_S(P))$ continuously depends on P. □

Theorem 6.2 *Let S be a surface in $\mathbb{H}$, $S = \partial\Omega$ and $S = \partial(\mathbb{H} - \overline{\Omega})$, where Ω is open in $\mathbb{H}$. Suppose that S is C^1 in the Euclidean sense. Then, $\nabla_{\mathbb{H}}\delta_S$ is a continuous function on $int(Unp)$.*

Proof It follows from Lemmata 6.1, 6.2 and 6.4. □

Proof of Theorem 6.1-(i) Part (i) of the theorem immediately follows from Theorem 6.2 and Theorem 5.3. Moreover the last part of the conclusion follows from Corollary 5.1.

Before giving the proof of Theorem 6.1-(ii) we need some preparation.

Let S be a surface in $\mathbb{H}$, which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$, and suppose that S is C^2 in the Euclidean sense. Suppose that in a neighborhood $\mathcal{U} \subset S$ of a point P_0 the surface is free of characteristic points and that it can be parametrized as

$$\mathcal{U} = \{(u, v, f(u, v)) : (u, v) \in A\},$$

where $A \subseteq \mathbb{R}^2$ is open and f is C^2. Let $F : A \times \mathbb{R} \to \mathbb{H}$,

$$F(u, v, s) = \exp_S((u, v, f(u, v)), s). \tag{13}$$

The function F is an expression of $\exp_S$ in local coordinates. It has an intrinsic geometric meaning, since the projection $proj : \mathbb{H} \to \mathbb{R}^2$, $proj(z, t) = z$, pushes the Carnot-Carathéodory metric of $\mathbb{H}$ onto the Euclidean metric of $\mathbb{R}^2$. Without loss of generality, assume that, if $P \in \mathcal{U}$, then $\mathcal{N}_P S$ points upward. In the sequel, we write $Xf = -\partial_u f + 2v$ and $Yf = -\partial_v f - 2u$. In other words, if $g(u, v, t) = t - f(u, v)$, $(u, v, t) \in \mathbb{H}$, then we write $Xf = Xg$ and $Yf = Yg$.

Lemma 6.5 *Let S be a C^2 surface in $\mathbb{H}$, $S = \partial\Omega$ and $S = \partial(\mathbb{H} - \overline{\Omega})$, where Ω is open in $\mathbb{H}$. Suppose that, as above, $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^2$, A open, gives a C^2 parametrization of a characteristic point free open portion $\mathcal{U} \subset S \setminus Char(S)$. Let F be the function defined in (13) and let $P = F(u, v, 0) \in S$. Then, the matrix representing $JF(u, v, 0)$ with respect to the basis $\{\partial_u, \partial_v, \partial_\sigma\}$ of $\mathbb{R}^2 \times \mathbb{R}$, and the basis $\{X, Y, \partial_t\}$ of $\mathbb{H} \equiv \mathbb{R}^3$ is*

$$\begin{pmatrix} 1 & 0 & \frac{Xf}{|\nabla_{\mathbb{H}} f|} \\ 0 & 1 & \frac{Yf}{|\nabla_{\mathbb{H}} f|} \\ -Xf & -Yf & 0 \end{pmatrix}. \tag{14}$$

Corollary 6.1 *With the assumptions of Lemma 6.5, suppose that C is the characteristic point of $\Pi_P S$. Then,*

$$\det JF(u, v, 0) = 2d(P, C) \neq 0. \tag{15}$$

Proof of Lemma 6.5 The map F can be written explicitly, since we know the expression of the normal metric and

$$F(u, v, \tau) = \mathcal{N}_{(u,v,f(u,v))} S(\tau). \tag{16}$$

We are going to use the coordinates

$$(x, y, t) = F(u, v, \tau) = (u, v, f(u, v)) \circ (x', y', t')$$

where $(x', y', t') = \gamma_{u,v}(\tau)$ and $\gamma_{u,v}$ is the metric normal's left translate by P^{-1}. The expression for $\gamma_{u,v}$ is given by (9):

