Limit sup and limit inf.

Introduction

In order to make us understand the information more on approaches of a given real sequence $\{a_n\}_{n=1}^{\infty}$, we give two definitions, thier names are upper limit and lower limit. It is fundamental but important tools in analysis.

Definition of limit sup and limit inf

Definition Given a real sequence $\{a_n\}_{n=1}^{\infty}$, we define

$$b_n = \sup\{a_m : m \geq n\}$$

and

$$c_n = \inf\{a_m : m \geq n\}.$$

Example $\{1 + (-1)^n\}_{n=1}^{\infty} = \{0, 2, 0, 2, \ldots\}$, so we have

$$b_n = 2 \text{ and } c_n = 0 \text{ for all } n.$$

Example $\{(-1)^n n\}_{n=1}^{\infty} = \{-1, 2, -3, 4, \ldots\}$, so we have

$$b_n = +\infty \text{ and } c_n = -\infty \text{ for all } n.$$

Example $\{-n\}_{n=1}^{\infty} = \{-1, -2, -3, \ldots\}$, so we have

$$b_n = -n \text{ and } c_n = -\infty \text{ for all } n.$$

Proposition Given a real sequence $\{a_n\}_{n=1}^{\infty}$, and thus define b_n and c_n as the same as before.

1 $b_n \neq -\infty$, and $c_n \neq \infty \ \forall n \in N$.
2 If there is a positive integer p such that $b_p = +\infty$, then $b_n = +\infty \ \forall n \in N$.
If there is a positive integer q such that $c_q = -\infty$, then $c_n = -\infty \ \forall n \in N$.
3 $\{b_n\}$ is decreasing and $\{c_n\}$ is increasing.

By property 3, we can give definitions on the upper limit and the lower limit of a given sequence as follows.

Definition Given a real sequence $\{a_n\}$ and let b_n and c_n as the same as before.

(1) If every $b_n \in R$, then

$$\inf\{b_n : n \in N\}$$

is called the upper limit of $\{a_n\}$, denoted by

$$\lim_{n\to\infty} \sup a_n.$$

That is,

$$\lim_{n\to\infty} \sup a_n = \inf_n b_n.$$

If every $b_n = +\infty$, then we define

$$\lim_{n\to\infty} \sup a_n = +\infty.$$

(2) If every $c_n \in R$, then

$$\sup\{c_n : n \in N\}$$

is called the lower limit of $\{a_n\}$, denoted by

$$\lim_{n\to\infty} \inf a_n.$$

That is,

$$\lim_{n\to\infty} \inf a_n = \sup_n c_n.$$

If every $c_n = -\infty$, then we define

$$\lim_{n\to\infty} \inf a_n = -\infty.$$

Remark The concept of lower limit and upper limit first appear in the book (**Analyse Alge'brique**) written by **Cauchy** in 1821. But until 1882, **Paul du Bois-Reymond** gave explanations on them, it becomes well-known.

Example $\{1 + (-1)^n\}_{n=1}^{\infty} = \{0,2,0,2,\dots\}$, so we have

$$b_n = 2 \text{ and } c_n = 0 \text{ for all } n$$

which implies that

$$\lim\sup a_n = 2 \text{ and } \lim\inf a_n = 0.$$

Example $\{(-1)^n n\}_{n=1}^{\infty} = \{-1,2,-3,4,\dots\}$, so we have

$$b_n = +\infty \text{ and } c_n = -\infty \text{ for all } n$$

which implies that

$$\lim\sup a_n = +\infty \text{ and } \lim\inf a_n = -\infty.$$

Example $\{-n\}_{n=1}^{\infty} = \{-1,-2,-3,\dots\}$, so we have

$$b_n = -n \text{ and } c_n = -\infty \text{ for all } n$$

which implies that

$$\lim\sup a_n = -\infty \text{ and } \lim\inf a_n = -\infty.$$

Relations with convergence and divergence for upper (lower) limit

Theorem Let $\{a_n\}$ be a real sequence, then $\{a_n\}$ converges if, and only if, the upper limit and the lower limit are real with

$$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} a_n.$$

Theorem Let $\{a_n\}$ be a real sequence, then we have

(1) $\lim_{n\to\infty} \sup a_n = +\infty \Leftrightarrow \{a_n\}$ has no upper bound.

