PTOLEMY'S INEQUALITY AND THE CHORDAL METRIC

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1. Introduction. Claudius Ptolemy, the celebrated mathematician, astronomer and geographer who flourished in Alexandria during the 2nd century A.D., is best known for his *Almagest*, a remarkable treatise on astronomy consisting of thirteen books. In Book I, Ptolemy calculated a table of chords which is equivalent to a five-place table of sines from 0 to 90 degrees, at intervals of quarter degrees. His calculations are based on a lemma, now known as Ptolemy's Theorem, which may be stated as follows:

PTOLEMY'S THEOREM. *The product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the products of the lengths of the opposite sides.*

In other words, if the vertices are labeled in cyclic order as a, b, c, d, we have

$$ac \cdot bd = ab \cdot cd + bc \cdot ad \tag{1}$$

where ab, cd, etc., denote distances. This formula contains, as special cases, the Pythagorean Theorem, the addition formulas for the sine and cosine, and the half-angle formula $2 \sin^2(x/2) = 1 - \cos x$, all of which can be obtained by specializing the quadrilateral. (See [**2**], p. 83.)

An English translation of Ptolemy's simple proof of (1) is given in [**4**], p. 225. A different proof, based on inversion with respect to a circle through one of the vertices, may be found in [**3**], p. 157, and in [**6**], p. 64. Inversion also provides a proof of the following extension of Ptolemy's Theorem to arbitrary convex quadrilaterals.

PTOLEMY'S INEQUALITY. *If abcd is a convex quadrilateral, then we have*

$$ac \cdot bd \leqq ab \cdot cd + bc \cdot ad \tag{2}$$

with equality if and only if the quadrilateral is inscribed in a circle.

I. J. Schoenberg [**7**] has shown that Ptolemy's inequality holds if a, b, c, d are any four points in a real inner-product space. Metric spaces in which Ptolemy's inequality (2) holds for all points a, b, c, d are called *ptolemaic*. In general, the triangle inequality for the metric neither implies nor is implied by Ptolemy's inequality. Schoenberg [**8**] also proved that every real seminormed space which is ptolemaic must arise from a real inner-product space.

In this note we show that Ptolemy's inequality in the plane is an immediate consequence of the triangle inequality for complex numbers. Then we show that the inequality in the plane implies the inequality in 3-space. Finally, we prove that the three-dimensional Ptolemy inequality is equivalent to the triangle inequality for the chordal metric of complex-variable theory.

2. Ptolemy's inequality deduced from the triangle inequality. Let a, b, c, d be any four complex numbers. Applying the triangle inequality to the algebraic identity

$$(a - b)(c - d) + (b - c)(a - d) = (a - c)(b - d) \tag{3}$$

we immediately obtain inequality (2). Equation (3) shows that the moduli of the three complex numbers

$$(a-b)(c-d), \qquad (b-c)(a-d), \qquad (a-c)(b-d) \tag{4}$$

are the lengths of the sides of a triangle. Therefore, we have equality in (2) if and only if the ratio

$$(a-b)(c-d)/(a-c)(b-d)$$

is real. But this ratio (the cross-ratio of a, b, c, d) is real if and only if a, b, c, d lie on a circle (see [1], p. 31). This proves the extended Ptolemy Theorem in the plane.

3. Ptolemy's inequality in 3-space. Now consider four noncoplanar points a, b, c, d in 3-space forming the vertices of a tetrahedron. We shall prove that we have the strict inequality

$$ab\cdot cd + bc\cdot ad > ac\cdot bd. \tag{5}$$

In other words, *the products of the lengths of the three pairs of opposite edges of a tetrahedron always form the lengths of the sides of a triangle.*

In the figure, imagine the edge bd as a taut flexible string, with the remaining edges of the tetrahedron being rigid. Rotate vertex d about the axis ac until it lies in the plane of the base abc at, say, d', choosing the direction of rotation so that $bd' > bd$. Applying Ptolemy's inequality (2) to the quadrilateral $abcd'$ and noting that $cd = cd'$, $ad = ad'$, we find

$$ab\cdot cd + bc\cdot ad \geqq ac\cdot bd' > ac\cdot bd,$$

which proves (5).

4. Ptolemy's inequality and the chordal metric. The chordal distance $\chi(a, b)$ between two complex numbers a and b (see [1], p. 81, or [5], p. 42) is given by the equation

$$\chi(a, b) = \frac{|a-b|}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}}. \tag{6}$$

In this section we prove that when a, b, c are three noncollinear points in the complex plane, the triangle inequality for the chordal metric,

$$\chi(a, b) < \chi(a, c) + \chi(c, b), \tag{7}$$

is equivalent to the tetrahedral theorem discussed in the foregoing section. Using (6) we see that (7) is equivalent to the inequality

$$|a-b|\sqrt{1+|c|^2} < |a-c|\sqrt{1+|b|^2} + |c-b|\sqrt{1+|a|^2}. \tag{8}$$

Construct a tetrahedron using as vertices the three points a, b, c in the complex plane and a fourth point d located at a distance 1 above the origin of the complex plane. The edges of this tetrahedron in the complex plane have lengths $|a-b|$, $|a-c|$, and $|c-b|$. The other three edges meeting at d have lengths

$\sqrt{1+|c|^2}$, $\sqrt{1+|b|^2}$, and $\sqrt{1+|a|^2}$. Therefore we see at once that Ptolemy's tetrahedral inequality implies (8). Conversely, inequality (8) implies Ptolemy's tetrahedral inequality for a tetrahedron with altitude 1. This, in turn, implies Ptolemy's inequality for a general tetrahedron.

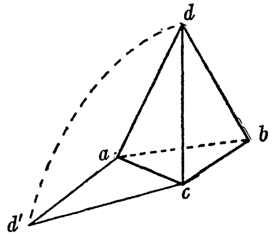

FIG. 1

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PASCAL-TYPE TRIANGLES FOR THE FOURIER EXPANSIONS OF $2^{n-1}\cos^n x$ AND $2^{n-1}\sin^n x$

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For positive integral n, the well known Pascal Triangle is a useful way of exhibiting coefficients of products of x and y in the expansion of $(x+y)^n$, and also provides a simple rule for computing these in terms of coefficients in the expansion of $(x+y)^{n-1}$, $n\geqq 1$. Here, after a short introduction to the Pascal Triangle, for the purpose of showing the similarities to, and differences from, our results, we present two new triangles, both of the Pascal type, which provide almost equally simple rules for rapid and easy computation of coefficients in the Fourier expansions of $2^{n-1}\cos^n x$ and $2^{n-1}\sin^n x$ in terms of coefficients in the Fourier expansions of $2^{n-2}\cos^{n-1} x$ and $2^{n-2}\sin^{n-1} x$, respectively. The procedures for entering numbers in the triangles are not entirely obvious, so they are derived for the Fourier expansion of $2^{n-1}\cos^n x$.