$$\begin{cases} x' = \frac{1}{4}(Xf\sin\alpha + Yf(1-\cos\alpha)) \\ y' = \frac{1}{4}(Yf\sin\alpha - Xf(1-\cos\alpha)) \\ t' = \frac{|\nabla_{\mathbb{H}}f|^2}{8}(\alpha - \sin\alpha), \end{cases}$$

where $\alpha = \frac{4\tau}{|\nabla_{\mathbb{H}}f|}$. Since

$$\begin{cases} x = u + x' \\ y = v + y' \\ t = f(u,v) + t' + 2(vx' - uy') \end{cases}$$

we have

$$\begin{cases} x_u = 1 + x'_u \\ y_u = y'_u \\ t_u = f_u(u,v) - 2y' + t'_u + 2(vx'_u - uy'_u), \end{cases}$$

$$\begin{cases} x_v = x'_v \\ y_v = 1 + y'_v \\ t_v = f_v(u,v) + 2x' + t'_v + 2(vx'_v - uy'_v) \end{cases}$$

and

$$\begin{cases} x_\tau = x'_\tau \\ y_\tau = y'_\tau \\ t_\tau = t'_\tau + 2(vx'_\tau - uy'_\tau). \end{cases}$$

We compute the derivatives of each coordinate,

$$\begin{aligned} 4x'_u &= (Xf)_u \sin\alpha + (Yf)_u(1-\cos\alpha) + \alpha_u Xf\cos\alpha + \alpha_u Yf\sin\alpha \\ &= (Xf)_u \sin\alpha + (Yf)_u(1-\cos\alpha) \\ &\quad - \cos\alpha\frac{Xf}{2}\frac{4\tau}{|\nabla_{\mathbb{H}}f|^3}(2Xf(Xf)_u + 2Yf(Yf)_u) \\ &\quad - \sin\alpha\frac{Xf}{2}\frac{4\tau}{|\nabla_{\mathbb{H}}f|^3}(2Xf(Xf)_u + 2Yf(Yf)_u), \end{aligned}$$

and so $(x'_u)_{|\tau=0} = 0$. Analogously

$$\begin{aligned} 4x'_v &= (Xf)_v \sin\alpha + (Yf)_v(1-\cos\alpha) \\ &\quad - \left(\cos\alpha Xf\frac{4\tau}{|\nabla_{\mathbb{H}}f|^3} + \sin\alpha Yf\frac{4\tau}{|\nabla_{\mathbb{H}}f|^3}\right)(Xf(Xf)_u + Yf(Yf)_u), \end{aligned}$$

and $(y'_u)_{|\tau=0} = 0$;

$$4x'_\tau = Xf\cos\alpha\frac{4}{|\nabla_{\mathbb{H}}f|} + Yf\sin\alpha\frac{4}{|\nabla_{\mathbb{H}}f|},$$

and$(x'_\tau)_{|\tau=0} = \frac{Xf}{|\nabla_{\mathbb{H}}f|}$;

$$\begin{aligned} 4y'_u &= (Yf)_u\sin\alpha - (Xf)_u(1-\cos\alpha) \\ &\quad - \frac{4\tau}{|\nabla_{\mathbb{H}}f|^3}(\cos\alpha Yf - \sin\alpha Xf)(Xf(Xf)_u + Yf(Yf)_u), \end{aligned}$$

and $(y'_u)_{|\tau=0} = 0$;

$$\begin{aligned} 4y'_v &= (Yf)_v \sin\alpha - (Xf)_v(1-\cos\alpha) \\ &\quad - \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid^3} \left(\cos\alpha\, Yf - \sin\alpha\, Xf\right) \left(Xf(Xf)_v + Yf(Yf)_v\right), \end{aligned}$$

and $(y'_v)_{|\tau=0} = 0$;

$$4y'_\tau = \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid^3} \left(\cos\alpha\, Yf - \sin\alpha\, Xf\right),$$

and $(y'_\tau)_{|\tau=0} = \frac{Yf}{|\nabla_{\mathbb{H}} f|}$.

