Tanvolutes: Generalized Involutes

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Abstract. The classical involute of a plane base curve intersects every tangent line at a right angle. This paper introduces a tanvolute, which intersects every tangent line at any given fixed angle. This minor change in the definition of a classical concept leads to a wealth of new examples and phenomena that go far beyond the original situation. Our treatment is based on two differential equations relating arclength functions for the base curve and its tanvolute, the tangent-length function from the base to the tanvolute, and the fixed angle. The parameters in the differential equations contribute many essential features to the solution curves. Even when the base curve is relatively simple, for example a circle, the variety in the shapes of the tanvolutes is remarkably rich. To illustrate, as a circle shrinks to a single point, its tanvolute becomes a logarithmic spiral! An application is given to a generalized pursuit problem in which a missile is fired at constant speed in an unknown tangent direction from an unknown point on a base curve. Surprisingly, it can always be intercepted by a faster constant-speed missile that follows a specific tanvolute of the base curve.

1. INTRODUCTION. This paper generalizes the concept of involute, illustrated in Figure 1a. Start with a curve τ, called the *tangency curve*, along which a tangent vector moves continuously. The free end of the tangent vector traces a curve σ, called the *free-end curve*, which depends on the length of the tangent vector. If σ cuts all tangents to τ at right angles it is called an *involute* of τ. Each tangency curve has infinitely many involutes parallel to one another. Every normal to σ is tangent to τ, so τ is the envelope of normals to the involute. An involute can be realized as the path traced by a point on a taut *inelastic string* unwrapped from τ, or in reverse, wrapped around τ.

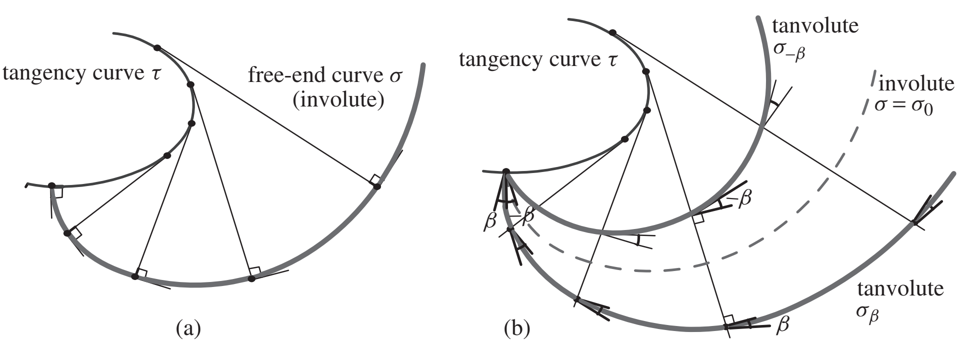

Figure 1. (a) Classical involute of τ. (b) Tanvolutes of τ formed by unwrapping an elastic string from τ so that its free end moves at a constant angle of attack $\pm\beta$, where $\beta > 0$.

In this paper we unwrap an *elastic string*, as suggested in Figure 1b, by changing its length in such a way that its free end moves at a *constant angle* β with the normal to the string, which is tangent to the involute through that point. We refer to angle β as the *angle of attack*, terminology borrowed from aerodynamics. In general, we allow β to vary from point to point. When β is constant we call the resulting free-end curve,

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which depends on β, a *tanvolute*, or more specifically, a β-*tanvolute*, and we denote it by σ_β. The word "tanvolute" is a blending of tangent and volute.

It suffices to assume that $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. When $\beta = 0$, tanvolute σ_0 is the classical involute of τ. As β increases from $-\pi/2$ to 0, the tanvolutes form a one-parameter family of curves intermediate to the tangency curve and its involute σ_0. As β continues to increase from 0 to $\pi/2$, the tanvolutes form another one-parameter family expanding outward beyond the involute as indicated in Figure 1b. As $\beta \to -\pi/2$, the tanvolute becomes a tangent ray, and then wraps around τ counterclockwise. As $\beta \to \pi/2$, the behavior is similar except that it wraps around τ clockwise.

Figure 1a shows a classical involute of τ, and Figure 1b shows two β-tanvolutes of τ corresponding to two values of β of opposite sign.

Figure 2 shows examples of tanvolutes that occur in a well-known pursuit problem. Here n ants start at the vertices of a regular n-gon, each ant pursuing its nearest clockwise neighbor at the same speed. At any time, the ants are at the vertices of a smaller similar rotated n-gon, shown for $n = 4$, 3, and 6 in Figure 2. In each case, ant 1 traces a logarithmic spiral τ toward the center, while its neighbor, ant 2, traces the β-tanvolute of τ with constant β (by similarity), given by $\beta = 2\pi/n - \pi/2$ radians. A new application of tanvolutes to pursuit problems is given in Section 7.

