Something around the number e

1. Show that the sequence $\{(1+\frac{1}{n})^n\}$ converges, and denote the limit by e.

Proof: Since

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k \\ &= 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+..+\frac{n(n-1)\cdots 1}{n!}\left(\frac{1}{n}\right)^n \\ &= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \qquad 1 \\ &\leq 1+1+\frac{1}{2}+\frac{1}{2^2}+..+\frac{1}{2^{(n-1)}}+\ldots \\ &= 3,\end{aligned}$$

and by (1), we know that the sequence is increasing. Hence, the sequence is convergent. We denote its limit e. That is,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Remark: 1. The sequence and e first appear in the mail that **Euler** wrote to **Goldbach.** It is a beautiful formula involving

$$e^{i\pi}+1 = 0.$$

2. Use the exercise, we can show that $\sum_{k=0}^{\infty}\frac{1}{k!} = e$ as follows.

Proof: Let $x_n = (1+\frac{1}{n})^n$, and let $k > n$, we have

$$1+1+\frac{1}{2!}\left(1-\frac{1}{k}\right)+..+\frac{1}{n!}\left(1-\frac{1}{k}\right)\cdot\cdot\left(1-\frac{n-1}{k}\right) \leq x_k$$

which implies that (let $k\to\infty$)

$$y_n := \sum_{i=0}^{n}\frac{1}{i!} \leq e. \qquad 2$$

On the other hand,

$$x_n \leq y_n \qquad 3$$

So, by (2) and (3), we finally have

$$\sum_{k=0}^{\infty}\frac{1}{k!} = e. \qquad 4$$

3. e is an irrational number.

Proof: Assume that e is a rational number, say $e = p/q$, where g.c.d. $(p,q) = 1$. Note that $q > 1$. Consider

$$\begin{aligned}(q!)e &= (q!)\left(\sum_{k=0}^{\infty}\frac{1}{k!}\right) \\ &= (q!)\left(\sum_{k=0}^{q}\frac{1}{k!}\right)+(q!)\left(\sum_{k=q+1}^{\infty}\frac{1}{k!}\right),\end{aligned}$$

and since $(q!)\left(\sum_{k=0}^{q}\frac{1}{k!}\right)$ and $(q!)e$ are integers, we have $(q!)\left(\sum_{k=q+1}^{\infty}\frac{1}{k!}\right)$ is also an integer. However,

$$
\begin{aligned}
(q!)\left(\sum_{k=q+1}^{\infty}\frac{1}{k!}\right) &= \sum_{k=q+1}^{\infty}\frac{q!}{k!} \\
&= \frac{1}{q+1}+\frac{1}{(q+1)(q+2)}+\ldots \\
&< \frac{1}{q+1}+\left(\frac{1}{q+1}\right)^2+\ldots \\
&= \frac{1}{q} \\
&< 1,
\end{aligned}
$$

a contradiction. So, we know that e is not a rational number.

4. Here is an estimate about $e=\sum_{k=0}^{n}\frac{1}{k!}+\frac{\theta}{n(n!)}$, where $0<\theta<1$. (In fact, we know that $e=2.71828\ 18284\ 59045\ldots.$)

Proof: Since $e=\sum_{k=0}^{\infty}\frac{1}{k!}$, we have

$$
\begin{aligned}
0<e-x_n &= \sum_{k=n+1}^{\infty}\frac{1}{k!},\text{ where } x_n=\sum_{k=0}^{n}\frac{1}{k!} \\
&\leq \frac{1}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+\ldots\right) \\
&\leq \frac{1}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)^2}+\ldots\right) \\
&\leq \frac{1}{(n+1)!}\cdot\frac{n+2}{n+1} \\
&\leq \frac{1}{n(n!)}\text{ since }\frac{n+2}{(n+1)^2}<\frac{1}{n}.
\end{aligned}
$$

So, we finally have

$$e=\sum_{k=0}^{n}\frac{1}{k!}+\frac{\theta}{n(n!)},\text{ where } 0<\theta<1.$$

Note: We can use the estimate dorectly to show e is an irrational number.

2. For continuous variables, we have the samae result as follows. That is,

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x=e.$$

Proof: (1) Since $(1+\frac{1}{n})^n\to e$ as $n\to\infty$, we know that for any sequence $\{a_n\}\subseteq N$, with $a_n\to\infty$, we have

$$\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n}=e. \qquad 5$$

(2) Given a sequence $\{x_n\}$ with $x_n\to+\infty$, and define $a_n=[x_n]$, then $a_n\leq x_n<a_n+1$, then we have

$$\left(1+\frac{1}{a_n+1}\right)^{a_n}\leq\left(1+\frac{1}{x_n}\right)^{x_n}\leq\left(1+\frac{1}{a_n}\right)^{a_n+1}.$$

Since

$$\left(1+\frac{1}{a_n+1}\right)^{a_n}\to e\text{ and }\left(1+\frac{1}{a_n}\right)^{a_n+1}\to e\text{ as } x\to+\infty\text{ by (5)}$$

we know that

$$\lim_{n\to+\infty}\left(1+\frac{1}{x_n}\right)^{x_n} = e.$$

Since $\{x_n\}$ is arbitrary chosen so that it goes infinity, we finally obtain that

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x} = e. \tag{6}$$

(3) In order to show $(1+\frac{1}{x})^x \to e$ as $x \to -\infty$, we let $x = -y$, then

$$\begin{aligned}\left(1+\frac{1}{x}\right)^{x} &= \left(1+\frac{1}{-y}\right)^{-y}\\ &= \left(\frac{y}{y-1}\right)^{y}\\ &= \left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right).\end{aligned}$$

Note that $x \to -\infty(\Leftrightarrow y \to +\infty)$, by (6), we have shown that

$$\begin{aligned}e &= \lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right)\\ &= \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}.\end{aligned}$$

3. Prove that as $x > 0$, we have$(1+\frac{1}{x})^x$ is strictly increasing, and $(1+\frac{1}{x})^{x+1}$ is dstrictly ecreasing.