$$8t'_u = 2\left(Xf(Xf)_u + Yf(Yf)_u\right)(\alpha - \sin\alpha) - \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid} \left(Xf(Xf)_u + Yf(Yf)_u\right)(1-\cos\alpha),$$

and $(t'_u)_{|\tau=0} = 0$;

$$8t'_v = 2\left(Xf(Xf)_v + Yf(Yf)_v\right)(\alpha - \sin\alpha) + \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid} \left(Xf(Xf)_v + Yf(Yf)_v\right)(1-\cos\alpha),$$

and $(t'_v)_{|\tau=0} = 0$, and, eventually

$$8t'_\tau = 4 \mid \nabla_{\mathbb{H}} f \mid (1-\cos\alpha),$$

and

$$t'_\tau = 0.$$

Hence

$$\begin{aligned} x_u &= 1 + \frac{1}{4}(Xf)_u \sin\alpha + \frac{1}{4}(Yf)_u(1-\cos\alpha) \\ &\quad - \frac{1}{4}\cos\alpha \frac{Xf}{2} \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid^3} \left(2Xf(Xf)_u + 2Yf(Yf)_u\right) \\ &\quad - \frac{1}{4}\sin\alpha \frac{Xf}{2} \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid^3} \left(2Xf(Xf)_u + 2Yf(Yf)_u\right) \end{aligned}$$

and $(x_u)_{|\tau=0} = 1$;

$$x_v = x'_v$$

and $(x_v)_{|\tau=0} = 0$;

$$x_\tau = x'_\tau$$

and $(x_\tau)_{|\tau=0} = \frac{Xf}{|\nabla_{\mathbb{H}} f|}$;

$$y_u = y'_u$$

and $(y_u)_{|\tau=0} = 0$;

$$\begin{aligned} y_v &= 1 + \frac{1}{4}(Yf)_v \sin\alpha - \frac{1}{4}(Xf)_v(1-\cos\alpha) \\ &\quad - \frac{1}{4}\frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid^3} \left(\cos\alpha\, Yf - \sin\alpha\, Xf\right) \left(Xf(Xf)_v + Yf(Yf)_v\right), \end{aligned}$$

and $(y_v)_{|\tau=0} = 1$;

$$y_\tau = y'_\tau,$$

and $(y_\tau)_{|\tau=0} \frac{Yf}{|\nabla_{\mathbb{H}} f|}$

$$\begin{aligned} t_u = f_u &+ \frac{1}{4}\left(Xf(Xf)_u + Yf(Yf)_u\right)(\alpha - \sin\alpha) \\ &- \frac{\tau}{2 \mid \nabla_{\mathbb{H}} f \mid}\left(Xf(Xf)_u + Yf(Yf)_u\right)(1 - \cos\alpha) \\ &+ 2vx'_u - 2uy'_u - 2y', \end{aligned}$$

with $(t_u)_{|\tau=0} = f_u$.

$$t_\tau = \frac{1}{2} \mid \nabla_{\mathbb{H}} f \mid (1 - \cos\alpha) + 2(vx'_\tau - uy'_\tau),$$

and $(t_\tau)_{|\tau=0} = 2v\frac{Xf}{|\nabla_{\mathbb{H}} f|} - 2u\frac{Yf}{|\nabla_{\mathbb{H}} f|}$.

