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CERTAIN FAMILY OF INTEGRAL OPERATORS PRESERVING SUBORDINATION AND SUPERORDINATION*

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Abstract We obtain subordination, superordination and sandwich-preserving new theorems for certain integral operators defined on the space of normalized analytic functions in the open unit disk. The sandwich-type theorem for these integral operators is also derived, and the results generalize some recently ones.

Key words analytic function, convex function; differential subordination and superordination; subordination chain; integral operator

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1 Introduction

Let $H(\mathrm{U})$ be the class of functions analytic in the open unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let denote by $H[a, n]$ the subclass of $H(\mathrm{U})$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots ,$$

with $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \cdots\}$. Also, we define the subclasses $\mathcal{A}$ and $\mathcal{P}$ of $H[0, 1]$ as follows:

$$\mathcal{A} = \{f \in H[0, 1] : f'(0) = 1\},$$
$$\mathcal{P} = \left\{h \in \mathcal{A} : h(z)h'(z) \neq 0,\ z \in \dot{\mathrm{U}} = \mathrm{U} \setminus \{0\}\right\}.$$

If f and F are members of $H(\mathrm{U})$, then the function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function $w \in H(\mathrm{U})$, with $w(0) = 0$ and

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$|w(z)| < 1$ for $z \in \mathrm{U}$, such that $f(z) = F(w(z))$ for all $z \in \mathrm{U}$, and in such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathrm{U}) \subset F(\mathrm{U})$ (see [1] and [2]).

Let $\Psi : \mathbb{C}^2 \times \mathrm{U} \to \mathbb{C}$ and let h be univalent in U. If $p \in H(\mathrm{U})$ and satisfies the first order differential subordination:

$$\Psi\left(p(z), zp'(z); z\right) \prec h(z), \tag{1.1}$$

then p is a solution of the differential subordination (1.1). The univalent function q is called a dominant of the solutions of the differential subordination (1.1) if $p(z) \prec q(z)$ for all p satisfying (1.1). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec q(z)$, for all the dominants of (1.1) is called the best dominant.

Similarly, if we let $\Phi : \mathbb{C}^2 \times \mathrm{U} \to \mathbb{C}$, such that p and $\Phi\left(p(z), zp'(z); z\right)$ are univalent in U and if p satisfies first order differential superordination

$$h(z) \prec \Phi\left(p(z), zp'(z); z\right), \tag{1.2}$$

then p is called a solution of the differential superordination (1.2). An analytic function q is called a subordinant of the solutions of the differential superordination (1.2) if $q(z) \prec p(z)$, for all p satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q(z) \prec \tilde{q}(z)$ for all the subordinants of (1.2) is called the best subordinant (see [1, 2]).

For a function $h \in \mathcal{P}$ and the parameters $\beta, \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$ with $\beta \neq 0$, we introduce the integral operators $\mathrm{I}^n_{h;\beta,\alpha_i}\left[f_i\right] : \mathcal{K} \to \mathcal{A}$, $\mathcal{K} \subset \mathcal{A}$, defined by

$$\mathrm{I}^n_{h;\beta,\alpha_i}\left[f_i\right](z) = \left[\frac{\sum\limits_{i=1}^{n}\alpha_i}{z^{\sum\limits_{i=1}^{n}\alpha_i - \beta}} \int_0^z \left(\prod_{i=1}^{n} f_i^{\alpha_i}(t)\right) h^{-1}(t) h'(t) \mathrm{d}t\right]^{\frac{1}{\beta}}, \tag{1.3}$$

where $f_i \in \mathcal{K}$, $i = 1, 2, \cdots, n$ (All powers are principal ones).

Remark 1.1 We emphasize the following significant special cases of the above-defined integral operator:

(i) For $n = 2$, $\alpha_1 = \beta$, $\alpha_2 = \gamma$, $f_1 = f$ and $f_2(t) = h(t) = t$ we obtain

$$\mathrm{I}_{\beta,\gamma}[f](z) = \left[\frac{\beta+\gamma}{z^{\gamma}} \int_0^z f^{\beta}(t) t^{\gamma-1} \mathrm{d}t\right]^{\frac{1}{\beta}}, \tag{1.4}$$

where $\mathrm{I}_{\beta,\gamma}$ is the integral operator introduced by Miller and Mocanu [2], and studied in [3–5] and more other articles.

(ii) Putting $n = 2$, $\alpha_1 = \beta$, $\alpha_2 = \gamma$, $f_1 = f$ and $f_2 = h$, we obtain

$$\mathrm{I}_{h;\beta,\gamma}[f](z) = \left[\frac{\beta+\gamma}{z^{\gamma}} \int_0^z f^{\beta}(t) h^{\gamma-1}(t) h'(t) \mathrm{d}t\right]^{\frac{1}{\beta}},$$

where $\mathrm{I}_{h;\beta,\gamma}$ is the integral operator recently studied in [6], that was previously introduced and studied for the special case $\gamma = 0$ in [7].

(iii) We mention that another recent general form of the above two integral operators was defined and studied in [8].

In the present paper we obtain sufficient conditions on the functions $g_{i,1}$, $g_{i,2}$ and h, and on the parameters $\alpha_1, \alpha_2, \cdots, \alpha_n$ and β, such that the following sandwich-type result holds:

$$z\prod_{i=1}^{n}\left(\frac{g_{i,1}(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)} \prec z\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)} \prec z\prod_{i=1}^{n}\left(\frac{g_{i,2}(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}$$

implies

$$z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}\left[g_{i,1}\right](z)}{z}\right)^{\beta} \prec z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}\left[f_i\right](z)}{z}\right)^{\beta} \prec z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}\left[g_{i,2}\right](z)}{z}\right)^{\beta}.$$

Moreover, the functions $z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}[g_{i,1}](z)}{z}\right)^{\beta}$ and $z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}[g_{i,2}](z)}{z}\right)^{\beta}$ are, respectively, the best subordinant and the best dominant.

