Our construction admits some extensions that we mention without detail. For example, the algorithm solves the *weighted envy-free* problem when we are given positive numbers $\alpha_1, \dots, \alpha_m$ summing up to 1 and we look for a partition $C = W_1 \cup \cdots \cup W_m$ such that $\mu_i(W_i)/\alpha_i \geq \mu_i(W_j)/\alpha_j$; we let $b_i = (b - d(1 - a_i)^2)\Sigma_{P_j \in G_i} \alpha_j$, etc. Furthermore, making $d$ small, we can additionally ensure that each $\mu_i(W_j)$ is arbitrarily close to $\alpha_j$. If each player wants to minimise his share, we let $b_i = m_i(b + d(1 - a_i)^2)$, $c_i = m_i(c + da_i^2)$, etc.

Our construction can be also reworded into what Even and Paz [**2**] call a *protocol*: playing 'fair', each player $P_i$ can guarantee that $\mu_i(W_i) \geq \mu_i(W_j)$ for all $j \in [m]$ even if the other players do not consistently stick to their measures.

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# Calculating Higher Derivatives of Inverses

**Tom M. Apostol**

**1. Introduction.** Suppose we are given a function $y = f(x)$ having a Taylor expansion in powers of $x$ convergent in some neighborhood of 0, with $f(0) = 0$. A classical inversion problem is to determine whether or not there exists one and only one inverse function $x = g(y)$ expressible as a power series in $y$ that converges in some neighborhood of 0 and satisfies $f[g(y)] = y$ in that neighborhood.

The answer is well known and remarkably simple. If the first derivative $f'(0) \neq 0$, then such a function $g$ exists and is unique. But the proof is not at all obvious. The problem is discussed (and completely solved) in Knopp [**4**; pp. 184–188]. As Knopp points out, you can try a power series for $g$ with undetermined coefficients, substitute into the equation $f[g(y)] = y$, and you get a triangular system of linear equations for the coefficients that can be solved one at a time in terms of the coefficients of the series for $f$. The difficult part is to show that this new series has a positive radius of convergence.

Lagrange [**5**] first solved the problem in 1770 and gave an explicit formula for the coefficients for $g$. His result can be stated as follows:

**Lagrange's Inversion Formula.** If $y = f(x)$, where $f(0) = 0$ and $f'(0) \neq 0$, then

$$x = \sum_{n=1}^{\infty} \frac{y^n}{n!} \left[ \frac{d^{n-1}}{dx^{n-1}} \left( \frac{x}{f(x)} \right)^n \right]_{x=0}.$$

Since this is the Taylor series for $g(y)$, the coefficient of $y^n/n!$ is also the $n$th derivative of $g$ evaluated at 0. It is natural to ask the more general question of whether the higher derivatives of $g$ at an arbitrary point $y = f(x)$ can be determined directly from the derivatives of the original function $f$ without introducing power series and without having to differentiate powers of the quotient $x/f(x)$. This question was generalized by Faà di Bruno **[2]** who gave an explicit formula for determining the $n$th derivative of the composition of two functions in terms of the derivatives of the functions being composed. His formula can be used to express the $n$th derivative of $g$ in the foregoing problem as a sum extended over all partitions of $n$, although this is not the simplest way to actually determine the derivatives in practice. The problem is treated along the same lines in Riordan [**7**; pp. 177–182] and in Comtet [**3**; p. 137 and p.151, Theorem E] where a general explicit formula is given in terms of exponential Bell polynomials.

J. B. Reynolds **[6]** gives explicit formulas for calculating these derivatives up to order 5. Reynolds starts with the formula for the first derivative, $dx/dy = 1/(dy/dx)$, and uses the straightforward approach of repeatedly differentiating the quotient on the right. But Reynolds' use of different letters to denote successive derivatives obscures any general pattern that might be revealed by his examples. This note revisits the problem and shows how to determine these formulas by recursion. Our method is accessible to beginning calculus students who know the chain rule and the product formula for derivatives. It does not require a knowledge of Bell polynomials or of partitions.

Throughout this note we denote the first derivative $f'(x)$ by $f_1$, the second derivative by $f_2$, etc., so that $f_k$ denotes the $k$th derivative, $f_k^2$ its square, etc. In this notation the five formulas given by Reynolds can be expressed as follows:

$$f_1 \frac{dx}{dy} = 1. \tag{1}$$

$$f_1^3 \frac{d^2x}{dy^2} = -f_2. \tag{2}$$

$$f_1^5 \frac{d^3x}{dy^3} = 3f_2^2 - f_1 f_3. \tag{3}$$

$$f_1^7 \frac{d^4x}{dy^4} = -15f_2^3 + 10f_1 f_2 f_3 - f_1^2 f_4. \tag{4}$$

$$f_1^9 \frac{d^5x}{dy^5} = 105f_2^4 - 105f_1 f_2^2 f_3 + 15f_1^2 f_2 f_4 + 10f_1^2 f_3^2 - f_1^3 f_5. \tag{5}$$

Differentiation of (5) leads to a corresponding result for order 6, which takes the form

$$f_1^{11} \frac{d^6x}{dy^6} = -945f_2^5 + 1260f_1 f_2^3 f_3 - 210f_1^2 f_2^2 f_4 - 280f_1^2 f_2 f_3^2 + 21f_1^3 f_2 f_5 + 35f_1^3 f_3 f_4 - f_1^4 f_6. \tag{6}$$

These examples begin to reveal a general pattern. It is clear that the right member is always a polynomial in $f_1, f_2, \ldots$ with integer coefficients. There is one term that is a constant times a power of $f_2$, and the remaining terms can be arranged in increasing powers of $f_1$. There is homogeneity in the terms: if you multiply each subscript by the corresponding exponent and add all the products in a given term, the result is a constant in each formula. For example, this constant is 10 in formula (6). It's not hard to show that the formula for the $(n+1)$st derivative looks like this:

$$f_1^{2n+1}\frac{d^{n+1}x}{dy^{n+1}} = (-1)^n\frac{n!}{2^n}\binom{2n}{n}f_2^n + \cdots - f_1^{n-1}f_{n+1}.$$

The problem is to determine the intermediate terms $+\cdots$. As might be expected, this can be done by use of recursion.