Moreover

$$\begin{aligned} t_v = f_v &+ \frac{1}{4}\left(Xf(Xf)_v + Yf(Yf)_v\right)(\alpha - \sin\alpha) \\ &- \frac{1}{2}\frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid}\left(Xf(Xf)_v + Yf(Yf)_v\right)(1 - \cos\alpha) \\ &+ 2vx'_v - 2uy'_v + 2x' \end{aligned}$$

and $(t_v)_{|\tau=0} = f_v$. As a consequence

$$J\begin{pmatrix} x & y & t \\ u & v & \tau \end{pmatrix}_{\tau=0} = \begin{pmatrix} 1 & 0 & \frac{Xf}{|\nabla_{\mathbb{H}} f|} \\ 0 & 1 & \frac{Yf}{|\nabla_{\mathbb{H}} f|} \\ f_u & f_v & \frac{2vXf - 2uYf}{|\nabla_{\mathbb{H}} f|} \end{pmatrix} \tag{17}$$

Changing the basis in the target space from $\partial_x, \partial_y, \partial_t$ to X, Y, ∂_t, we obtain (14). □

Corollary 6.1 follows immediately,

$$\det JF(u, v, 0) = \mid \nabla_{\mathbb{H}} f \mid . \tag{18}$$

Proof of Theorem 6.1-(ii) Suppose that $S = \{(x, y, t) : g(x, y, t) = 0\}$, with $g \in C^k$, $k \geq 2$. Let P_0 be a noncharacteristic point on S such that $\partial_t g(P_0) \neq 0$. Then, in a neighborhood of P_0, we can assume that $g(x, y, t) = t - f(x, y)$, with f as in Lemma 6.5, $f \in C^k$. Observe that F is of class C^{k-1} in the Euclidean sense, since in its definition all the Euclidean derivatives of f appear. Now, if $Q = F(u, v, \tau)$, then $\tau = \delta_S(Q)$. By Lemma 6.5, Corollary 6.1 and the Inverse Function Theorem in $\mathbb{R}^3$, δ_S is a C^{k-1} function in a neighborhood of P_0. Since δ_S satisfies (11), δ_S is C^{k-1}, by Lemma 6.4 we have that $\nabla_{\mathbb{H}}\delta_S$ is C^{k-1} as well.

We now consider the set S_0 of those points (x_0, y_0, t_0) in S where $\partial_t g(x_0, y_0, t_0) = 0$. We consider two cases. Suppose that (x_0, y_0, t_0) has a neighborhood $\mathcal{U}$ in S such that the metric normal at any point of $\mathcal{U}$ is a Euclidean straight line, which is normal in the Euclidean sense to S. Hence, in an open neighborhood of (x_0, y_0, t_0), δ_S is the Euclidean distance, and the required smoothness of δ_S and $\nabla_{\mathbb{H}}\delta_S$ follows.

The second case is that where (x_0, y_0, t_0) is the limit of points (x, y, t) of S where $\partial_t g(x, y, t) \neq 0$. Since $\nabla g(x_0, y_0, t_0) \neq 0$, we can assume that, for (x, y, t) in a neighborhood $\mathcal{U}$ of (x_0, y_0, t_0), $\partial_y g(x, y, t) \neq 0$. By the Implicit Function Theorem, restricting $\mathcal{U}$ if necessary, we can assume that $g(x, y, t) = 0$ if and only if $y = h(x, t)$, where h is defined in an open neighborhood of (x_0, t_0) in $\mathbb{R}^2$. We consider a function $H : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{H}$, defined in an open neighborhood of $(x_0, t_0, 0)$,

$$H(x, t, \tau) = \mathcal{N}^+_{(x,h(x,t),t)} S(\tau). \tag{19}$$

Observe that H is a smooth map, in the Euclidean sense, at points where $\partial_t g = 0$ as well, by (9) and (10). Now,