2 Preliminaries

The following definitions and lemmas will be required in our present investigation.

Definition 2.1 ([2]) Denote by $\mathcal{Q}$ the set of all functions q that are analytic and injective on $\overline{\mathrm{U}} \setminus E(q)$, where

$$E(q) = \left\{\zeta \in \partial\mathrm{U} : \lim_{z\to\zeta} q(z) = \infty\right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathrm{U} \setminus E(q)$. Further, let denote by $\mathcal{Q}(a)$ the subclass of $\mathcal{Q}$ for which $q(0) = a$.

A function $L(z;t) : \mathrm{U} \times [0, +\infty) \to \mathbb{C}$ is called a subordination (or a Loewner) chain if $L(\cdot;t)$ is analytic and univalent in U for all $t \geq 0$, and $L(z;s) \prec L(z;t)$ when $0 \leq s \leq t$.

The next well-known lemma gives a sufficient condition so that the $L(z;t)$ function will be a subordination chain.

Lemma 2.2 ([9, p.159]) Let $L(z;t) = a_1(t)z + a_2(t)z^2 + \cdots$, with $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim\limits_{t\to+\infty}|a_1(t)| = +\infty$. Suppose that $L(\cdot;t)$ is analytic in U for all $t \geq 0$, $L(z;\cdot)$ is continuously differentiable on $[0, +\infty)$ for all $z \in \mathrm{U}$. If $L(z;t)$ satisfies

$$\mathrm{Re}\left[z\frac{\partial L(z;t)/\partial z}{\partial L(z;t)/\partial t}\right] > 0,\ z \in \mathrm{U},\ t \geq 0$$

and

$$|L(z;t)| \leq K_0\left|a_1(t)\right|,\ |z| < r_0 < 1,\ t \geq 0$$

for some positive constants K_0 and r_0, then $L(z;t)$ is a subordination chain.

Lemma 2.3 ([10]) Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition

$$\mathrm{Re}\, H(\mathrm{i}s, t) \leq 0,\ s \in \mathbb{R},\ t \leq -\frac{n\left(1+s^2\right)}{2},$$

where $n \in \mathbb{N}$. If the function $p(z) = 1 + p_n z^n + p_{n+1}z^{n+1} + \cdots$ is analytic in U and

$$\mathrm{Re}\, H(p(z), zp'(z)) > 0,\ z \in \mathrm{U},$$

then $\mathrm{Re}\, p(z) > 0$ for $z \in \mathrm{U}$.

The next result deals with the solutions of the Briot–Bouquet differential equation (2.1), and more general forms of the following lemma may be found in [11, Theorem 1].

Lemma 2.4 ([11]) Let $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq 0$ and let $k \in H(\mathrm{U})$, with $k(0) = c$. If $\operatorname{Re}[\lambda k(z) + \mu] > 0$, $z \in \mathrm{U}$, then the solution of the differential equation

$$q(z) + \frac{zq'(z)}{\lambda q(z) + \mu} = k(z), \tag{2.1}$$

with $q(0) = c$, is analytic in U and satisfies $\operatorname{Re}[\lambda q(z) + \mu] > 0$, $z \in \mathrm{U}$.

Lemma 2.5 ([2]) Let $p \in \mathcal{Q}(a)$ and let $q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ be analytic in U, with $q(z) \neq a$ and $n \geq 1$. If q is not subordinate to p, then there exists two points $z_0 = r_0 \mathrm{e}^{\mathrm{i}\theta} \in \mathrm{U}$ and $\zeta_0 \in \partial\mathrm{U} \setminus E(q)$ and a number $m \geq n$, such that

$$q(\mathrm{U}_{r_0}) \subset p(\mathrm{U}), \quad q(z_0) = p(\zeta_0) \ \text{ and } \ z_0 p'(z_0) = m\zeta_0 p(\zeta_0),$$

where $\mathrm{U}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$.

Lemma 2.6 ([9]) Let $\Phi : \mathbb{C}^2 \to \mathbb{C}$, $q \in H[a, 1]$, and set $\Phi(q(z), zq'(z)) = h(z)$. If $L(z; t) = \Phi(q(z), tzq'(z))$ is a subordination chain and $q \in H[a, 1] \cap \mathcal{Q}(a)$, then

$$h(z) \prec \Phi(p(z), zp'(z))$$

implies that $q(z) \prec p(z)$. Furthermore, if $\Phi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}(a)$, then q is the best subordinant.

Let $c \in \mathbb{C}$ with $\operatorname{Re} c > 0$, let $n \in \mathbb{N}^*$ and let

$$C_n = C_n(c) = \frac{n}{\operatorname{Re} c}\left[|c|\sqrt{1 + 2\operatorname{Re}\left(\frac{c}{n}\right)} + \operatorname{Im} c\right].$$

If R is the univalent function $R(z) = \frac{2C_n z}{1 - z^2}$, then the open door function $R_{c,n}$ is defined by

$$R_{c,n}(z) = R\left(\frac{z + b}{1 + \bar{b}z}\right), \ z \in \mathrm{U},$$

where $b = R^{-1}(c)$.