**2. General solution of the derivative problem.** The following theorem is easily proved by induction on $n$.

**Theorem.** *Assume existence of all derivatives involved. Then for every integer $n \geq 1$ we have*

$$f_1^{2n-1}\frac{d^n x}{dy^n} = P_n,$$

*where $P_n$ is a polynomial in $f_1, f_2, \dots, f_n$ with integer coefficients. These polynomials can be determined successively by the recursion formula*

$$P_{n+1} = f_1 P_n' - (2n-1)f_2 P_n,$$

*where $P_1 = 1$ and $P'$ denotes differentiation with respect to $x$.*

The explicit formulas increase in complexity as $n$ increases, and some of the coefficients can be quite large. For example, we have

$$\begin{aligned} f_1^{13}\frac{d^7x}{dy^7} &= 10395f_2^6 - 17325f_1f_2^4f_3 + 6300f_1^2f_2^2f_3^2 + 3150f_1^2f_2^3f_4 \\ &\quad - 378f_1^3f_2^2f_5 - 1260f_1^3f_2f_3f_4 - 280f_1^3f_3^3 \\ &\quad + 28f_1^4f_2f_6 + 56f_1^4f_3f_5 + 35f_1^4f_4^2 - f_1^5f_7 \end{aligned} \tag{7}$$

$$\begin{aligned} f_1^{15}\frac{d^8x}{dy^8} &= -135135f_2^7 + 270270f_1f_2^5f_3 - 138600f_1^2f_2^3f_3^2 - 51975f_1^2f_2^4f_4 \\ &\quad + 6930f_1^3f_2^3f_5 + 34650f_1^3f_2^2f_3f_4 + 15400f_1^3f_2f_3^3 \\ &\quad - 630f_1^4f_2^2f_6 - 2520f_1^4f_2f_3f_5 - 1575f_1^4f_2f_4^2 - 2100f_1^4f_3^2f_4 \\ &\quad + 36f_1^5f_2f_7 + 84f_1^5f_3f_6 + 126f_1^5f_4f_5 - f_1^6f_8 \end{aligned} \tag{8}$$

$$\begin{aligned} f_1^{17}\frac{d^9x}{dy^9} &= 2027025f_2^8 - 4729725f_1f_2^6f_3 + 3153150f_1^2f_2^4f_3^2 + 945945f_1^2f_2^5f_4 \\ &\quad - 135135f_1^3f_2^4f_5 - 900900f_1^3f_2^2f_3f_4 - 600600f_1^3f_2^2f_3^3 \\ &\quad + 13860f_1^4f_2^3f_6 + 83160f_1^4f_2^2f_3f_5 + 51975f_1^4f_2^2f_4^2 \\ &\quad + 138600f_1^4f_2f_3^2f_4 + 15400f_1^4f_3^4 - 990f_1^5f_2^2f_7 - 4620f_1^5f_2f_3f_6 \\ &\quad - 6930f_1^5f_2f_4f_5 - 4620f_1^5f_3^2f_5 - 5775f_1^5f_3f_4^2 \\ &\quad + 120f_1^6f_3f_7 + 45f_1^6f_2f_8 + 210f_1^6f_4f_6 + 126f_1^6f_5^2 - f_1^7f_9. \end{aligned} \tag{9}$$

Table 5.2 of Riordan [**7**; p. 181] contains a modified form of formulas (1) through (9). Formulas (1) through (8) also occur in Comtet [**3**; p. 151]. Because considerable arithmetical calculation is involved in determining the coefficients, errors are likely to occur. Our results in (1) through (6) agree with those in Reynolds and Riordan, but Comtet has two misprints in (6)—the algebraic signs of the coefficients 945 and 1260 are reversed. Our formula (7) agrees with both Comtet and Riordan, except that Riordan has a misprint—the coefficient 378 appears incorrectly as 278. Our formula (8) agrees with both Comtet and Riordan, except that the term with coefficient 2100 is missing in Riordan. Also, the term in (9) with coefficient 15400 is missing in Riordan's table. All these are probably typographical errors, because some of the other coefficients depend on these missing terms. Because of these discrepancies with Comtet and Riordan, we developed a second set of recursion formulas that were used to check the calculation of the coefficients. Except for the discrepancies just mentioned, all the coefficients in our formulas agree with those given by Comtet and Riordan.

**Example.** These formulas are especially useful in determining Taylor coefficients of the inverse of $f$ when the successive derivatives of $f$ are easy to calculate. For example, if $f(x) = \sin x$, the successive derivatives are $f_1 = \cos x$, $f_2 = -\sin x$, $f_3 = -\cos x$, $f_4 = \sin x$, and so on. At the origin, the derivatives of even order vanish and those of odd order are alternately 1 and $-1$. If $g(x) = \arcsin x$ our formulas (1) through (9) show that the even order derivatives of $g$ vanish at the origin, while those of odd order are given by $g_1(0) = 1$, $g_3(0) = 1$, $g_5(0) = 9$, $g_7(0) = 225$, $g_9(0) = 11025$. These values quickly give

$$\arcsin x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots .$$

Calculation of these terms by standard methods requires a great deal more effort; see [**1**; Exercise 22 on p. 444].

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