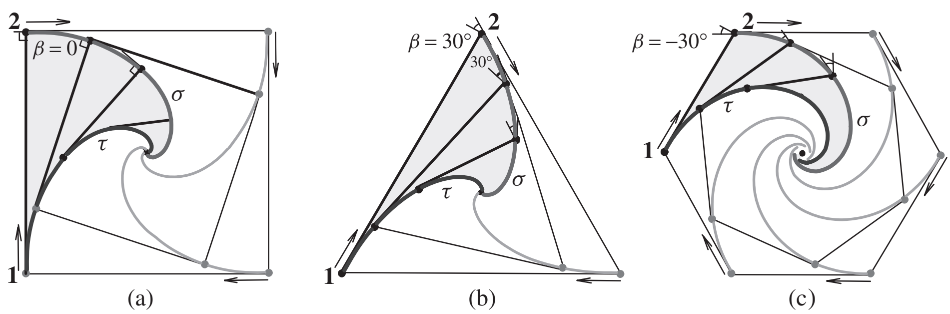

Figure 2. Pursuit problem with n ants at the vertices of a regular n-gon. While ant 1 pursues ant 2, ant 2 traces the β-tanvolute of the path of ant 1.

As we show later, a logarithmic spiral can be regarded as a tanvolute of a single point. Thus, β-tanvolutes also generalize the logarithmic spiral, with its pole replaced by a curve.

2. THE BASIC FUNCTIONS AND THREE BASIC PROBLEMS. Our analysis of tanvolutes is framed in the context of the method of sweeping tangents. We used this method in earlier papers [**2**–**5**, **7**] to calculate areas of many classical regions described by geometric properties rather than by equations. In [**5**] we extended the method to determine arclengths and intrinsic equations of many classical curves. To keep the present paper self-contained, we reformulate the basic ideas here.

As in Section 1, we begin with a tangency curve τ, together with a moving tangent line. Now we assume the moving tangent point on τ is described by a position vector $\mathbf{x}$, as indicated in Figure 3. Denote by $l = l(\alpha)$ the arclength function of τ, where α is the angle between the moving tangent and some initial direction, chosen so that $l(0) = 0$. As the point of tangency moves continuously along τ, we assume the tangent vector angle α changes monotonically, increasing when the tangent turns counterclockwise. As α varies from $-\infty$ to $+\infty$, the arclength function can be positive or negative. The absolute value $|l(\alpha)|$ tells us how far the point of tangency moves along τ when the turning angle changes from 0 to α. It is common terminology to say that l, as a function of α, provides a *natural* or *intrinsic description* of τ. The pair $(\alpha, l(\alpha))$ represents intrinsic coordinates of τ.

The derivative $d\mathbf{x}/dl$ is equal to $\mathbf{T}$, the unit tangent vector, as indicated in Figure 3. The derivative of $\mathbf{T}$, in turn, is given by $d\mathbf{T}/dl = \kappa\mathbf{N}$, where $\kappa = d\alpha/dl$ is the curvature, and $\mathbf{N}$ is the principal unit normal, as in Figure 3.

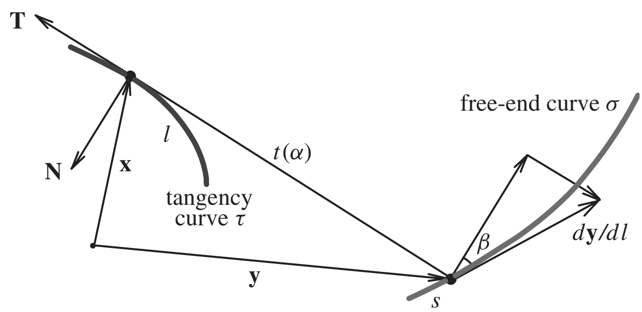

Figure 3. Tangency curve τ described by position vector $\mathbf{x}$, and free-end curve σ described by position vector $\mathbf{y} = \mathbf{x} - t\mathbf{T}$, where $\mathbf{T}$ is the unit tangent vector of τ.

Denote by $t = t(\alpha)$ the function whose absolute value is the length of the tangent vector from τ to the free-end curve σ. The vector $\mathbf{y} = \mathbf{x} - t\mathbf{T}$ is the position vector of curve σ, as shown in Figure 3. Denote by $s = s(\alpha)$ the corresponding arclength function of σ with respect to angle α, measured so that $s(0) = 0$. Unlike $l(\alpha)$, which gives an intrinsic description of τ, the function $s(\alpha)$ does not always provide an intrinsic description of σ because α is not necessarily the angle swept by the tangent to σ. In this paper we assume that each of the functions l, t, and s is continuously differentiable.

At each point where the tangent intersects σ, let β denote the angle of attack as described in Section 1, with $|\beta| < \pi/2$. For tanvolutes, β is constant. But in Sections 3 and 10, we allow β to vary from point to point and regard β as a function of α.

This paper relates the four functions β, t, l, and s through two differential equations in Theorem 1. When β is constant, we treat three problems in which we specify one of t, l, s and determine the other two.