Remark that $R_{c,n}$ is univalent in U, $R_{c,n}(0) = c$ and $R_{c,n}(\mathrm{U}) = R(\mathrm{U})$ is the complex plane slit along the half-lines $\operatorname{Re} w = 0$, $\operatorname{Im} w \geq C_n$ and $\operatorname{Re} w = 0$, $\operatorname{Im} w \leq -C_n$. Moreover, if $c > 0$, then $C_{n+1} > C_n$ and $\lim_{n\to\infty} C_n = \infty$, hence $R_{c,n}(z) \prec R_{c,n+1}(z)$ and $\lim_{n\to\infty} R_{c,n}(\mathrm{U}) = \mathbb{C}$. In our paper we will use the notation $R_c := R_{c,1}$.

If we denote by $\mathcal{A}_n$ the class of functions

$$\mathcal{A}_n = \{f \in H(\mathrm{U}) : f(z) = z + a_{n+1} z^{n+1} + \cdots\},$$

then $\mathcal{A} = \mathcal{A}_1$.

Lemma 2.7 (Integral Existence Theorem [12], [13]) Let $\phi, \Phi \in H[1, n]$ with $\phi(z) \neq 0$, $\Phi(z) \neq 0$ for $z \in \mathrm{U}$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ and $\operatorname{Re}(\alpha + \delta) > 0$. If the function $f(z) = z + a_{n+1} z^{n+1} + \cdots \in \mathcal{A}_n$ and satisfies

$$\alpha \frac{zf'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta \prec R_{\alpha+\delta,n}(z),$$

then

$$F(z) = \left[\frac{\beta + \gamma}{z^{\gamma}\Phi(z)} \int_0^z f^{\alpha}(t)\phi(t)t^{\delta-1}\,\mathrm{d}\,t\right]^{\frac{1}{\beta}} = z + b_{n+1} z^{n+1} + \cdots \in \mathcal{A}_n,$$

$\frac{F(z)}{z} \neq 0$, $z \in \mathrm{U}$, and

$$\operatorname{Re}\left[\beta \frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0, \ z \in \mathrm{U}$$

(All powers are principal ones).

Finally, we denote by $K(\rho)$, $\rho<1$, the class of convex functions of order ρ in the unit disk U, i.e.,

$$K(\alpha)=\left\{f\in\mathcal{A}:\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>\rho,\ z\in\mathrm{U}\right\}.$$

In particular, the class $K:=K(0)$ represents the class of convex (and univalent) functions in the unit disk.

3 Main Results

Unless otherwise mentioned, we assume throughout this section that $h\in\mathcal{P}$, and β, $\alpha_1,\alpha_2,\cdots,\alpha_n\in\mathbb{C}$ with $\beta\neq0$ and $\operatorname{Re}\sum\limits_{i=1}^{n}\alpha_i\geq1$, and all powers are principal ones.

First we will use the last lemma to determine the subset $\mathcal{K}\subset\mathcal{A}$ such that the integral operator $\mathrm{I}^n_{h;\beta,\alpha_i}$ given by (1.3) will be well-defined.

Lemma 3.1 Let $h\in\mathcal{P}$ and $\beta,\alpha_1,\alpha_2,\cdots,\alpha_n\in\mathbb{C}$ with $\beta\neq0$, such that $\operatorname{Re}\sum\limits_{i=1}^{n}\alpha_i>0$. If $f_i\in\mathcal{A}_{h;\beta,\alpha_i}$, where

$$\mathcal{A}_{h;\beta,\alpha_i}=\left\{f_i\in\mathcal{A}:\sum_{i=1}^{n}\alpha_i\frac{zf_i'(z)}{f_i(z)}+1+\frac{zh''(z)}{h'(z)}-\frac{zh'(z)}{h(z)}\prec R_{\sum\limits_{i=1}^{n}\alpha_i}(z)\right\},$$

then $\mathrm{I}^n_{h;\beta,\alpha_i}[f_i]\in\mathcal{A}$, $\frac{\mathrm{I}^n_{h;\beta,\alpha_i}[f_i](z)}{z}\neq0$ for all $z\in\mathrm{U}$, and

$$\operatorname{Re}\left[\beta\frac{z\left(\mathrm{I}^n_{h;\beta,\alpha_i}[f_i](z)\right)'}{\mathrm{I}^n_{h;\beta,\alpha_i}[f_i](z)}+\sum_{i=1}^{n}\alpha_i-\beta\right]>0,\ z\in\mathrm{U},$$

where $\mathrm{I}^n_{h;\beta,\alpha_i}[f_i]$ is the integral operator defined by (1.3).

Proof Taking in the above lemma $n:=1$, $\alpha:=1$, $\gamma:=\sum\limits_{i=1}^{n}\alpha_i-\beta$ and $\delta:=\sum\limits_{i=1}^{n}\alpha_i-1$, it follows that $\alpha+\delta=\beta+\gamma$ and $\operatorname{Re}(\alpha+\delta)=\operatorname{Re}\sum\limits_{i=1}^{n}\alpha_i>0$.

Let consider the functions $\Phi(z)\equiv1$, $\phi(t):=\frac{th'(t)}{h(t)}$ and

$$f(t):=\frac{\prod\limits_{i=1}^{n}f_i^{\alpha_i}(t)}{t^{\sum\limits_{i=1}^{n}\alpha_i-1}}=t\prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\alpha_i}.$$

Since $h\in\mathcal{P}$ we have $\phi\in H[1,1]$, and the assumption $f_i\in\mathcal{A}_{h;\beta,\alpha_i}$, $i=1,2,\cdots,n$, implies that $f_i(z)\neq0$ for all $z\in\dot{\mathrm{U}}$ and $i=1,2,\cdots,n$, hence $f\in\mathcal{A}$. Moreover, $\phi(z)\neq0$, $\Phi(z)\neq0$ for $z\in\mathrm{U}$, and our result follows immediately from Lemma 2.7. □

