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Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function

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Abstract

In this paper we introduce and study two subclasses $\Sigma_{p,q,s}(\alpha_1; A, B, \lambda)$ and $\Sigma^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$ of meromorphic p-valent functions of order $\lambda\,(0 \leq \lambda < p)$ defined by certain linear operator involving the generalized hypergeometric function. We investigate the various important properties and characteristics of the problem. We extend the familiar concept of neighborhoods of analytic functions to these subclasses of meromorphically multivalent functions. We also derive many interesting results for the Hadamard products of functions belonging to the class $\Sigma^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$.
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1. Introduction

Let Σ_p be the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in N = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = U/\{0\}$. For functions $f(z) \in \Sigma_p$ given by (1.1) and $g(z) \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (p \in N), \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z). \tag{1.3}$$

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For complex parameters

$$\alpha_1, \ldots, \alpha_q \quad \text{and} \quad \beta_1, \ldots, \beta_s \quad (\beta_j \notin Z_0^- = \{0, -1, -2, \ldots\};\ j = 1, 2, \ldots, s),$$

we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by (see, for example, [1, p. 19])

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k} . \frac{z^k}{k!} \quad (q \le s + 1;\ q, s \in N_0 = N \cup \{0\};\ z \in U), \tag{1.4}$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in C \setminus \{0\}), \\ \theta(\theta + 1) \ldots (\theta + \nu - 1) & (\nu \in N; \theta \in C). \end{cases} \tag{1.5}$$

Corresponding to the function $h_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s; z)$, defined by

$$h_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s; z) = z^{-p}\, {}_qF_s(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s; z), \tag{1.6}$$

we consider a linear operator

$$H_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s) : \Sigma_p \to \Sigma_p,$$

which is defined by the following Hadamard product (or convolution):

$$H_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s) f(z) = h_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z). \tag{1.7}$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$H_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s) f(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k} . \frac{a_{k-p}}{k!} z^{k-p}. \tag{1.8}$$

If, for convenience, we write

$$H_{p,q,s}(\alpha_1) = H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s), \tag{1.9}$$

then one can easily verify from the definition (1.7) that

$$z(H_{p,q,s}(\alpha_1) f(z))' = \alpha_1 H_{p,q,s}(\alpha_1 + 1) f(z) - (\alpha_1 + p) H_{p,q,s}(\alpha_1) f(z). \tag{1.10}$$

The linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [2]. In particular, for $s = 1, q = 2$ and $\alpha_2 = 1$, we obtain the linear operator:

$$\ell_p(\alpha_1, \beta_1) f(z) = H_p(\alpha_1, 1; \beta_1) f(z),$$

which was introduced and studied by Liu and Srivastava [3].

Some interesting subclasses of analytic functions associated with the generalized hypergeometric function, were considered recently by (for example) Dziok and Srivastava [4,5], Gangadharan et al. [6] and Liu [7].

For fixed parameters A, B and λ, with $(-1 \le B < A \le 1; 0 \le \lambda < p;\ p \in N)$, we say that a function $f(z) \in \Sigma_p$ is in the class $\Sigma_{p,q,s}(\alpha_1; A, B, \lambda)$ of meromorphically p-valent functions in U if it also satisfies the following inequality:

$$\left| \frac{z^{p+1}(H_{p,q,s}(\alpha_1) f(z))' + p}{B z^{p+1}(H_{p,q,s}(\alpha_1) f(z))' + [pB + (A - B)(p - \lambda)]} \right| < 1 \quad (z \in U). \tag{1.11}$$

Furthermore, we say that a function $f(z) \in \Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda)$ whenever $f(z)$ is of the form [cf. Eq. (1.1)]:

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} |a_k| z^k \quad (p \in N). \tag{1.12}$$

We note that:

(i)

$$\Sigma^{+}_{p,q,s}(\alpha_1;\beta,-\beta,\lambda)=\Sigma^{+}_{p,q,s}(\alpha_1;\lambda,\beta)=\left\{f(z)\in\Sigma_p:\left|\frac{z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'+p}{z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'-p+2\lambda}\right|<\beta,\right.$$
$$\left.(z\in U;0\le\lambda<p;p\in N;0<\beta\le 1)\right\}; \tag{1.13}$$