Proof: Since, by **Mean Value Theorem**

$$\frac{1}{x+1} < \log\left(1+\frac{1}{x}\right) = \log(x+1) - \log(x) = \frac{1}{\xi} < \frac{1}{x} \text{ for all } x > 0,$$

we have

$$\left[x\log\left(1+\frac{1}{x}\right)\right]' = \log\left(1+\frac{1}{x}\right) - \frac{1}{x+1} > 0 \text{ for all } x > 0$$

and

$$\left[(x+1)\log\left(1+\frac{1}{x}\right)\right]' = \log\left(1+\frac{1}{x}\right) - \frac{1}{x} < 0 \text{ for all } x > 0.$$

Hence, we know that

$$x\log\left(1+\frac{1}{x}\right) \text{ is strictly increasing on } (0,\infty)$$

and

$$(x+1)\log\left(1+\frac{1}{x}\right) \text{ is strictly decreasing on } (0,\infty).$$

It implies that

$$\left(1+\frac{1}{x}\right)^{x} \text{ is strictly increasing } (0,\infty), \text{ and } \left(1+\frac{1}{x}\right)^{x+1} \text{ is strictly decreasing on } (0,\infty).$$

Remark: By exercise 2, we know that

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x} = e = \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x+1}.$$

4. Follow the Exercise 3 to find the smallest a such that $(1+\frac{1}{x})^{x+a} > e$ and strictly decreasing for all $x \in (0,\infty)$.

Proof: Let $f(x) = (1+\frac{1}{x})^{x+a}$, and consider

$$\log f(x) = (x+a)\log\left(1+\frac{1}{x}\right) := g(x),$$

Let us consider

$$
\begin{aligned}
g'(x) &= \log\left(1+\frac{1}{x}\right) - \frac{x+a}{x^2+x} \\
&= -\log(1-y) + [-y+(1-a)y^2]\frac{1}{1-y}, \text{ where } 0 < y = \frac{1}{1+x} < 1 \\
&= \sum_{k=1}^{\infty}\frac{y^k}{k} + [-y+(1-a)y^2]\sum_{k=0}^{\infty} y^k \\
&= \left(\frac{1}{2}-a\right)y^2 + \left(\frac{1}{3}-a\right)y^3 + \ldots + \left(\frac{1}{n}-a\right)y^n + \ldots
\end{aligned}
$$

It is clear that for $a \geq 1/2$, we have $g'(x) < 0$ for all $x \in (0,\infty)$. Note that for $a < 1/2$, if there exists such a so that f is strictly decreasing for all $x \in (0,\infty)$. Then $g'(x) \leq 0$ for all $x \in (0,\infty)$. However, it is impossible since

$$
\begin{aligned}
g'(x) &= \left(\frac{1}{2}-a\right)y^2 + \left(\frac{1}{3}-a\right)y^3 + \ldots + \left(\frac{1}{n}-a\right)y^n + \ldots \\
&\to \frac{1}{2} - a > 0 \text{ as } y \to 1^-.
\end{aligned}
$$

So, we have proved that **the smallest value of a is** 1/2.

Remark: There is another proof to show that $(1+\frac{1}{x})^{x+1/2}$ is strictly decreasing on $(0,\infty)$.

Proof: Consider $h(t) = 1/t$, and two points $(1,1)$ and $\left(1+\frac{1}{x}, \frac{1}{1+\frac{1}{x}}\right)$ lying on the graph From three areas, the idea is that

The area of lower rectangle $<$ The area of the curve $<$ The area of trapezoid

So, we have

$$
\frac{1}{1+x} = \frac{1}{x}\left(\frac{1}{1+\frac{1}{x}}\right) < \log\left(1+\frac{1}{x}\right) < \frac{1}{2x}\left(1+\frac{1}{1+\frac{1}{x}}\right) = \left(x+\frac{1}{2}\right)\left(\frac{1}{x(x+1)}\right). \quad 7
$$

Consider

$$
\begin{aligned}
\left[\left(1+\frac{1}{x}\right)^{x+1/2}\right]' &= \left[\left(1+\frac{1}{x}\right)^{x+1/2}\right]\left[\log\left(1+\frac{1}{x}\right) - \left(x+\frac{1}{2}\right)\left(\frac{1}{x(x+1)}\right)\right] \\
&< 0 \text{ by (7);}
\end{aligned}
$$

hence, we know that $(1+\frac{1}{x})^{x+1/2}$ is strictly decreasing on $(0,\infty)$.

Note: Use the method of remark, we know that $(1+\frac{1}{x})^{x}$ is strictly increasing on $(0,\infty)$.