$$|\det JH| = |h_t \cdot JF|$$

and the right hand side of the inequality, when $\tau = 0$, equals

$$2|h_t| d(P, C) = |h_t| \frac{|\nabla_{\mathbb{H}} g|}{|\partial_t g|}.$$

Since

$$0 = g_y h_t + g_t,$$

we have that

$$\frac{|\nabla_{\mathbb{H}} g|}{|\partial_y g|} = |\nabla_{\mathbb{H}} h|.$$

Hence, we have an expression analogous to (18),

$$\det JH(x, h(x, t), t) = | \nabla_{\mathbb{H}} h(x, t) | . \tag{20}$$

Here, $Xh = X(h(x, t) - y) = h_x + 2hh_t$ and $Yh = Y(h(x, t) - y) = -1 - 2xh_t$. On vertical points P, in particular,

$$|\det JH(P)| = \sqrt{h_x^2 + 1} > 0.$$

Now, the proof that δ_S and $\nabla_{\mathbb{H}} \delta_S$ have the required smoothness proceeds as in the previous cases. □

As a byproduct, we have the proof of a special case of a theorem of Monti and Serra-Cassano [13].

Corollary 6.2 *Let $S = \partial\Omega$ be the boundary of an open C^2 subset of $\mathbb{H}$ and let d_S be the distance function for S. Then, d_S satisfies the eikonal equation*

$$|\nabla_{\mathbb{H}} d_S| = 1,$$

on the surface S. Moreover if $\Omega = \{g < 0\}$, where g is a smooth function, then

$$\nabla_{\mathbb{H}} \delta_S = \frac{\nabla_{\mathbb{H}} g}{|\nabla_{\mathbb{H}} g|}.$$

From Lemma 6.5 and the analyticity of the geodesics' equations (2), it follows that δ_S is analytic in a neighborhood of $S - \mathrm{Char}(S)$ if S itself is analytic.

7 Cutlocus

Next, we give an elementary description of the set of the metric normals' endpoints. Let $S = \partial\Omega$ be the boundary of an open, connected set in $\mathbb{H}$ and assume that S is at least C^2. For $P \in S$, let Q be endpoint in $\overline{\Omega}$, other than P, of the metric normal $\mathcal{N}_P S$. When $\mathcal{N}_P S$ reduces to the point P, we set $Q = P$. The *cut-locus* in $\overline{\Omega}$ of S (the *skeleton* of Ω) is the set K_S of the points Q as P varies over S.

Below, $\mathcal{N}_P S$ refers to the portion of the metric normal at P which lies inside $\overline{\Omega}$. Here are some properties of K_S.

Lemma 7.1 *Let $P \in S$ be non characteristic. If $\mathcal{N}_P S$ is not a straight line, then it is a proper subarc of a maximal length minimizing geodesic starting at P.*

Proof Without loss of generality, assume that $P = O$. Let $\mathcal{N}_P S = \gamma|_{[0,b]}$, where γ is a maximal length-minimizing geodesic starting at P and having length $a \geq b$. If $b = a$, then $\gamma(b) = Q$ belongs to the t-axis, hence $R_\theta\gamma$ satisfies $l_{\mathbb{H}}(R_\theta\gamma) = l_{\mathbb{H}}(\gamma) = d(P,Q) = d(Q,S)$. i.e., for all θ, $R_\theta\gamma$ lies on $\mathcal{N}_P S$. This implies that $R_\theta\dot{\gamma}(0)$ is perpendicular to $T_O S$, for all θ, which is absurd. □

Lemma 7.2 *Let C be a characteristic point of S. Then, $\mathcal{N}_C S = \{C\}$.*

Proof Suppose that $Q \in \mathcal{N}_C S = \{C\}$ is different from C. Then, S does not intersect $B(Q, d(Q,C))$ and meets its closure in C. Hence, S is not smooth at C. □

Proposition 7.1 *The cut locus K_S of S has the following properties.*

(i) *K_S has empty interior.*
(ii) *K_S contains the characteristic points of S and each characteristic point of S is an accumulation point of $K_S - S$.*

Proof The assertion in (i) is a consequence of the following.