(ii)

$$\Sigma^{+}_{p,q,s}(\alpha_1;\beta,-(2\gamma-1)\beta,\lambda)=\Sigma^{+}_{p,q,s}(\alpha_1;\lambda,\beta,\gamma)$$
$$=\left\{f(z)\in\Sigma_p:\left|\frac{z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'+p}{(2\gamma-1)z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'-p+2\gamma\lambda}\right|<\beta,\right.$$
$$\left.\left(z\in U;0\le\lambda<p;p\in N;\frac{1}{2}\le\gamma\le 1;0<\beta\le 1\right)\right\}. \tag{1.14}$$

Meromorphically multivalent functions have been extensively studied by (for example) Mogra [8,9], Uralegaddi and Ganigi [10], Uralegaddi and Somanatha [11], Aouf [12,13], Srivastava et al. [14], Owa et al. [15], Joshi and Aouf [16], Joshi and Srivastava [17], Aouf et al. [18], Raina and Srivastava [19] and Yang [20].

In this paper we investigate the various important properties and characteristics of the classes $\Sigma_{p,q,s}(\alpha_1;A,B,\lambda)$ and $\Sigma^{+}_{p,q,s}(\alpha_1;A,B,\lambda)$. Following the recent investigations by Altintas et al. [21, p. 1668], we extend the concept of neighborhoods of analytic functions, which was considered earlier by (for example) Goodman [22] and Ruscheweyh [23], to meromorphically multivalent functions belonging to the classes $\Sigma_{p,q,s}(\alpha_1;A,B,\lambda)$ and $\Sigma^{+}_{p,q,s}(\alpha_1;A,B,\lambda)$. We also derive many interesting results on the Hadamard products of functions belonging to the p-valently meromorphic function class $\Sigma^{+}_{p,q,s}(\alpha_1;A,B,\lambda)$.

2. Properties of the class $\Sigma_{p,q,s}(\alpha_1;A,B,\lambda)$

We begin by recalling the following result (Jack's lemma), which we shall apply in proving our first inclusion theorem.

Lemma 1 (*[24]*). *Let the (non-constant) function $w(z)$ be analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in U$, then*

$$z_0w'(z_0)=\gamma w(z_0), \tag{2.1}$$

where γ is a real number and $\gamma\ge 1$.

Theorem 1. *The following inclusion property holds true for the class $\Sigma_{p,q,s}(\alpha_1;A,B,\lambda)$, $\alpha_1>0$:*

$$\Sigma_{p,q,s}(\alpha_1+1;A,B,\lambda)\subset\Sigma_{p,q,s}(\alpha_1;A,B,\lambda). \tag{2.2}$$

Proof. Let $f(z)\in\Sigma_{p,q,s}(\alpha_1+1;A,B,\lambda)$ and suppose that

$$z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'=-\frac{p+[pB+(A-B)(p-\lambda)]w(z)}{1+Bw(z)}, \tag{2.3}$$

where the function $w(z)$ is either analytic or meromorphic in U, with $w(0)=0$. Then, by using (1.10) and (2.3), we have

$$z^{p+1}(H_{p,q,s}(\alpha_1+1)f(z))'=-\frac{p+[pB+(A-B)(p-\lambda)]w(z)}{1+Bw(z)}-\frac{(A-B)(p-\lambda)}{\alpha_1}\cdot\frac{zw'(z)}{(1+Bw(z))^2}. \tag{2.4}$$

We claim that $|w(z)| < 1$ for $z \in U$. Otherwise there exists a point $z_0 \in U$ such that $\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$. Applying Jack's lemma, we have $z_0 w'(z_0) = \gamma w(z_0) (\gamma \geq 1)$. Writing $w(z_0) = e^{i\theta} (0 \leq \theta \leq 2\pi)$ and taking $z = z_0$ in (2.4), we get

$$\begin{aligned}
&\left| \frac{z_0^{p+1} (H_{p,q,s}(\alpha_1 + 1) f(z_0))' + p}{B z_0^{p+1} (H_{p,q,s}(\alpha_1 + 1) f(z_0))' + [pB + (A - B)(p - \lambda)]} \right|^2 - 1 \\
&= \frac{\left|\alpha_1 + \gamma + \alpha_1 B e^{i\theta}\right|^2 - \left|\alpha_1 + B(\alpha_1 - \gamma) e^{i\theta}\right|^2}{\left|\alpha_1 + B(\alpha_1 - \gamma) e^{i\theta}\right|^2} \\
&= \frac{\gamma^2 (1 - B^2) + 2\alpha_1 \gamma (1 + B^2 + 2B \cos\theta)}{\left|\alpha_1 + B(\alpha_1 - \gamma) e^{i\theta}\right|^2} \geq 0,
\end{aligned}$$

which obviously contradicts our hypothesis that $f(z) \in \Sigma_{p,q,s}(\alpha_1 + 1; A, B, \lambda)$. Thus we must have $|w(z)| < 1 (z \in U)$, and so from (2.3), we conclude that $f(z) \in \Sigma_{p,q,s}(\alpha_1; A, B, \lambda)$, which evidently completes the proof of Theorem 1. □