Theorem 3.2 Let $h\in\mathcal{P}$ and $\beta,\alpha_1,\alpha_2,\cdots,\alpha_n\in\mathbb{C}$ with $\beta\neq0$, such that $\operatorname{Re}\sum\limits_{i=1}^{n}\alpha_i\geq1$. For $f_i,g_i\in\mathcal{A}_{h;\beta,\alpha_i}$, $i=1,2,\cdots,n$, suppose that the function ϕ defined by

$$\phi(z)=z\prod_{i=1}^{n}\left(\frac{g_i(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}\tag{3.1}$$

satisfies the inequality

$$\operatorname{Re}\left[1+\frac{z\phi''(z)}{\phi'(z)}\right]>-\delta_0,\ z\in\mathrm{U},\tag{3.2}$$

where δ_0 is given by

$$\delta_0 = \begin{cases} \dfrac{1+|a|^2 - \left|1-a^2\right|}{4\,\mathrm{Re}\,a}, & \text{if } \mathrm{Re}\,a > 0, \\ 0, & \text{if } \mathrm{Re}\,a = 0, \end{cases} \tag{3.3}$$

with

$$a = \sum_{i=1}^{n} \alpha_i - 1.$$

Then, the subordination condition

$$z\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)} \prec z\prod_{i=1}^{n}\left(\frac{g_i(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)} \tag{3.4}$$

implies that

$$z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}\left[f_i\right](z)}{z}\right)^{\beta} \prec z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}\left[g_i\right](z)}{z}\right)^{\beta},$$

and the function $z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}[g_i](z)}{z}\right)^{\beta}$ is the best dominant.

Proof Let define the functions F and G by

$$F(z) = z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}\left[f_i\right](z)}{z}\right)^{\beta}, \quad G(z) = z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}\left[g_i\right](z)}{z}\right)^{\beta},\ z \in \mathrm{U}, \tag{3.5}$$

respectively. From Lemma 3.1 it follows that these two functions are well defined and $F, G \in \mathcal{A}$.

First, we show that, if

$$q(z) = 1 + \frac{zG''(z)}{G'(z)},\ z \in \mathrm{U}, \tag{3.6}$$

then

$$\mathrm{Re}\,q(z) > 0,\ z \in \mathrm{U}. \tag{3.7}$$

From (1.3) and the definitions of the functions G and ϕ, we obtain that

$$\phi(z) = \left(1 - \frac{1}{\sum\limits_{i=1}^{n}\alpha_i}\right)G(z) + \frac{1}{\sum\limits_{i=1}^{n}\alpha_i}\, zG'(z). \tag{3.8}$$

Differentiating both side of (3.8) with respect to z, we have

$$\phi'(z) = G'(z) + \frac{zG''(z)}{\sum\limits_{i=1}^{n}\alpha_i}, \tag{3.9}$$

and combining (3.6) and (3.9) we easily get

$$k(z) := 1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + \sum\limits_{i=1}^{n}\alpha_i - 1}. \tag{3.10}$$

According to (3.10), from (3.2) it follows that

$$\mathrm{Re}\left[k(z) + \sum_{i=1}^{n}\alpha_i - 1\right] > -\delta_0 + \mathrm{Re}\,a \geq 0,\ z \in \mathrm{U},$$

whenever

$$\delta_0 \leq \mathrm{Re}\,a. \tag{3.11}$$

Supposing that the inequality (3.11) holds, according to Lemma 2.4 we conclude that the differential equation (3.10) has a solution $q \in H(\mathrm{U})$, with $k(0) = q(0) = 1$.

If we let

$$H(u,v) = u + \frac{v}{u+a} + \delta,$$

from (3.10) and (3.3) we obtain

$$\mathrm{Re}\, H(q(z), zq'(z)) > 0,\ z \in \mathrm{U}.$$

To verify the condition

$$\mathrm{Re}\, H(\mathrm{i}s, t) \le 0 \quad \text{for} \quad s \in \mathbb{R},\ t \le -\frac{1+s^2}{2}, \tag{3.12}$$

first we see that

$$\mathrm{Re}\, H(\mathrm{i}s,t) = \frac{t\,\mathrm{Re}\, a}{|a+\mathrm{i}s|^2} + \delta = \delta \le \delta_0 = 0 \quad \text{for} \ \ \mathrm{Re}\, a = 0.$$

If $\mathrm{Re}\, a > 0$, then

$$\mathrm{Re}\, H(\mathrm{i}s,t) = \mathrm{Re}\left[\mathrm{i}s + \frac{t}{\mathrm{i}s+a} + \delta_0\right] = \frac{t\,\mathrm{Re}\, a}{|a+\mathrm{i}s|^2} + \delta \le -\frac{K_{a,\delta}(s)}{2\,|a+\mathrm{i}s|^2},\ \text{for} \ \ t \le -\frac{1+s^2}{2},$$

where

$$K_{a,\delta}(s) = (\mathrm{Re}\, a - 2\delta)\ s^2 - 4\delta\,\mathrm{Im}\, a\, s - 2\delta\,|a|^2 + \mathrm{Re}\, a.$$

We need to determine the value

$$\delta_0 = \sup\left\{\delta :\ K_{a,\delta}(s) \ge 0,\ s \in \mathbb{R},\ \delta \le \delta_0\right\}.$$

(i) If $\mathrm{Re}\, a - 2\delta = 0$, then

$$K_{a,\delta}(s) = \mathrm{Re}\, a\left(-2\,\mathrm{Im}\, a\, s + 1 - |a|^2\right) \ge 0,\ s \in \mathbb{R},$$

if and only if $a \in (0,1]$, and in this case $\delta_0 = a/2$. Thus, it is easy to see that the definition relation (3.3) could be used for this special case.

