MORE ON LOWER SEMI-CONTINUITY

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Whyburn [1] and Smithson [2] presented characterizations of upper and lower semi-continuity of multifunctions in terms of directed families (i.e., filter bases). It is sometimes desirable to use a characterization of semi-continuity in terms of accumulation and adherent points.

Let f be a multifunction on a topological space X onto a space Y and for $B \subseteq Y$, define $f^{-}(B) = \{x \in X \mid f(x) \text{ meets } B\}$. Then f is lower semi-continuous provided $f^{-}(B)$ is open in X, whenever B is open in Y.

THEOREM. *A multifunction f: $X \to Y$ is lower semi-continuous (lsc) if and only if whenever p is an accumulation point of $A \subseteq X$, then every point of $f(p)$ is an adherent point of $f[A] = \bigcup_{a \in A} f(a)$.*

Proof. Assume f to be lsc. Let p be an accumulation point of $A \subseteq X$. Without loss of generality assume $p \notin A$. Suppose $y \in f(p) - f[A]$ and assume there is an open neighborhood 0 of y such that $0 \cap f[A] = \emptyset$. Because f is lsc, $f^{-}(0)$ is open. Clearly $p \in f^{-}(0)$, hence $f^{-}(0) \cap A \neq \emptyset$. Let $x \in f^{-}(0) \cap A$; then $f(x)$ meets 0 and $f(x) \subseteq f[A]$, which is a contradiction. Conversely let V be an open subset of Y and suppose $f^{-}(V)$ is not open in X. Hence there exists $p \in f^{-}(V)$ such that p is an accumulation point of $X - f^{-}(V)$. According to the hypothesis, every point of $f(p)$ is an adherent point of $B = \cup (f(a) \mid a \in X - f^{-}(V)\}$. But there is a point $y \in f(p) \cap V$, and y cannot be an adherent point of B, a contradiction. Thus $f^{-}(V)$ must be open, i.e., f is lsc.

References

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A NOTE ON PERIODIC COMPLETELY MULTIPLICATIVE ARITHMETICAL FUNCTIONS

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A complex-valued function f defined on the positive integers and not identically zero is called *completely multiplicative* if

(1) $$f(mn) = f(m)f(n) \qquad \text{for all } m \text{ and } n.$$

Such a function satisfies $f(1) = 1$. An arithmetical function f is called *periodic* mod k if there is a positive integer k such that

(2) $$f(m) = f(n), \qquad \text{whenever} \qquad m \equiv n \pmod{k}.$$

Properties (1) and (2) imply that $f(n)$ is a root of unity whenever n is relatively prime to k. In fact, if $(n, k) = 1$ then $n^{\varphi(k)} \equiv 1 \pmod{k}$ so $f(n)^{\varphi(k)} = 1$. Here $\varphi(k)$ is Euler's totient.

A completely multiplicative function which is periodic mod k is called a *Dirichlet character* mod k if it has the additional property that

(3) $$f(n) = 0, \qquad \text{whenever} \qquad (n, k) > 1.$$

There are exactly $\varphi(k)$ distinct Dirichlet characters mod k. (See [1] or [2].) One of them is the *principal character* defined by (3) and the relation $f(n) = 1$ if $(n, k) = 1$.

The purpose of this note is to show that property (3) follows easily from (1) and (2) if k is the smallest positive period of f. This simple fact does not seem to be well known, or at least is not mentioned in the standard texts on number theory which treat Dirichlet characters. We also show that a Dirichlet character mod k has no positive period smaller than k if k is squarefree.

THEOREM 1. *Let f be periodic and completely multiplicative, and let k be the smallest positive period of f. Then $f(n)=0$ whenever $(n,k)>1$, so f is a Dirichlet character* mod k.

Proof. Choose n so that $(n,k)>1$ and let $d=(n,k)$. Then $n=dn_1$ and $k=dk_1$ where $1\leqq k_1<k$. Since k_1 is not a period of f there exists an m such that

$$f(m+k_1)-f(m)\neq 0.$$

But

$$f(d)\{f(m+k_1)-f(m)\}=f(dm+k)-f(dm)=f(dm)-f(dm)=0,$$

so $f(d)=0$. Therefore $f(n)=f(d)f(n_1)=0$.

The converse of Theorem 1 is not always true. That is, there are Dirichlet characters mod k for which k is not the smallest positive period. For example, if p is prime, the principal character mod p^2 also has period p. The next theorem provides a partial converse.

THEOREM 2. *Let f be a Dirichlet character* mod k. *If k is squarefree then k is the smallest positive period of f.*

Proof. Let k' denote the smallest positive period of f. Then $k=tk'$ for some integer $t\geqq 1$. We assume that $t>1$ and arrive at a contradiction.

First we note that f is also a character mod k', because f is periodic mod k', and if $(n,k')>1$ then $(n,k)>1$ and (3) implies $f(n)=0$.

Now since k is squarefree, t has a prime factor p which does not divide k'. Hence $(p,k')=1$ so $f(p)$ is a root of unity since f is a character mod k'. On the other hand $p\mid k$ so by (3) we have $f(p)=0$. This contradiction shows that $t=1$ and $k=k'$.

References

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PYTHAGORAS AND THE CAUCHY-SCHWARZ INEQUALITY

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Most current textbook introductions to coordinate vector algebra appeal to the students' knowledge of Euclidean geometry and geometric intuition to justify the introduction of Cartesian coordinates and the usual distance formula. Similarly, vector addition, scalar multiplication, and length of a vector are accompanied by discussions of their geometric significance. Frequently however, the usual inner (dot) product then appears abruptly, the Cauchy-Schwarz and triangle inequalities are proved as technical exercises, and the cosine of the angle between vectors is defined with no reference to the ratio of sides of a right triangle. The following remarks indicate one way that elementary geometric considerations lead naturally to the introduction of the inner product, to trivial proofs of the inequalities, and to the geometric law of cosines.

We assume that the distance between points (vectors) A and B in R^n and the length of $B-A$ may be computed as $|B-A|=(\Sigma(b_i-a_i)^2)^{1/2}$, and that this has been justified by appeal to the