Theorem 2. *Let μ be a complex number such that* $\text{Re}(\mu) > 0$. *If $f(z) \in \Sigma_{p,q,s}(\alpha_1; A, B, \lambda)$, then the function*

$$F(z) = \frac{\mu}{z^{\mu+p}} \int_0^z t^{\mu+p-1} f(t) dt \tag{2.5}$$

is also in the same class $\Sigma_{p,q,s}(\alpha_1; A, B, \lambda)$.

Proof. From (2.5), we have

$$\mu H_{p,q,s}(\alpha_1) f(z) = (\mu + p) H_{p,q,s}(\alpha_1) F(z) + z(H_{p,q,s}(\alpha_1) F(z))'. \tag{2.6}$$

Put

$$z^{p+1} (H_{p,q,s}(\alpha_1) F(z))' = -\frac{p + [pB + (A - B)(p - \lambda)] w(z)}{1 + B w(z)}, \tag{2.7}$$

where $w(z)$ is either analytic or meromorphic in U with $w(0) = 0$. Then, by using (2.6) and (2.7), we have

$$z^{p+1} (H_{p,q,s}(\alpha_1) f(z))' = -\frac{p + [pB + (A - B)(p - \lambda)] w(z)}{1 + B w(z)} - \frac{(A - B)(p - \lambda)}{\mu} \cdot \frac{z w'(z)}{(1 + B w(z))^2}. \tag{2.8}$$

The remaining part of the proof is similar to that of Theorem 1 and so is omitted. □

Theorem 3. $f(z) \in \Sigma_{p,q,s}(\alpha_1; A, B, \lambda)$ *if and only if*

$$F(z) = \frac{\alpha_1}{z^{\alpha_1+p}} \int_0^z t^{\alpha_1+p-1} f(t) dt \in \Sigma_{p,q,s}(\alpha_1 + 1; A, B, p, \lambda). \tag{2.9}$$

Proof. In view of the definition of $F(z)$, we have

$$\alpha_1 f(z) = (\alpha_1 + p) F(z) + z F'(z). \tag{2.10}$$

By using (1.10) and (2.10), we have

$$\begin{aligned}
\alpha_1 H_{p,q,s}(\alpha_1) f(z) &= (\alpha_1 + p) H_{p,q,s}(\alpha_1) F(z) + z(H_{p,q,s}(\alpha_1) F(z))' \\
&= \alpha_1 H_{p,q,s}(\alpha_1 + 1) F(z).
\end{aligned}$$

The desired result follows immediately. □

3. Properties of the class $\Sigma^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$

In this section we assume further that $\alpha_j > 0\,(j = 1, \ldots, q)$, $\beta_j > 0\,(j = 1, \ldots, s)$, $-1 \leq B < A \leq 1$, $-1 \leq B \leq 0$, $0 \leq \lambda < p$ and $p \in N$.

Theorem 4. *Let $f(z) \in \Sigma_p$ be given by (1.12). Then $f(z) \in \Sigma^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$ if and only if*

$$\sum_{k=p}^{\infty} k(1-B)\Gamma_{k+p}(\alpha_1)|a_k| \leq (A-B)(p-\lambda), \tag{3.1}$$

where, for convenience,

$$\Gamma_m(\alpha_1) = \frac{(\alpha_1)_m \ldots (\alpha_q)_m}{m!(\beta_1)_m \ldots (\beta_s)_m} \quad (m \in N). \tag{3.2}$$

Proof. Let $f(z) \in \Sigma^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$ be given by (1.12). Then, from (1.11) and (1.12), we have

$$\left| \frac{z^{p+1}(H_{p,q,s}(\alpha_1) f(z))' + p}{Bz^{p+1}(H_{p,q,s}(\alpha_1) f(z))' + [pB + (A-B)(p-\lambda)]} \right|$$
$$= \left| \frac{\sum_{k=p}^{\infty} k\Gamma_{k+p}(\alpha_1)\,|a_k|\, z^{k+p}}{(A-B)(p-\lambda) + \sum_{k=p}^{\infty} B\, k\Gamma_{k+p}(\alpha_1)\ |a_k|\, z^{k+p}} \right| < 1 \quad (z \in U). \tag{3.3}$$

Since $|\text{Re}(z)| \leq |z|$ for any z, choosing z to be real and letting $z \to 1^{-}$ through real values, (3.3) yields

$$\sum_{k=p}^{\infty} k(1-B)\Gamma_{k+p}(\alpha_1)|a_k| \leq (A-B)(p-\lambda), \tag{3.4}$$

which leads us at once to (3.1).