(ii) If $\mathrm{Re}\, a - 2\delta \ne 0$, then $K_{a,\delta}(s) \ge 0$ for all $s \in \mathbb{R}$, if and only if

$$\mathrm{Re}\, a - 2\delta \ge 0 \tag{3.13}$$

and

$$4\delta^2\,\mathrm{Im}^2\, a - (\mathrm{Re}\, a - 2\delta)\left(\mathrm{Re}\, a - 2\delta|a|^2\right) \le 0. \tag{3.14}$$

Using the fact that the inequality (3.14) is equivalent to

$$\chi(\delta) = -4\,\mathrm{Re}\, a\ \delta^2 + 2\left(1+|a|^2\right)\ \delta - \mathrm{Re}\, a \le 0,$$

a simple computation shows that the function χ has the positive zeroes $0 < \delta_0 \le \delta_1$, where δ_0 is given by (3.3). Since $\chi(\delta) \le 0$ for all $\delta \le \delta_0$, and $\chi\left(\mathrm{Re}\, a/2\right) = \mathrm{Re}\, a\,\mathrm{Im}^2\, a \ge 0$, it follows that $\delta_0 \le \mathrm{Re}\, a/2$, i.e., the condition (3.13) holds for $\delta = \delta_0$.

Moreover, because

$$\delta_0 \le \frac{\mathrm{Re}\, a}{2} < \mathrm{Re}\, a,\ \text{if} \ \ \mathrm{Re}\, a > 0,$$

we conclude that the inequality (3.11) holds whenever $\mathrm{Re}\, a > 0$. Obviously, it holds also for $\mathrm{Re}\, a = 0$, since in this case $\delta_0 = 0$.

In conclusion, for the assumed value of δ_0 given by (3.3), we proved that $K_{a,\delta}(s) \geq 0$ for all $s \in \mathbb{R}$, which implies that (3.12) holds. Using Lemma 2.3 we conclude that the inequality (3.7) holds, and from the definition (3.6) it follows that G is convex, i.e., $G \in K$, hence G is a univalent function in U.

Next, we prove that the subordination condition (3.4) implies that $F(z) \prec G(z)$, where the functions F and G are defined by (3.5). For this purpose, let define the function $L(z;t)$ by

$$L(z;t) = \left(1 - \frac{1}{\sum\limits_{i=1}^{n} \alpha_i}\right) G(z) + \frac{1+t}{\sum\limits_{i=1}^{n} \alpha_i}\, zG'(z),\ z \in \mathrm{U},\ t \geq 0. \tag{3.15}$$

If we denote $L(z;t) = a_1(t)z + \cdots$, then

$$a_1(t) = \frac{\partial L(0;t)}{\partial z} = \left(1 + \frac{t}{\sum\limits_{i=1}^{n} \alpha_i}\right) G'(0) = 1 + \frac{t}{\sum\limits_{i=1}^{n} \alpha_i},$$

hence $\lim\limits_{t \to +\infty} |a_1(t)| = +\infty$, and using the fact that $\mathrm{Re} \sum\limits_{i=1}^{n} \alpha_i > 0$ we obtain $a_1(t) \neq 0$ for all $t \geq 0$.

Since $\mathrm{Re}\, q(z) > 0$, $z \in \mathrm{U}$, and $\mathrm{Re} \sum\limits_{i=1}^{n} \alpha_i \geq 1$, we deduce that

$$\mathrm{Re}\left[z \frac{\partial L(z;t)/\partial z}{\partial L(z;t)/\partial t}\right] = \mathrm{Re}\left[\sum_{i=1}^{n} \alpha_i - 1 + (1+t) q(z)\right] > 0,\ z \in \mathrm{U},\ t \geq 0.$$

From the definition (3.15), since $\mathrm{Re} \sum\limits_{i=1}^{n} \alpha_i \geq 1$, for all $t \geq 0$ we have that

$$\frac{|L(z;t)|}{|a_1(t)|} = \frac{\left|\left(\sum\limits_{i=1}^{n} \alpha_i - 1\right) G(z) + (1+t) zG'(z)\right|}{\left|\sum\limits_{i=1}^{n} \alpha_i + t\right|} \leq \frac{\left|\sum\limits_{i=1}^{n} \alpha_i - 1\right| |G(z)| + (1+t)\, |zG'(z)|}{\left|\sum\limits_{i=1}^{n} \alpha_i + t\right|}. \tag{3.16}$$

Since $G \in K$, the following well-known growth and distortion sharp inequalities (see [14]) are true

$$\frac{r}{1+r} \leq |G(z)| \leq \frac{r}{1-r},\ \text{if } |z| \leq r, \tag{3.17}$$
$$\frac{1}{(1+r)^2} \leq |G'(z)| \leq \frac{1}{(1-r)^2},\ \text{if } |z| \leq r.$$

Using the right-hand sides of these inequalities in (3.16), we obtain

$$\frac{|L(z;t)|}{|a_1(t)|} \leq \frac{r}{(1-r)^2}\ \frac{t + 1 + \left|\sum\limits_{i=1}^{n} \alpha_i - 1\right| (1-r)}{\left|\sum\limits_{i=1}^{n} \alpha_i + t\right|},\ |z| \leq r,\ t \geq 0. \tag{3.18}$$

The assumption $\mathrm{Re} \sum\limits_{i=1}^{n} \alpha_i \geq 1$ implies

$$\left|t + \sum_{i=1}^{n} \alpha_i\right| \geq \left|\sum_{i=1}^{n} \alpha_i\right|, \quad \left|t + \sum_{i=1}^{n} \alpha_i\right| \geq |t+1|,\ t \geq 0,$$

and from (3.18) we conclude that

$$\frac{|L(z;t)|}{|a_1(t)|} \le \frac{r}{(1-r)^2}\left[1+\frac{\left|\sum\limits_{i=1}^{n}\alpha_i-1\right|(1-r)}{\left|\sum\limits_{i=1}^{n}\alpha_i\right|}\right],\ |z|\le r,\ t\ge 0.$$

Thus, the second assumption of Lemma 2.2 holds, and according to this Lemma we obtain that the function $L(z;t)$ is a subordination chain.