3. BASIC DIFFERENTIAL EQUATIONS.

Theorem 1. *The four functions l, s, t, and β are related by the differential equations*

$$\frac{dt}{d\alpha} - \frac{dl}{d\alpha} = t \tan\beta \tag{1}$$

and

$$\frac{ds}{d\alpha} = \frac{t}{\cos\beta}. \tag{2}$$

This theorem was proved in [**5**] by a graphical method using linear approximations to the curves. Here we give a brief analytic argument.

Proof. Refer to Figure 3, where $\mathbf{x}$ is the position vector of the moving point of tangency on τ, and $\mathbf{y} = \mathbf{x} - t\mathbf{T}$ is the corresponding position vector of the free-end curve

σ. Differentiation with respect to l gives

$$\frac{d\mathbf{y}}{dl} = \left(1 - \frac{dt}{dl}\right)\mathbf{T} - t\kappa\mathbf{N} = \kappa\left[\left(\frac{dl}{d\alpha} - \frac{dt}{d\alpha}\right)\mathbf{T} - t\mathbf{N}\right]. \tag{3}$$

The vector $d\mathbf{y}/dl$, shown tangent to σ in Figure 3, is the sum of two perpendicular vectors, one parallel to $\mathbf{T}$ and another parallel to $\mathbf{N}$ in the directions indicated. The three vectors form a right triangle, with angle β adjacent to the hypotenuse. From this triangle and (3) we deduce (1). Because the length of $d\mathbf{y}/dl$ is ds/dl, the same triangle shows that $(ds/dl)\cos\beta = \kappa t$. But $ds/dl = \kappa ds/d\alpha$, and we obtain (2). ■

4. CONSTANT ANGLE OF ATTACK: β-TANVOLUTES. In general, β can vary with α, but for tanvolutes we keep β constant. Then as the tangent to τ sweeps through angle α, the corresponding tangent to the β-tanvolute sweeps through the same angle, as shown in Figure 4. This geometric property plays an important role in the analysis because, for constant β, arclength s yields an intrinsic description of σ_β.

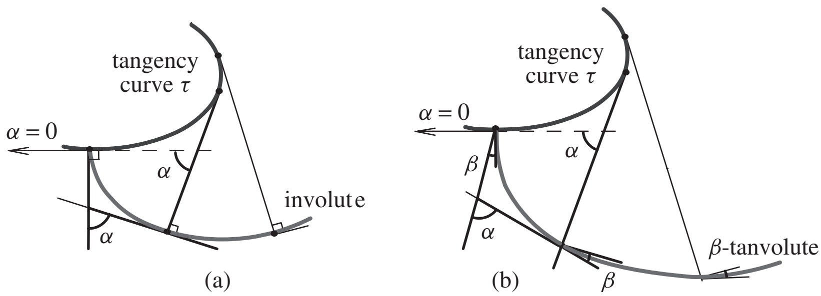

Figure 4. For constant β, when the tangent to τ sweeps through any angle α, the corresponding tangent to the β-tanvolute sweeps through the same angle α.

For constant β, we introduce constants c and γ, where

$$c = \tan\beta, \quad \gamma = \frac{1}{\cos\beta}.$$

Then differential equations (1) and (2) become linear in t, l, and s:

$$\frac{dt}{d\alpha} - \frac{dl}{d\alpha} = ct, \quad \frac{ds}{d\alpha} = \gamma t.$$

As mentioned earlier, we treat three problems, in which we specify one of these functions and determine the other two. The first specifies l and determines t and s, which we denote as t_β and s_β. The other two are discussed in Sections 8 and 9.

5. PROBLEM 1. FINDING β-TANVOLUTES OF A GIVEN CURVE.

Problem 1. *For constant β and a given function l with $l(0) = 0$, and a given real T, determine t_β and s_β satisfying* (1) *and* (2) *with initial conditions*

$$t_\beta(0) = T, \quad s_\beta(0) = 0.$$

Solution. Differential equation (1) is first-order linear for t_β with solution

$$t_\beta(\alpha) = l(\alpha) + Te^{c\alpha} + ce^{c\alpha}\int_0^\alpha l(\theta)e^{-c\theta}d\theta. \tag{4}$$

Once t_β is known we can determine s_β by integrating γt_β, as suggested by (2). However, to avoid integrating an integral, we determine s_β somewhat differently. First use (2) to replace $t_\beta(\alpha)$ on the right of (1) and obtain the following version of (1):

$$\frac{dt_\beta}{d\alpha} = \frac{dl}{d\alpha} + \frac{c}{\gamma}\frac{ds_\beta}{d\alpha}.$$

Integrate this over the interval $[0, \alpha]$ and use the prescribed initial conditions to get

$$t_\beta(\alpha) - T = l(\alpha) + \frac{c}{\gamma}s_\beta(\alpha). \tag{5}$$

If $\beta = 0$, then $c = 0$, $\gamma = 1$, (5) becomes $t_0(\alpha) = T + l(\alpha)$, and $s_0(\alpha)$ is obtained by integration:

$$s_0(\alpha) = T\alpha + \int_0^\alpha l(\theta)d\theta. \tag{6}$$

If $\beta \neq 0$, then $c \neq 0$ and we can solve for $s_\beta(\alpha)$ in (5), and use (4) to obtain

$$s_\beta(\alpha) = \gamma T\frac{e^{c\alpha}-1}{c} + \gamma e^{c\alpha}\int_0^\alpha l(\theta)e^{-c\theta}d\theta. \tag{7}$$

When $c \to 0$ in (7) we get (6) as a limiting case. Thus, (4) and (7) provide the solution to Problem 1.