(2) $\lim_{n\to\infty} \sup a_n = -\infty \Leftrightarrow$ for any $M > 0$, there is a positive integer n_0 such that as $n \geq n_0$, we have

$$a_n \leq -M.$$

(3) $\lim_{n\to\infty} \sup a_n = a$ if, and only if, (a) given any $\varepsilon > 0$, there are infinite many numbers n such that

$$a - \varepsilon < a_n$$

and (b) given any $\varepsilon > 0$, there is a positive integer n_0 such that as $n \geq n_0$, we have

$$a_n < a + \varepsilon.$$

Similarly, we also have

Theorem Let $\{a_n\}$ be a real sequence, then we have

(1) $\lim_{n\to\infty} \inf a_n = -\infty \Leftrightarrow \{a_n\}$ has no lower bound.

(2) $\lim_{n\to\infty} \inf a_n = +\infty \Leftrightarrow$ for any $M > 0$, there is a positive integer n_0 such

that as $n \geq n_0$, we have

$$a_n \geq M.$$

(3) $\lim_{n\to\infty} \inf a_n = a$ if, and only if, (a) given any $\varepsilon > 0$, there are infinite many numbers n such that

$$a + \varepsilon > a_n$$

and (b) given any $\varepsilon > 0$, there is a positive integer n_0 such that as $n \geq n_0$, we have

$$a_n > a - \varepsilon.$$

From Theorem 2 an Theorem 3, the sequence is divergent, we give the following definitios.

Definition Let $\{a_n\}$ be a real sequence, then we have

(1) If $\lim_{n\to\infty} \sup a_n = -\infty$, then we call the sequence $\{a_n\}$ diverges to $-\infty$, denoted by

$$\lim_{n\to\infty} a_n = -\infty.$$

(2) If $\lim_{n\to\infty} \inf a_n = +\infty$, then we call the sequence $\{a_n\}$ diverges to $+\infty$, denoted by

$$\lim_{n\to\infty} a_n = +\infty.$$

Theorem Let $\{a_n\}$ be a real sequence. If a is a limit point of $\{a_n\}$, then we have

$$\lim_{n\to\infty} \inf a_n \leq a \leq \lim_{n\to\infty} \sup a_n.$$

Some useful results

Theorem Let $\{a_n\}$ be a real sequence, then

(1) $\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n$.
(2) $\lim_{n\to\infty} \inf(-a_n) = -\lim_{n\to\infty} \sup a_n$ and $\lim_{n\to\infty} \sup(-a_n) = -\lim_{n\to\infty} \inf a_n$
(3) If every $a_n > 0$, and $0 < \lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n < +\infty$, then we have

$$\lim_{n\to\infty} \sup \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} \inf a_n} \text{ and } \lim_{n\to\infty} \inf \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} \sup a_n}.$$

Theorem Let $\{a_n\}$ and $\{b_n\}$be two real sequences.

(1) If there is a positive integer n_0 such that $a_n \leq b_n$, then we have

$$\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \inf b_n \text{ and } \lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sup b_n.$$

(2) Suppose that $-\infty < \lim_{n\to\infty} \inf a_n$, $\lim_{n\to\infty} \inf b_n$, $\lim_{n\to\infty} \sup a_n$, $\lim_{n\to\infty} \sup b_n < +\infty$, then

$$\begin{aligned} &\lim_{n\to\infty} \inf a_n + \lim_{n\to\infty} \inf b_n \\ &\leq \lim_{n\to\infty} \inf(a_n + b_n) \\ &\leq \lim_{n\to\infty} \inf a_n + \lim_{n\to\infty} \sup b_n \text{ (or } \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \inf b_n \text{)} \\ &\leq \lim_{n\to\infty} \sup(a_n + b_n) \\ &\leq \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \sup b_n. \end{aligned}$$