Lemma 7.3 *Let $Q \in K_S$ and let γ be a geodesic from Q to S such that $d(Q,S) = l_{\mathbb{H}}(\gamma)$. Then $\gamma - \{Q\}$ is free of points of K_S.*

Proof the lemma Let P be the endpoint of γ in S. Let R be point of K_S on γ, other than Q. By definition of K_S, there exists a maximal geodesic η through R, having length $a > d(R,S)$, such that $\eta(0) = P' \in S$, $R = \eta(b)$, where $b = d(R,S) > a$. For $b < c \leq a$, we have $d(\eta(c),S) < c$ (otherwise R would not be the endpoint of $\mathcal{N}_{P'}S$). Clearly, $d(R,P') = d(R,P)$.

By the triangle inequality, then,

$$d(Q,S) \leq d(Q,P') \leq d(Q,R) + d(R,P') = d(Q,R) + d(R,P) = d(Q,P) = d(Q,S).$$

This is possible only if R lies on the geodesic between Q and P', and so $Q \in \eta$, contradicting the fact that η's length does not realize the distance of its points from S past R. This proves the lemma. □

We finish the proof of Proposition 7.1. The first assertion in (ii) was proved in Lemma 7.2. Let now C be a characteristic point of S and let P_n be a sequence of noncharacteristic points of S tending to C. (Such sequence exists because the horizontal distribution is not closed under Lie brackets and by Frobenius Theorem). The formula for the geodesic normal in Theorem 4.1 implies that the length of $\mathcal{N}_{P_n}S$ is no more that $d(P_n, C_n)$, where C_n is the characteristic point of $T_{P_n}S$. By continuity, $d(P_n, C_n) \to d(C, C) = 0$ as $n \to \infty$. Let $R_n \in K_S$ be the endpoint other than P_n of $\mathcal{N}_{P_n}S$. Then,

$$d(R_n, C) \leq d(R_n, P_n) + d(P_n, C) \leq d(R_n, C_n) + d(P_n, C) \to 0$$

and this proves (ii). □

Proposition 7.2 *Let $R \in \mathbb{H} - K_S$. Then there is a unique geodesic γ from R to S such that $l_{\mathbb{H}}\gamma = d(R, S)$. i.e., there exists a unique $P \in S$ such that $R \in \mathcal{N}_P S$.*

Proof Suppose there are two such geodesics, γ and γ', having the other endpoint, resp. P and P', on S. We can not have $P = P'$, otherwise, with a reasoning similar to that of Lemma 7.1, one shows that P is characteristic, hence that γ reduces to a point and $R \in S$.

Suppose first that γ is not a straight line. Let $Q \in K_S$ be the endpoint of $\mathcal{N}_P \subseteq \gamma$ other than P. If γ' dos not lie in the (possibly non length-minimizing) extension of γ, then, as in the proof of Proposition 7.1, we have that $d(P', Q) < d(P, Q) = d(Q, S)$, a contradiction. The other possibility is that γ' and γ have in common the arc between Q and R. This leads to a contradiction, too, since it would imply that

$$d(Q, P') = d(P', R) - d(Q, R) < d(P, R) + d(Q, R) = d(P, Q),$$

hence that Q can not belong to $\mathcal{N}_P$.

If $\gamma = [P, R]$ and $\gamma' = [P', R]$ are straight lines, either they lie on the extension of each other (but this is not possible, otherwise $R \in K_S$), or they have different directions. In the second case, there exists a point Q such that $R \in [Q, P]$ and $d(Q, S) = l_{\mathbb{H}}([P, Q])$ (otherwise R would be the endpoint of $\mathcal{N}_P$, then $R \in K_S$) and, as above, we would have $d(Q, P') < d(P, Q)$, contradicting the fact that $Q \in \mathcal{N}_P$. □

A detailed study of the cut-locus K_S and of its properties will be the object of further research. For the case when S is analytic, [1] contains a detailed study of the cut locus, which is defined, however, in a different way.

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