The rightmost term in each of (4) and (7) contains the convolution integral

$$I(\alpha) = \int_0^\alpha l(\theta)e^{c(\alpha-\theta)}d\theta; \tag{8}$$

hence (7) and (4) can also be written, respectively, as

$$s_\beta(\alpha) = \gamma T\frac{e^{c\alpha}-1}{c} + \gamma I(\alpha), \tag{9}$$

and

$$t_\beta(\alpha) = l(\alpha) + Te^{c\alpha} + cI(\alpha). \tag{10}$$

When $\alpha = 0$, these give us $s_\beta(0) = 0$ and $t_\beta(0) = T$, as expected. The free endpoint of the string starts at distance $|T|$ from the tangency curve. We always take $\alpha = 0$ in the direction of the initial tangent to the tanvolute.

6. EXAMPLES ILLUSTRATING PROBLEM 1. Next we consider examples for which the convolution integral $I(\alpha)$ in (8) is easily evaluated. We begin with a circle, whose arclength function is linear in α.

Example 1 (Tanvolutes of a circle). For a circle of radius r the arclength is $l(\alpha) = r\alpha$, and integral (8) becomes

$$I(\alpha) = re^{c\alpha}\int_0^\alpha \theta e^{-c\theta}d\theta = r\frac{e^{c\alpha}-1}{c^2} - \frac{r\alpha}{c},$$

which, when used in (9) and (10), gives us the formulas

$$s_\beta(\alpha) = \gamma\left(T + \frac{r}{c}\right)\frac{e^{c\alpha}-1}{c} - \frac{\gamma r\alpha}{c}, \tag{11}$$

$$t_\beta(\alpha) = Te^{c\alpha} + r\frac{e^{c\alpha}-1}{c}. \tag{12}$$

Classical case: Involute of a circle. When $\beta = 0$ we have, from (5) and (6),

$$t_0(\alpha) = T + r\alpha, \quad s_0(\alpha) = T\alpha + \frac{1}{2}r\alpha^2. \tag{13}$$

Figure 5 shows how the involute of a circle depends on T.

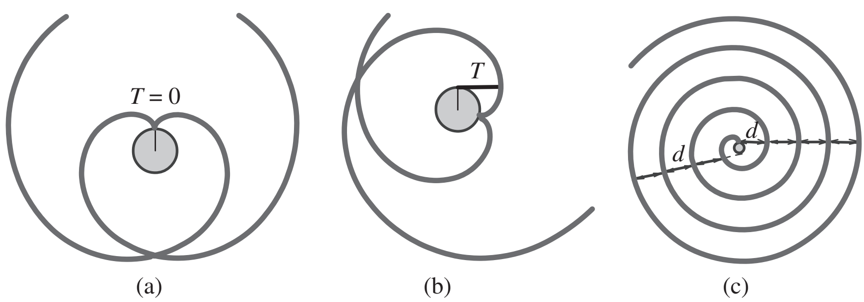

Figure 5. Involutes of a circle depending on T. (a) When $T = 0$ the involute has two branches spiraling outward symmetrically from the point of contact on the circle. (b) When $T > 0$, the involute is a rotated version of that in (a), spiraling outward from a different point on the circle. (c) Bird's eye view showing one of the spirals with constant spacing between its arms.

Special tanvolute: No exponential influence. In both (11) and (12), the exponential term $e^{c\alpha}$ disappears when $T = -r/c$. For this special value of T, which we label T_*, we denote the arclength function in (11) by s_β^* and the corresponding tanvolute by σ_β^*, which we call a *special tanvolute*. Because $c/\gamma = \sin\beta$, from (11) we find

$$s_\beta^*(\alpha) = R_\beta\alpha,$$

where $R_\beta = -r/\sin\beta$. Hence the special tanvolute σ_β^* is a concentric circle, shown in Figure 6b, of radius

$$R_* = \frac{r}{|\sin\beta|}. \tag{14}$$

General tanvolute of a circle. Next we analyze the asymptotic behavior of (11) for large $|\alpha|$ when the exponential term is present (which implies $\beta \neq 0$). Either the exponential term dominates (when $c\alpha > 0$), or it tends to 0 (when $c\alpha < 0$), in which case the linear term dominates. Geometrically, this means that a general tanvolute spirals away from the original circle in unbounded fashion in one direction, and tends asymptotically to the special circular tanvolute in the other direction.

Figure 6 gives examples with $\beta = -30°$, so $c < 0$.