In order to prove the converse, we assume that the inequality (3.1) holds true. Then, if we let $z \in \partial U$, we find from (1.12) and (3.1) that

$$\left| \frac{z^{p+1}(H_{p,q,s}(\alpha_1) f(z))' + p}{Bz^{p+1}(H_{p,q,s}(\alpha_1) f(z))' + [pB + (A-B)(p-\lambda)]} \right| \leq \frac{\sum_{k=p}^{\infty} k\Gamma_{k+p}(\alpha_1)|a_k|}{(A-B)(p-\lambda) + \sum_{k=p}^{\infty} Bk\Gamma_{k+p}(\alpha_1)|a_k|}$$
$$< 1 \quad (z \in \partial U = \{z : z \in C \text{ and } |z| = 1\}). \tag{3.5}$$

Hence, by the maximum modulus theorem, we have $f(z) \in \Sigma^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$. This completes the proof of Theorem 4. □

Corollary 1. *Let $f(z) \in \Sigma_p$ be given by (1.12). If $f(z) \in \Sigma^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$, then*

$$|a_k| \leq \frac{(A-B)(p-\lambda)}{k(1-B)\Gamma_{k+p}(\alpha_1)} \quad (k \geq p;\ p \in N). \tag{3.6}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(p-\lambda)}{k(1-B)\Gamma_{k+p}(\alpha_1)} z^k \quad (k \geq p;\ p \in N). \tag{3.7}$$

The following property is an easy consequence of Theorem 4.

Theorem 5. *Let each of the functions $f_j(z)$ defined by*

$$f_j(z) = z^{-p} + \sum_{k=p}^{\infty} \left|a_{k,j}\right| z^k \quad (j = 1, 2, \ldots, m) \tag{3.8}$$

be in the class $\Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda)$. Then the function $h(z)$ defined by

$$h(z) = \sum_{j=1}^{m} \zeta_j f_j(z) \quad \left(\zeta_j \geq 0 \text{ and } \sum_{j=1}^{m} \zeta_j = 1\right)$$

is also in the class $\Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda)$.

Next we prove the following growth and distortion properties for the class $\Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda)$.

Theorem 6. *Let a function $f(z)$ defined by (1.12) be in the class $\Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda)$. If sequence $\{C_k\} = \{k\Gamma_{k+p}(\alpha_1)\}$ $(k \geq p;\ p \in N)$ is nondecreasing, then*

$$\begin{aligned} &\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(A-B)(p-\lambda)}{(1-B)C_p} \cdot \frac{p!}{(p-m)!} r^{2p}\right\} r^{-(p+m)} \leq \left|f^{(m)}(z)\right| \\ &\quad \leq \left\{\frac{(p+m-1)!}{(p-1)!} + \frac{(A-B)(p-\lambda)}{(1-B)C_p} \cdot \frac{p!}{(p-m)!} r^{2p}\right\} r^{-(p+m)} \\ &\qquad (0 < |z| = r < 1;\ 0 \leq \lambda < p;\ m \in N_0 = N \cup \{0\};\ p \in N;\ p > m), \end{aligned} \tag{3.9}$$

where $\Gamma_m(\alpha_1)$ is given by (3.2). The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(p-\lambda)}{p(1-B)\Gamma_{2p}(\alpha_1)} z^p \quad (p \in N). \tag{3.10}$$

Proof. In view of Theorem 4, we have

$$\Gamma_{2p}(\alpha_1)\frac{p}{p!}\sum_{k=p}^{\infty} k!|a_k| \leq \sum_{k=p}^{\infty} k\Gamma_{k+p}(\alpha_1)|a_k| \leq \frac{(A-B)(p-\lambda)}{(1-B)},$$

which yields

$$\sum_{k=p}^{\infty} k!|a_k| \leq \frac{(A-B)(p-\lambda)p!}{(1-B)p\Gamma_{2p}(\alpha_1)} \quad (p \in N). \tag{3.11}$$

Now, by differentiating both sides of (1.12) m times with respect to z, we have

$$f^{(m)}(z) = (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-(p+m)} + \sum_{k=p}^{\infty} \frac{k!}{(k-m)!}|a_k| z^{k-m}, \quad (m \in N_0;\ p \in N;\ p > m) \tag{3.12}$$

and Theorem 6 follows easily from (3.11) and (3.12).