Using Lemma 2.5 we will show that $F(z)\prec G(z)$. Without loss of generality, we can assume that ϕ and G are analytic and univalent in $\overline{\mathrm{U}}$ and $G'(\zeta)\neq 0$ for $|\zeta|=1$. If not, then we could replace ϕ with $\phi_\rho(z)=\phi(\rho z)$ and G with $G_\rho(z)=G(\rho z)$, where $\rho\in(0,1)$. These new functions have the desired properties on $\overline{\mathrm{U}}$ and we can use them in our proof. Therefore, the results would follow by letting $\rho\to 1$.

From the definition of the subordination chain it follows

$$\phi(z)=\left(1-\frac{1}{\sum\limits_{i=1}^{n}\alpha_i}\right)G(z)+\frac{1}{\sum\limits_{i=1}^{n}\alpha_i}\,zG'(z)=L(z;0),$$

and

$$L(z;0)\prec L(z;t),\ \ t\ge 0,$$

which implies

$$L(\zeta;t)\notin L(\mathrm{U};0)=\phi(\mathrm{U}),\ \zeta\in\partial\mathrm{U},\ t\ge 0. \tag{3.19}$$

If $F(z)\not\prec G(z)$, then according to Lemma 2.5 there exist two points $z_0\in\mathrm{U}$ and $\zeta_0\in\partial\mathrm{U}$ and a number $m=1+t_0\ge 1$, such that

$$F(z_0)=G(\zeta_0)\quad\text{and}\quad z_0F'(z_0)=(1+t_0)\,\zeta_0G'(\zeta_0),\ t_0\ge 0. \tag{3.20}$$

Hence, by virtue of (3.20), we have

$$\begin{aligned}L(\zeta_0;t_0)&=\left(1-\frac{1}{\sum\limits_{i=1}^{n}\alpha_i}\right)G(\zeta_0)+\frac{1+t_0}{\sum\limits_{i=1}^{n}\alpha_i}\,\zeta_0G'(\zeta_0)\\&=\left(1-\frac{1}{\sum\limits_{i=1}^{n}\alpha_i}\right)F(z_0)+\frac{1}{\sum\limits_{i=1}^{n}\alpha_i}\,z_0F'(z_0)\\&=z_0\prod_{i=1}^{n}\left(\frac{f_i(z_0)}{z_0}\right)^{\alpha_i}\frac{z_0h'(z_0)}{h(z_0)}\in\phi(\mathrm{U}),\end{aligned}$$

which contradicts the above remark (3.19), i.e., $L(\zeta_0;t_0)\notin\phi(\mathrm{U})$. Consequently, the subordination condition (3.4) implies that $F(z)\prec G(z)$, and considering $F=G$ we conclude that the function G is the best dominant, which completes our proof. □

Remark 3.3 (i) Taking $n=2$, $\alpha_1=\beta$, $\alpha_2=\gamma$, $f_1=f$ and $f_2=h$ in Theorem 3.2, we obtain the subordination result of [6, Theorem 2.1];

(ii) Taking $n=2$, $\alpha_1=\beta$, $\alpha_2=\gamma$, $f_1=f$ and $f_2(t)=h(t)=t$ in Theorem 3.2, we obtain a subordination results for the class integral operators studied in [3, 4]. Note that in [3, Theorem 1] the author supposed that $0<\beta+\gamma\le 1$, in [4, Theorem 3.1] the assumption was extended to $0 < \beta+\gamma \le 2$, while the above theorem extends the range of these parameters to $\mathrm{Re}(\beta+\gamma) \ge 1$.

According to this last remark, for the special case $\beta+\gamma > 0$, combining [4, Theorem 3.1] and Theorem 3.2 we obtain the following result.

Corollary 3.4 Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, such that $\beta+\gamma > 0$. For $f, g \in \mathcal{F}_{\beta,\gamma} = \left\{\varphi \in \mathcal{A} : \beta\frac{z\varphi'(z)}{\varphi(z)} + \gamma \prec R_{\beta+\gamma}(z)\right\}$, suppose that the function ϕ defined by

$$\phi(z) = z\left[\frac{g(z)}{z}\right]^{\beta}$$

satisfies the inequality

$$\mathrm{Re}\left[1 + \frac{z\phi''(z)}{\phi'(z)}\right] > \widetilde{\delta},\ z \in \mathrm{U},$$

where $\widetilde{\delta}$ is given by

$$\widetilde{\delta} = \begin{cases} 1-(\beta+\gamma), & \text{if } 0 < \beta+\gamma \le 1, \\ \dfrac{1-(\beta+\gamma)}{2}, & \text{if } 1 \le \beta+\gamma \le 2, \\ -\dfrac{1}{2(\beta+\gamma-1)}, & \text{if } \beta+\gamma \ge 2. \end{cases}$$

Then, the subordination condition

$$z\left[\frac{f(z)}{z}\right]^{\beta} \prec z\left[\frac{g(z)}{z}\right]^{\beta} \quad \text{implies} \quad z\left[\frac{\mathrm{I}_{\beta,\gamma}[f](z)}{z}\right]^{\beta} \prec z\left[\frac{\mathrm{I}_{\beta,\gamma}[g](z)}{z}\right]^{\beta},$$

where the integral operator $\mathrm{I}_{\beta,\gamma}$ is given by (1.4). Moreover, the function $z\left[\frac{\mathrm{I}_{\beta,\gamma}[g](z)}{z}\right]^{\beta}$ is the best dominant.