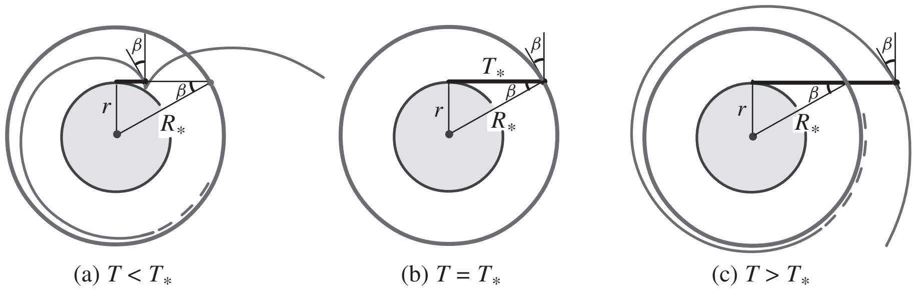

Figure 6. Here $\beta = -30°$. (a) When $T < T_*$, the tanvolute spirals outward, toward the special circular tanvolute if $\alpha > 0$, but in unbounded fashion if $\alpha < 0$. (b) When $T = T_*$, the tanvolute is the special tanvolute. (c) For $T > T_*$, the tanvolute spirals inward toward the special tanvolute if $\alpha > 0$, but spirals outward in unbounded fashion if $\alpha < 0$.

Example 2 (Tanvolutes of a single point are logarithmic spirals). Now we take the tangency curve to be a single point O, which can be regarded as the limiting case of a small circular arc τ whose radius shrinks to zero. As $r \to 0$ in (11) we find that the arclength function of the tanvolute is

$$s_\beta(\alpha) = \frac{T}{\sin\beta}(e^{c\alpha} - 1). \tag{15}$$

The coefficient $T/\sin\beta$ in (15) has a simple geometric meaning. Let L denote the arclength of the tanvolute between O ($\alpha = -\infty$) and the point on the tanvolute where $\alpha = 0$. From (15) we see that $L = s_\beta(0) - s_\beta(-\infty) = T/\sin\beta$, and

$$s_\beta(\alpha) = L(e^{c\alpha} - 1).$$

Similarly, (12) gives the limiting case $t_\beta(\alpha) = Te^{c\alpha} = L\sin\beta\, e^{c\alpha}$. In this limiting case, $t_\beta(\alpha)$ becomes the radial distance $r_\beta(\alpha)$ from point O to σ_β. In general, the tanvolute has polar equation

$$r_\beta(\alpha) = Te^{c\alpha},$$

where $T = L\sin\beta$. When $\beta = 0$, $s_0(\alpha) = T\alpha$, giving a circle of radius T, as in Figure 7a. But when $\beta \neq 0$, this is the polar equation of a logarithmic spiral, shown in Figure 7b for $\beta > 0$. This agrees with the geometric definition of a logarithmic spiral as a curve cutting all its polar radial lines at a constant angle of attack.

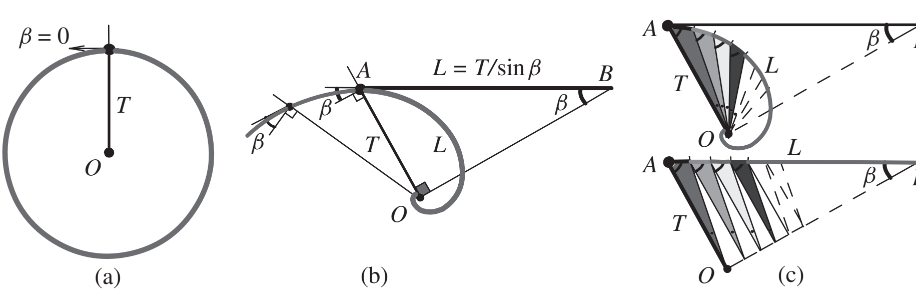

Figure 7. Tanvolutes of a single point: (a) Circle as involute. (b) Logarithmic spiral as a β-tanvolute. (c) Arclength and area obtained geometrically by an unfolding process.

Figure 7c shows the region between the spiral arc OA of length L and the initial segment OA, dissected into tiny triangles with equal vertex angles (as Archimedes did with a circular disk) and unfolded to fill part of right triangle AOB. This shows geometrically why AB has length L and why the region between spiral arc OA and the initial tangent has area equal to half that of triangle OAB, a known result that can also be verified analytically.

7. APPLICATIONS OF TANVOLUTES TO PURSUIT PROBLEMS. We turn now to an application generalizing a remarkable pursuit problem that appears in the literature in various forms and under various names, e.g., the trawler problem, the rum-runner problem, or the anti-missile problem (Exercise 16 in [**1**, p. 544]).

The problem is of special interest because at first glance it seems that the given information is insufficient to solve it. We shall generalize all forms as an anti-missile problem and solve it with the help of tanvolutes.

Generalized pursuit problem. A military base is equipped with a horizontal convex track around a mountain (Figure 8a), along which a missile can fly (clockwise) at constant speed v and can be aimed, at the same speed v, toward a target along a line tangent to the track (Figure 8b). A missile, mistakenly fired, starts at point S and flies along the track to some unknown point on the track from which it proceeds along the tangent direction, which is also unknown (Figure 8b). To destroy the first missile, an anti-missile missile is fired from S exactly two seconds later at constant speed $V > v$. Unlike the first missile, the second missile can fly along any path.