Finally, it is easy to see that the bounds in (3.9) are attained for the function $f(z)$ given by (3.10).

Next we determine the radii of meromorphically p-valent starlikeness of order δ $(0 \leq \delta < p)$ and meromorphically p-valent convexity of order δ $(0 \leq \delta < p)$ for functions in the class $\Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda)$. □

Theorem 7. *Let the function $f(z)$ defined by (1.12) be in the class $\Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda)$, then we have:*

(i) *$f(z)$ is meromorphically p-valent starlike of order δ $(0 \leq \delta < p)$ in the disc $|z| < r_1$, that is,*

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \delta \quad (|z| < r_1;\ 0 \leq \delta < p;\ p \in N), \tag{3.13}$$

where

$$r_1 = \inf_{k\geq p}\left\{\frac{(p-\delta)k(1-B)\Gamma_{k+p}(\alpha_1)}{(k+\delta)(A-B)(p-\lambda)}\right\}^{\frac{1}{k+p}}. \tag{3.14}$$

(ii) *$f(z)$ is meromorphically p-valent convex of order δ $(0 \leq \delta < p)$ in the disc $|z| < r_2$, that is,*

$$\operatorname{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \delta \quad (|z| < r_2;\, 0 \leq \delta < p;\, p \in N), \tag{3.15}$$

where

$$r_2 = \inf_{k\geq p}\left\{\frac{p(p-\delta)(1-B)\Gamma_{k+p}(\alpha_1)}{(k+\delta)(A-B)(p-\lambda)}\right\}^{\frac{1}{k+p}}. \tag{3.16}$$

Each of these results is sharp for the function $f(z)$ given by (3.7).

Proof. (i) From the definition (1.12), we easily get

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \leq \frac{\sum_{k=p}^{\infty}(k+p)|a_k||z|^{k+p}}{2(p-\delta)-\sum_{k=p}^{\infty}(k-p+2\delta)|a_k||z|^{k+p}}. \tag{3.17}$$

Thus, we have the desired inequality

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \leq 1 \quad (0 \leq \delta < p;\; p \in N), \tag{3.18}$$

if

$$\sum_{k=p}^{\infty}\left(\frac{k+\delta}{p-\delta}\right)|a_k||z|^{k+p} \leq 1. \tag{3.19}$$

Hence, by Theorem 4, (3.19) will be true if

$$\left(\frac{k+\delta}{p-\delta}\right)|z|^{k+p} \leq \left\{\frac{k(1-B)\Gamma_{k+p}(\alpha_1)}{(A-B)(p-\lambda)}\right\} \quad (k \geq p;\; p \in N). \tag{3.20}$$

The last inequality (3.20) leads us immediately to the disc $|z| < r_1$, where r_1 is given by (3.14).

(ii) In order to prove the second assertion of Theorem 7, we find from the definition (1.12) that

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\delta}\right| \leq \frac{\sum_{k=p}^{\infty}k(k+p)|a_k||z|^{k+p}}{2p(p-\delta)-\sum_{k=p}^{\infty}k(k-p+2\delta)|a_k||z|^{k+p}}. \tag{3.21}$$

Thus we have the desired inequality

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\delta}\right| \leq 1 \quad (0 \leq \delta < p;\; p \in N), \tag{3.22}$$

if

$$\sum_{k=p}^{\infty}\frac{k(k+\delta)}{p(p-\delta)}|a_k|\,|z|^{k+p} \leq 1. \tag{3.23}$$

Hence, by Theorem 4, (3.23) will be true if

$$\frac{k(k+\delta)}{p(p-\delta)}|z|^{k+p} \leq \frac{k(1-B)\Gamma_{k+p}(\alpha_1)}{(A-B)(p-\lambda)} \quad (k \geq p;\ p \in N). \tag{3.24}$$

The last inequality (3.24) readily yields the disc $|z| < r_2$, where r_2 is defined by (3.16), and the proof of Theorem 7 is completed by merely verifying that each assertion is sharp for the function $f(z)$ given by (3.7). □