In particular, if $\{a_n\}$ converges, we have

$$\lim_{n\to\infty} \sup(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} \sup b_n$$

and

$$\lim_{n\to\infty}\inf(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty}\inf b_n.$$

(3) Suppose that $-\infty < \lim_{n\to\infty}\inf a_n,\ \lim_{n\to\infty}\inf b_n,\ \lim_{n\to\infty}\sup a_n,$ $\lim_{n\to\infty}\sup b_n < +\infty$, and $a_n > 0,\ b_n > 0\ \forall n$, then

$$\begin{aligned}
&\left(\lim_{n\to\infty}\inf a_n\right)\left(\lim_{n\to\infty}\inf b_n\right)\\
&\le \lim_{n\to\infty}\inf(a_n b_n)\\
&\le \left(\lim_{n\to\infty}\inf a_n\right)\left(\lim_{n\to\infty}\sup b_n\right)\ \left(\text{or } \left(\lim_{n\to\infty}\inf b_n\right)\left(\lim_{n\to\infty}\sup a_n\right)\right)\\
&\le \lim_{n\to\infty}\sup(a_n b_n)\\
&\le \left(\lim_{n\to\infty}\sup a_n\right)\left(\lim_{n\to\infty}\sup b_n\right).
\end{aligned}$$

In particular, if $\{a_n\}$ converges, we have

$$\lim_{n\to\infty}\sup(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\lim_{n\to\infty}\sup b_n$$

and

$$\lim_{n\to\infty}\inf(a_n+b_n) = \left(\lim_{n\to\infty} a_n\right)\lim_{n\to\infty}\inf b_n.$$

Theorem Let $\{a_n\}$ be a **positive** real sequence, then

$$\lim_{n\to\infty}\inf\frac{a_{n+1}}{a_n} \le \lim_{n\to\infty}\inf(a_n)^{1/n} \le \lim_{n\to\infty}\sup(a_n)^{1/n} \le \lim_{n\to\infty}\sup\frac{a_{n+1}}{a_n}.$$

Remark We can use the inequalities to show

$$\lim_{n\to\infty}\frac{(n!)^{1/n}}{n} = 1/e.$$

Theorem Let $\{a_n\}$ be a real sequence, then

$$\lim_{n\to\infty}\inf a_n \le \lim_{n\to\infty}\inf\frac{a_1+\ldots+a_n}{n} \le \lim_{n\to\infty}\sup\frac{a_1+\ldots+a_n}{n} \le \lim_{n\to\infty}\sup a_n.$$

Exercise Let $f : [a,d] \to R$ be a continuous function, and $\{a_n\}$ is a real sequence. If f is increasing and for every n, $\lim_{n\to\infty}\inf a_n,\ \lim_{n\to\infty}\sup a_n \in [a,d]$, then

$$\lim_{n\to\infty}\sup f(a_n) = f\left(\lim_{n\to\infty}\sup a_n\right) \text{ and } \lim_{n\to\infty}\inf f(a_n) = f\left(\lim_{n\to\infty}\inf a_n\right).$$

Remark: (1) The condition that f is increasing cannot be removed. For example,

$$f(x) = |x|,$$

and

$$a_k = \begin{cases} 1/k \text{ if } k \text{ is even} \\ -1-1/k \text{ if } k \text{ is odd.}\end{cases}$$

(2) The proof is easy if we list the definition of limit sup and limit inf. So, we omit it.

Exercise Let $\{a_n\}$ be a real sequence satisfying $a_{n+p} \le a_n + a_p$ for all n, p. Show that $\{\frac{a_n}{n}\}$ converges.

Hint: Consider its limit inf.