Figure 8. (a) Horizontal missile track on a mountain. (b) Top view showing the track as a convex tangency curve. The tangent vectors indicate possible directions of missile 1.

What path should the second missile follow to overtake the first one?

First, we try to determine where missile 1 could be at any given moment. In Figure 9a, suppose that, at a particular moment, missile 1 has moved a distance a along the base track plus a distance b along the unknown direction of tangency. Then at that moment missile 1 must lie somewhere on the involute to the tangency track along which $a + b$ is constant, as though a string of length $a + b$ is unwrapped from the track. These involutes expand uniformly with time as indicated by the dashed curves in Figure 9a.

There are several strategies available to overtake missile 1. The simplest assumes that missile 1 travels from starting point S and moves a distance SQ along the track to some unknown point Q, then proceeds along the tangent at Q, as in Figure 9b. Then missile 2 travels along the track from S to the point P where missile 1 would have been had it stayed on the track. Point P can be determined because the missile speeds v and V are known.

If missile 2 hits missile 1 at P, the pursuit is over. Otherwise, missile 2 changes course and moves from P along a β-tanvolute, which will intersect all the tangent lines

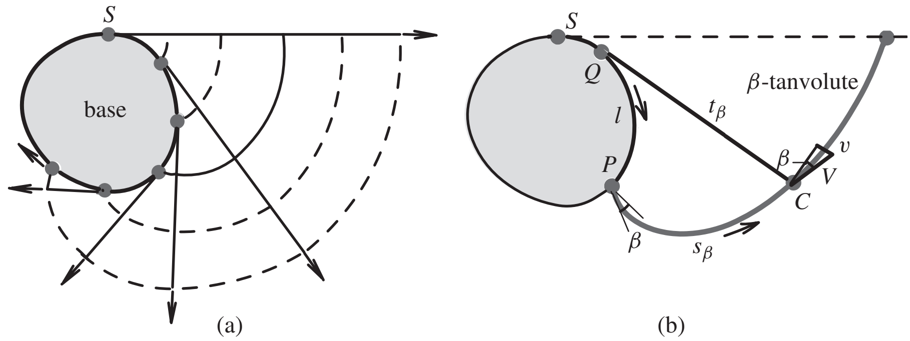

Figure 9. (a) Possible time-loci of missile 1 are involutes of the base track, expanding uniformly with time. (b) Missile 2 proceeds to point P on the base track where interception would occur if missile 1 was still on the track, and then follows the β-tanvolute of the track.

along which missile 1 might travel. Of course, we need to choose β so that missile 2 collides with missile 1 at some point, denoted by C in Figure 9b. The small triangle suggests that β should be chosen so that $\sin\beta = v/V$. The reason is that missile 1 moves tangentially a distance v in unit time while missile 2 moves along the β-tanvolute a distance V, forming a leg and hypotenuse of a small right triangle with adjacent angle whose sine is v/V. The adjacent angle is the angle of attack β.

We can also obtain this choice of β analytically. Missiles 1 and 2 are to reach C simultaneously. Missile 1 travels distance SQ along the track plus distance $t_\beta(\alpha)$ along the tangent ray. The time required to cover this distance at constant speed v is

$$\frac{SQ + t_\beta(\alpha)}{v}.$$

Missile 2 travels distance SP from S to P, then distance $s_\beta(\alpha)$ from P to C, where α is measured so that $\alpha = 0$ at P. The time required to cover this distance at constant speed V is

$$\frac{SP + s_\beta(\alpha)}{V}.$$

Because missile 2 is fired later than missile 1, say with time delay Δ, collision will occur at C if

$$\frac{SP + s_\beta(\alpha)}{V} - \frac{SQ + t_\beta(\alpha)}{v} = \Delta. \tag{16}$$

But Δ is the difference in times required for the two missiles to travel from S to P:

$$\Delta = \frac{SP}{v} - \frac{SP}{V}.$$

Use this in (16), together with $SQ = SP - l(\alpha)$. Then (16) simplifies to

$$\frac{s_\beta(\alpha)}{V} = \frac{t_\beta(\alpha) - l(\alpha)}{v}.$$

Comparing this with (5), with $T = 0$, we find

$$\frac{v}{V} = \frac{c}{\gamma} = \sin\beta,$$

the same result suggested geometrically.

The pursuit problem can be solved similarly if missile 2 starts at any point off the track (now the time delay is not necessary). Missile 2 proceeds directly to the track and then follows the track to the point where missile 2 would hit missile 1 had missile 1 stayed on the track. If it hits, the pursuit is over. Otherwise, missile 2 follows the β-tanvolute at that point with $\sin\beta = v/V$. Collision will occur at the latest at the tangent line through S, shown dashed in Figure 9b.