We now derive the following superordination result:

Theorem 3.5 Let $h \in \mathcal{P}$ and $\beta, \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$ with $\beta \neq 0$, such that $\mathrm{Re}\sum\limits_{i=1}^{n}\alpha_i > 1$.

For $f_i, g_i \in \mathcal{A}_{h;\beta,\alpha_i}$, $i = 1, 2, \cdots, n$, suppose that the function ϕ defined by (3.1) satisfies the condition (3.2), where δ_0 is given by (3.3).

If the function $z\prod\limits_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}$ is univalent in U and $z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}[f_i](z)}{z}\right)^{\beta} \in \mathcal{Q}(0)$, then the superordination condition

$$z\prod_{i=1}^{n}\left(\frac{g_i(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)} \prec z\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)} \tag{3.21}$$

implies that

$$z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}\left[g_i\right](z)}{z}\right)^{\beta} \prec z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}\left[f_i\right](z)}{z}\right)^{\beta},$$

and the function $z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}[g_i](z)}{z}\right)^{\beta}$ is the best subordinant.

Proof Like in the proof of Theorem 3.2, suppose that the functions F, G and q are defined by (3.5) and (3.6), respectively. Applying a similar method as in the proof of Theorem 3.2 we get that the inequality (3.7) holds, and from the definition (3.6) it follows that G is convex, i.e., $G \in K$, hence G is a univalent function in U.

Next, we will prove that the superordination condition (3.21) implies that $G(z) \prec F(z)$. For this, we define the function $L(z;t)$ by

$$L(z;t) = \left(1 - \frac{1}{\sum\limits_{i=1}^{n} \alpha_i}\right) G(z) + \frac{t}{\sum\limits_{i=1}^{n} \alpha_i} zG'(z),\ z \in \mathrm{U},\ t \geq 0. \tag{3.22}$$

If we denote $L(z;t) = a_1(t)z + \cdots$, then

$$a_1(t) = \frac{\partial L(0;t)}{\partial z} = \left(1 + \frac{t-1}{\sum\limits_{i=1}^{n} \alpha_i}\right) G'(0) = 1 + \frac{t-1}{\sum\limits_{i=1}^{n} \alpha_i},$$

hence $\lim\limits_{t \to +\infty} |a_1(t)| = +\infty$, and using the assumption $\mathrm{Re} \sum\limits_{i=1}^{n} \alpha_i > 1$ we obtain $a_1(t) \neq 0$ for all $t \geq 0$.

Using the facts that $\mathrm{Re}\, q(z) > 0$, $z \in \mathrm{U}$, and $\mathrm{Re} \sum\limits_{i=1}^{n} \alpha_i > 1$, we obtain

$$\mathrm{Re} \left[z \frac{\partial L(z;t)/\partial z}{\partial L(z;t)/\partial t} \right] = \mathrm{Re} \left[\sum_{i=1}^{n} \alpha_i - 1 + tq(z) \right] > 0,\ z \in \mathrm{U},\ t \geq 0.$$

From the definition (3.22), since $\mathrm{Re} \sum\limits_{i=1}^{n} \alpha_i > 1$, for all $t \geq 0$ we have

$$\frac{|L(z;t)|}{|a_1(t)|} = \frac{\left| \left(\sum\limits_{i=1}^{n} \alpha_i - 1 \right) G(z) + tzG'(z) \right|}{\left| \sum\limits_{i=1}^{n} \alpha_i + t - 1 \right|} \leq \frac{\left| \sum\limits_{i=1}^{n} \alpha_i - 1 \right| |G(z)| + t\, |zG'(z)|}{\left| \sum\limits_{i=1}^{n} \alpha_i + t - 1 \right|}.$$

Since $G \in K$, using in the above relation the right-hand sides of the inequalities (3.17), we obtain

$$\frac{|L(z;t)|}{|a_1(t)|} \leq \frac{r}{(1-r)^2} \, \frac{t + \left| \sum\limits_{i=1}^{n} \alpha_i - 1 \right| (1-r)}{\left| \sum\limits_{i=1}^{n} \alpha_i + t - 1 \right|},\ |z| \leq r,\ t \geq 0. \tag{3.23}$$

The assumption $\mathrm{Re} \sum\limits_{i=1}^{n} \alpha_i > 1$ implies

$$\left| t - 1 + \sum_{i=1}^{n} \alpha_i \right| \geq \left| \sum_{i=1}^{n} \alpha_i - 1 \right|, \quad \left| t - 1 + \sum_{i=1}^{n} \alpha_i \right| > |t|,\ t \geq 0,$$

and from (3.23) we conclude that

$$\frac{|L(z;t)|}{|a_1(t)|} < \frac{r(2-r)}{(1-r)^2},\ |z| \leq r,\ t \geq 0.$$

Hence, all the assumptions of Lemma 2.2 hold and we conclude that the function $L(z;t)$ is a subordination chain.

