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Author(s): Tom M. Apostol

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NOTE ON THE TRIVIAL ZEROS OF DIRICHLET L-FUNCTIONS

TOM M. APOSTOL

ABSTRACT. The trivial zeros of Dirichlet L-functions are located without use of the functional equation.

If χ is a Dirichlet character mod k, then the Dirichlet L-function $L(s,\chi)$ has trivial zeros whose location is determined by the number $\chi(-1)$ which is ± 1. If $\chi(-1)=1$ there are zeros at the points $s=0,\ -2,-4,-6,\ldots$, and if $\chi(-1)=-1$ there are zeros at the points $s=-1,\ -3,-5,-7,\ldots$. In other words, for every integer $n\geqslant 0$,

$$\chi(-1)=(-1)^n \quad \text{implies} \quad L(-n,\chi)=0. \tag{1}$$

This result is usually derived from the functional equation for the L-functions which is valid only for primitive characters. A separate argument is then needed to treat the nonprimitive characters (see [**2**, pp. 215–216]).

This note gives a short proof of (1) which does not require the functional equation for L-functions and which applies to all Dirichlet characters, primitive or not. It makes use of the representation [**1**, p. 255]

$$L(s,\chi)=k^{-s}\sum_{r=1}^{k-1}\chi(r)\zeta(s,r/k) \tag{2}$$

which holds for all complex s and all Dirichlet characters χ mod k. Here $\zeta(s,a)$ is the Hurwitz zeta function defined for $R(s)>1$ by the series

$$\zeta(s,a)=\sum_{n=0}^{\infty}(n+a)^{-s},$$

where $0<a\leqslant 1$; it can be extended as a meromorphic function to the entire s-plane by the contour integral

$$\zeta(s,a)=\frac{\Gamma(1-s)}{2\pi i}\int_C\frac{z^{s-1}e^{az}}{1-e^z}\,dz, \tag{3}$$

where C is a loop around the negative real axis [**1**, p. 253]. The only property of $\zeta(s,a)$ we need to prove (1) is the formula

$$\zeta(-n,1-a)=(-1)^{n+1}\zeta(-n,a) \tag{4}$$

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for integer n. This follows at once by replacing z by $-z$ in the contour integral (3).

Taking $s = -n$ in (2) and using (4) we find

$$L(-n,\chi) = k^n \sum_{r=1}^{k-1} \chi(r)\zeta\left(-n,\frac{r}{k}\right) = k^n \sum_{r=1}^{k-1} \chi(k-r)\zeta\left(-n,\frac{k-r}{k}\right)$$
$$= k^n\chi(-1)(-1)^{n+1} \sum_{r=1}^{k-1} \chi(r)\zeta\left(-n,\frac{r}{k}\right)$$
$$= -\chi(-1)(-1)^n L(-n,\chi).$$

If $\chi(-1) = (-1)^n$ we get $L(-n,\chi) = -L(-n,\chi)$, which proves (1).

References

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Department of Mathematics, California Institute of Technology, Pasadena, California 91125

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