In the limiting case when the tangency curve τ reduces to a single point, the expanding involutes are concentric circles (see Figure 7), and the problem reduces to the classical anti-missile problem, or to the other two classical problems mentioned earlier. In each of these classical problems, the β-tanvolute is a logarithmic spiral, the tanvolute of a single point.

8. PROBLEM 2. FINDING THE TANGENCY CURVE WHEN ITS β-TANVOLUTE IS KNOWN. In Section 4, we specified the function l in differential equations (1) and (2) and solved for s and t, keeping β constant. Now we specify s, and determine t and l, which we denote by t_β and l_β. In other words, Problem 2 specifies the β-tanvolute σ_β, because it prescribes s and β, and asks for the tangency curve τ (through its intrinsic equation $l = l(\alpha)$), as well as the corresponding tangent length function t. Figure 1b also illustrates the geometric meaning of Problem 2.

Problem 2. *For constant β and a given function s with $s(0) = 0$, determine t_β and l_β satisfying* (1) *and* (2) *with initial condition*

$$l_\beta(0) = 0.$$

Solution. We obtain t_β at once from (2) which gives

$$t_\beta = \cos\beta\,\frac{ds}{d\alpha}.$$

Using this in (1) and integrating over the interval $[0, \alpha]$ we find

$$l_\beta(\alpha) = t_\beta(\alpha) - t_\beta(0) - \sin\beta\, s(\alpha).$$

Expressed in terms of s, this becomes

$$l_\beta(\alpha) = \cos\beta(s'(\alpha) - s'(0)) - \sin\beta\, s(\alpha).$$

When $\beta = 0$ this gives $l_0(\alpha) = s'(\alpha) - s'(0)$, and the foregoing equation becomes

$$l_\beta(\alpha) = -s(\alpha)\sin\beta + l_0(\alpha)\cos\beta. \tag{17}$$

Thus, the arclength function l_β is a linear combination of the arclength functions for σ and for the curve whose involute is σ.

Similarly, we find $t_\beta(\alpha) = t_0(\alpha)\cos\beta$, which tells us that the length of the tangent segment from τ to the β-tanvolute is smaller than the tangent segment from τ to the involute by a factor $\cos\beta$. The maximum tangent length occurs when $\beta = 0$. Each example illustrating Problem 1 also illustrates Problem 2.

Evolutoids. The tangency curve τ is the envelope of inclined normals to the given tanvolute σ_β, the constant angle of inclination being β. The problem of finding the envelope of normals turned by a fixed angle to a given curve σ appeared in the literature in work by Reaumur in 1709 and by Lancret in 1811. (See [**6**, p. 297].) In this earlier work, the envelope is referred to as the *evolutoid* or *developpoid* of σ. For $\beta = 0$ the evolutoid becomes the classical evolute of σ.

Reaumur was the first to show that the evolutoid of a circle is a concentric circle. But apparently no one realized that many other curves have a given circle as evolutoid. This is illustrated in Figure 6, which shows a family of β-tanvolutes of a given circle, each tanvolute obtained by varying the initial tangent length T but keeping β fixed. The circle is the evolutoid of each of these tanvolutes.

In the same way, for any tangency curve τ and any given β, there is an infinite family of β-tanvolutes, obtained by varying the initial tangent length T. All members of this family have the same tangency curve τ as β-evolutoid but with different values of T. The simplest tanvolute is the special β-tanvolute, for which $T = T_*$, so that there is no exponential influence.

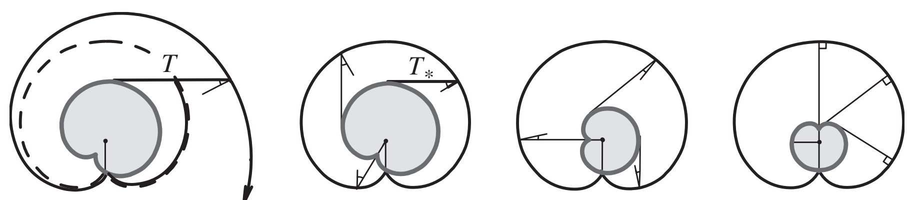

Figure 10. (a) General β-tanvolute of the shaded cardioid wraps asymptotically around the special tanvolute (shown dashed). (b) Special tanvolute of the same cardioid for the same β. In (b), (c), and (d): the large cardioid is the special β-tanvolute of three different smaller cardioids.

In Figures 10a and 10b the cardioid with interior shading is the evolutoid of two different β-tanvolutes for the same β, the second being the special β-tanvolute. In Figures 10b, 10c, and 10d, each of the smaller cardioids is an evolutoid of the same larger cardioid for different values of β.

9. PROBLEM 3. FINDING β-TANVOLUTES WHEN t IS KNOWN. We turn next to the third in our trilogy of problems. For fixed β, this one specifies t and determines both s and l, which we denote by s_β and l_β.