According to Lemma 2.6, the supeordination condition (3.21) implies that $G(z) \prec F(z)$, and since the differential equation

$$\phi(z) = \left(1 - \frac{1}{\sum\limits_{i=1}^{n} \alpha_i}\right) G(z) + \frac{1}{\sum\limits_{i=1}^{n} \alpha_i} zG'(z) = \Phi(G(z), zG'(z))$$

has a univalent solution G, the function G is the best subordinant. $\square$

Remark 3.6 (i) Taking $n=2$, $\alpha_1=\beta$, $\alpha_2=\gamma$, $f_1=f$ and $f_2=h$ in Theorem 3.5, we obtain the superordination result of [6, Theorem 2.2];

(ii) Taking $n=2$, $\alpha_1=\beta$, $\alpha_2=\gamma$, $f_1=f$ and $f_2(t)=h(t)=t$ in Theorem 3.5, we obtain a superordination result that generalizes the result from [5, Theorem 3.1], where a similar implication was obtained for $1<\beta+\gamma\le 2$. In the present paper this result was extended, by assuming that $\mathrm{Re}(\beta+\gamma)>1$.

Combining the above-mentioned subordination and superordination results involving the operator $\mathrm{I}^n_{h;\beta,\alpha_i}$, the following sandwich-type result is derived.

Theorem 3.7 Let $h\in\mathcal{P}$ and $\beta,\alpha_1,\alpha_2,\cdots,\alpha_n\in\mathbb{C}$ with $\beta\neq 0$, such that $\mathrm{Re}\sum\limits_{i=1}^{n}\alpha_i>1$. For $f_{i,j},g_{i,j}\in\mathcal{A}_{h;\beta,\alpha_i}$, $i=1,2,\cdots,n$, suppose that the functions ϕ_j defined by

$$\phi_j(z)=z\prod_{i=1}^{n}\left(\frac{g_{i,j}(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)},$$

satisfy the inequalities

$$\mathrm{Re}\left[1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right]>-\delta_0,\ z\in\mathrm{U},\ j=1,2,$$

where δ_0 is given by (3.3).

If the function $z\prod\limits_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}$ is univalent in U and $z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}[f_i](z)}{z}\right)^{\beta}\in\mathcal{Q}(0)$, then the condition

$$z\prod_{i=1}^{n}\left(\frac{g_{i,1}(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}\prec z\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}\prec z\prod_{i=1}^{n}\left(\frac{g_{i,2}(z)}{z}\right)^{\alpha_i}\frac{zh'(z)}{h(z)}$$

implies that

$$z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}\left[g_{i,1}\right](z)}{z}\right)^{\beta}\prec z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}\left[f_i\right](z)}{z}\right)^{\beta}\prec z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}\left[g_{i,2}\right](z)}{z}\right)^{\beta},$$

and the functions $z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}[g_{i,1}](z)}{z}\right)^{\beta}$ and $z\left(\frac{\mathrm{I}^n_{h;\beta,\alpha_i}[g_{i,2}](z)}{z}\right)^{\beta}$ are, respectively, the best subordinant and the best dominant.

Remark 3.8 (i) Taking $n=2$, $\alpha_1=\beta$, $\alpha_2=\gamma$, $f_1=f$ and $f_2=h$ in Theorem 3.7, we obtain the sandwich result of [6, Theorem 2.3];

(ii) Taking $n=2$, $\alpha_1=\beta$, $\alpha_2=\gamma$, $f_1=f$ and $f_2(t)=h(t)=t$ in Theorem 3.7, we obtain the sandwich superordination result that generalizes the result from [5, Theorem 3.2]. While in this previous article the assumption for the parameters $\beta,\gamma\in\mathbb{C}$ was $1<\beta+\gamma\le 2$, we proved now that the implication holds for $\mathrm{Re}(\beta+\gamma)>1$.

Thus, for the special case $\beta+\gamma>1$ we deduce the following sandwich-type result:

Corollary 3.9 Let $\beta,\gamma\in\mathbb{C}$ with $\beta\neq 0$, such that $\beta+\gamma>1$. For $f,g_1,g_2\in\mathcal{F}_{\beta,\gamma}$, suppose that the functions ϕ_k, $k=1,2$, defined by

$$\phi_k(z)=z\left[\frac{g_k(z)}{z}\right]^{\beta},\ k=1,2,$$

satisfy the inequality

$$\mathrm{Re}\left[1+\frac{z\phi_k''(z)}{\phi_k'(z)}\right]>\widehat{\delta},\ z\in\mathrm{U},\ k=1,2,$$

where $\widehat{\delta}$ is given by

$$\widehat{\delta}=\begin{cases}\dfrac{1-(\beta+\gamma)}{2}, & \text{if } 1<\beta+\gamma\le 2,\\ -\dfrac{1}{2(\beta+\gamma-1)}, & \text{if } \beta+\gamma\ge 2.\end{cases}$$

If the function $z\left[\frac{f(z)}{z}\right]^{\beta}$ is univalent in U and $z\left[\frac{\mathrm{I}_{\beta,\gamma}[f](z)}{z}\right]^{\beta}\in\mathcal{Q}(0)$, then the condition

$$z\left[\frac{g_1(z)}{z}\right]^{\beta}\prec z\left[\frac{f(z)}{z}\right]^{\beta}\prec z\left[\frac{g_2(z)}{z}\right]^{\beta}$$

implies that

$$z\left[\frac{\mathrm{I}_{\beta,\gamma}\left[g_1\right](z)}{z}\right]^{\beta} z\left[\frac{\mathrm{I}_{\beta,\gamma}[f](z)}{z}\right]^{\beta}\prec z\left[\frac{\mathrm{I}_{\beta,\gamma}\left[g_2\right](z)}{z}\right]^{\beta},$$

where the integral operator $\mathrm{I}_{\beta,\gamma}$ is given by (1.4). Moreover, the functions $z\left[\frac{\mathrm{I}_{\beta,\gamma}[g_1](z)}{z}\right]^{\beta}$ and $z\left[\frac{\mathrm{I}_{\beta,\gamma}[g_2](z)}{z}\right]^{\beta}$ are, respectively, the best subordinant and the best dominant.

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