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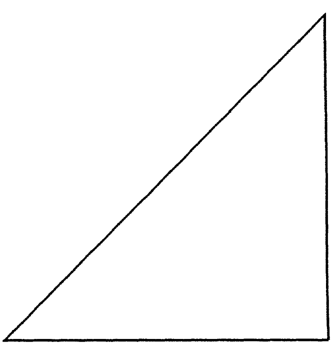
Irrationality of The Square Root of Two —A Geometric Proof

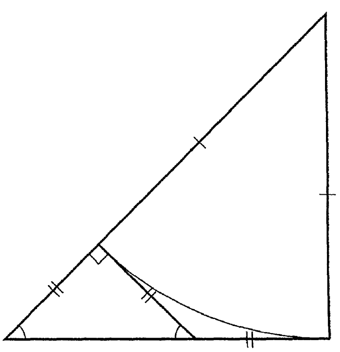
Tom M. Apostol

This note presents a remarkably simple proof of the irrationality of $\sqrt{2}$ that is a variation of the classical Greek geometric proof.

By the Pythagorean theorem, an isosceles right triangle of edge-length 1 has hypotenuse of length $\sqrt{2}$. If $\sqrt{2}$ is rational, some positive integer multiple of this triangle must have three sides with integer lengths, and hence there must be a *smallest* isosceles right triangle with this property. But inside any isosceles right triangle whose three sides have integer lengths we can always construct a smaller one with the same property, as shown below. Therefore $\sqrt{2}$ cannot be rational.



If this is an isosceles right triangle with integer sides,



then there is a smaller one with the same property.

Figure 1

Construction. A circular arc with center at the uppermost vertex and radius equal to the vertical leg of the triangle intersects the hypotenuse at a point, from which a perpendicular to the hypotenuse is drawn to the horizontal leg. Each line segment in the diagram has integer length, and the three segments with double tick marks have equal lengths. (Two of them are tangents to the circle from the same point.) Therefore the smaller isosceles right triangle with hypotenuse on the horizontal base also has integer sides.

The reader can verify that similar arguments establish the irrationality of $\sqrt{n^2+1}$ and $\sqrt{n^2-1}$ for any integer $n > 1$. For $\sqrt{n^2+1}$ use a right triangle with legs of lengths 1 and n. For $\sqrt{n^2-1}$ use a right triangle with hypotenuse n and one leg of length 1.

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Fundamental Theorem of Algebra—Yet Another Proof

Anindya Sen

Theorem. The Fundamental Theorem of Algebra. *Let*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_k z^{n-k} + \cdots + a_n$$

be a polynomial of degree $n \geq 1$ with complex numbers a_i as coefficients. Then P has a root, i.e., there is a $\zeta \in \mathbf{C}$ such that $P(\zeta) = 0$.

We prove the theorem by showing that Image(P) = $\mathbf{C}$.

We assume the standard result that a complex polynomial $P : \mathbf{C} \to \mathbf{C}$ is a *proper map*, i.e., $P^{-1}(A)$ is compact whenever $A \subseteq \mathbf{C}$ is compact. (P is continuous, and $|P(x)| \to \infty$ as $|x| \to \infty$. Hence, if $A \subseteq \mathbf{C}$ is closed and bounded, so is $P^{-1}(A)$. Hence, P is proper.)

Let $f : U \to R^2$ be a differentiable map of an open set $U \subseteq R^2$ to R^2. A point $x \in U$ is said to be a *regular point* of f if $Df(x) : R^2 \to R^2$ is invertible. Otherwise, x is said to be a *critical point* of f. A point $y \in R^2$ is said to be a *critical value* of f if it is the image of a critical point.

With this notation in mind, we first prove

Lemma 1. *Let K be the set of critical values of P. Then K and $P^{-1}(K)$ are both finite subsets of* $\mathbf{C}$.

Proof: The critical points of P are the points at which $P'(z) = 0$. Since P' is a polynomial of degree $n - 1$, there are at most $n - 1$ critical points. Since each critical value is the image of a critical point, K has at most $n - 1$ points. Now each critical value has at most n inverse images, hence, $P^{-1}(K)$ has at most $n(n-1)$ points. (We use the fact that a complex polynomial of degree k has at most k roots. The proof of this result does not use the fundamental theorem of algebra.) ■