Problem 3. *For constant β and a given function t with $t(0) = T$, determine s_β and l_β satisfying* (1) *and* (2) *with initial conditions*

$$s_\beta(0) = 0, \quad l_\beta(0) = 0.$$

Solution. By integrating (1) and (2) we find

$$s_\beta(\alpha) = \gamma \int_0^\alpha t(\theta)d\theta, \quad l_\beta(\alpha) = t(\alpha) - T - c\int_0^\alpha t(\theta)d\theta,$$

where $\gamma = 1/\cos\beta$ and $c = \tan\beta$.

This simple solution reveals a surprising geometric fact: a knowledge of the tangent length function t alone determines intrinsically both the tangency curve and its β-

tanvolute, except for their position and orientation in the plane. Examples illustrating Problems 1 and 2 also illustrate Problem 3.

In particular, if t is constant, say $t = T$, we find $s_\beta(\alpha) = \gamma T\alpha$ and $l_\beta(\alpha) = -cT\alpha$, each of which represents the arclength of a circle. If $t(\alpha)$ is a linear function of α, say $t(\alpha) = T + r\alpha$, where T and r are constants, then $s_\beta(\alpha) = \gamma T\alpha + \gamma r\alpha^2/2$, $l_\beta(\alpha) = r\alpha - cr\alpha^2/2$, each of which is the arclength function of an involute of a circle.

10. VARIABLE ANGLE OF ATTACK β. Beginning in Section 4 we considered only constant angle of attack β. But the basic differential equations (1) and (2) in Theorem 1 also hold when β is not constant. This opens the door to further research by choosing β to be a specific function of α.

In the general discussion it is important to establish the tangency angle ψ of the curve σ so that (ψ, s) provides intrinsic coordinates of σ. From a diagram similar to that in Figure 4 it is easy to see that in the most general case the tangency angle ψ (with respect to some initial direction) is given by

$$\psi = \alpha - \beta,$$

where α and β are as in Figure 4b, with β not necessarily constant.

In [**5**] we treat several examples with $\beta = \alpha$, in which case $\psi = 0$ and the free-end curve σ is a straight line. This leads to simple derivations of known formulas for the arclength $l = l(\alpha)$ of a cycloid (proportional to $\sin\alpha$), catenary (proportional to $\tan\alpha$), and tractrix (proportional to $\log(\cos\alpha)$). More complicated arclength functions are given for an exponential curve, a parabola, and generalized pursuit curves, all with $\beta = \alpha$.

The arclength of an epicycloidal arc traced by a point on the boundary of a disk of radius r rolling along the outer circumference of a fixed circle of radius R corresponds to $\beta = \alpha/\kappa$, where $\kappa = 1 + 2r/R$. A corresponding result is obtained for the arclength of a hypocycloidal arc in which the circle of radius r rolls inside the circumference of the fixed circle of radius R. In this case $\beta = \alpha/\kappa$, where $\kappa = 1 - 2r/R$. In both cases, $\psi = \alpha(1 - 1/\kappa)$ and the arclength is proportional to $\sin\beta$.

11. CONCLUDING REMARKS. The results in this paper and in [**5**] underscore the importance of intrinsic equations in the study of curves. Every plane curve can be regarded as the path of a moving particle. The intrinsic equation connects the distance covered by the particle with the angle through which the tangent line turns, and provides the most natural description of the curve because it does not rely on any external coordinate system. Our treatment of tanvolutes was considerably simplified by using intrinsic equations.

In Section 6 we considered examples of tanvolutes in which the arclength function l of the tangency curve τ is such that the convolution integral in (8) can be evaluated in closed form. The authors have also investigated many other such examples and were surprised to find that the initial value T could always be chosen so that the exponential influence in (7) disappears. We call the resulting tanvolute a special tanvolute.

In general, classical involutes are more complicated curves than their tangency curves. For example, the arclength function of a circle is linear in α but that of its involute (obtained by integration) is quadratic in α. The arclength function of the involute of the involute of a circle is cubic in α, but its special tanvolute is quadratic, which means it is the involute of a different circle. If the arclength function of a tangency curve τ is a polynomial in α of degree k, then that of its involute is a polynomial of degree $k + 1$. In contrast, the arclength function of the special tanvolute of τ involves a polynomial of degree k.

In all our examples, the special tanvolute is a scaled and rotated version of the tangency curve itself. Particular examples are the special tanvolute of a circle, a concentric circle as in Figure 6b, and the special tanvolute of a cardioid, a scaled and rotated cardioid, as in Figure 10. In Figure 2, the special β-tanvolute of a logarithmic spiral is the same spiral rotated by $\beta + \pi/2$.

Finally, we remark that elsewhere we have shown (see for example [**5**] or [**7**]) that a knowledge of $t(\alpha)$ allows us to determine the area of the *tangent sweep*, the region between the tangency curve and the free-end curve swept by the tangent segment. As α increases from α_1 to α_2, this area is given by the formula

$$\text{area of tangent sweep} = \frac{1}{2}\int_{\alpha_1}^{\alpha_2} t^2(\alpha)\,d\alpha.$$

In particular, in all three problems discussed in this paper, this gives the area of the tangent sweep.

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