4. Neighborhoods and partial sums

In this section, we also assume that

$$\alpha_j > 0 \quad (j = 1, \ldots, q) \quad \text{and} \quad \beta_j > 0 \quad (j = 1, \ldots, s).$$

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [22] and Ruscheweyh [23], and (more recently) by Altintas et al. [21,26], Altintas and Owa [25], Liu [7,27] and Liu and Srivastava [2,3], we begin by introducing here the δ-neighborhood of a function $f(z) \in \Sigma_p$ of the form (1.1) by means of the definition given below:

$$N_\delta(f) = \left\{ g \in \Sigma_p : g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \text{ and } \sum_{k=1}^{\infty} \frac{(k+p)(1+|B|)\Gamma_k(\alpha_1)}{(A-B)(p-\lambda)} \left|b_{k-p} - a_{k-p}\right| \leq \delta, \right.$$
$$\left. (-1 \leq B < A \leq 1;\ \delta > 0;\ 0 \leq \lambda < p;\ p \in N) \right\}. \tag{4.1}$$

Making use of the definition (4.1), we now prove Theorem 8 below.

Theorem 8. *Let the function $f(z)$ defined by (1.1) be in the class $\Sigma_{p,q,s}(\alpha_1; A, B, \lambda)$. If $f(z)$ satisfies the following condition:*

$$\frac{f(z) + \epsilon z^{-p}}{1+\epsilon} \in \Sigma_{p,q,s}(\alpha_1; A, B, \lambda) \quad (\epsilon \in C,\ |\epsilon| < \delta,\ \delta > 0), \tag{4.2}$$

then

$$N_\delta(f) \subset \Sigma_{p,q,s}(\alpha_1; A, B, \lambda). \tag{4.3}$$

Proof. It is easily seen from (1.11) that $g(z) \in \Sigma_{p,q,s}(\alpha_1; A, B, \lambda)$ if and only if for any complex number σ with $|\sigma| = 1$,

$$\frac{z^{p+1}(H_{p,q,s}(\alpha_1)g(z))' + p}{Bz^{p+1}(H_{p,q,s}(\alpha_1)g(z))' + [pB + (A-B)(p-\lambda)]} \neq \sigma \quad (z \in U), \tag{4.4}$$

which is equivalent to

$$\frac{(g*h)(z)}{z^{-p}} \neq 0 \quad (z \in U), \tag{4.5}$$

where, for convenience,

$$h(z) = z^{-p} + \sum_{k=1}^{\infty} c_{k-p} z^{k-p}$$
$$= z^{-p} + \sum_{k=1}^{\infty} \frac{(k-p)(1-B\sigma)\Gamma_k(\alpha_1)}{(B-A)(p-\lambda)\sigma} z^{k-p}. \tag{4.6}$$

From (4.6), we have

$$\left|c_{k-p}\right| = \left|\frac{(k-p)(1-B\sigma)\Gamma_k(\alpha_1)}{(B-A)(p-\lambda)\sigma}\right| \le \frac{(k+p)(1+|B|)\Gamma_k(\alpha_1)}{(A-B)(p-\lambda)} \quad (k, p \in N). \tag{4.7}$$

Now, if $f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \in \Sigma_p$ satisfies the condition (4.2), then (4.5) yields

$$\left|\frac{(f*h)(z)}{z^{-p}}\right| \ge \delta \quad (z \in U;\ \delta > 0). \tag{4.8}$$

By letting

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \in N_\delta(f), \tag{4.9}$$

so that

$$\left|\frac{[g(z)-f(z)]*h(z)}{z^{-p}}\right| = \left|\sum_{k=1}^{\infty}(b_{k-p}-a_{k-p})c_{k-p}z^k\right| \le |z| \sum_{k=1}^{\infty} \frac{(k+p)(1+|B|)\Gamma_k(\alpha_1)}{(A-B)(p-\lambda)} \left|b_{k-p}-a_{k-p}\right| < \delta \quad (z \in U;\ \delta > 0). \tag{4.10}$$

Thus we have (4.5), and hence also (4.4) for any $\sigma \in C$ such that $|\sigma| = 1$, which implies that $g(z) \in \Sigma_{p,q,s}(\alpha_1; A, B, \lambda)$. This evidently proves the assertion (4.3) of Theorem 8.

We now define the δ-neighborhood of a function $f(z) \in \Sigma_p$ of the form (1.12) as follows:

$$N_\delta^+(f) = \left\{ g \in \Sigma_p : g(z) = z^{-p} + \sum_{k=p}^{\infty} |b_k| z^k \text{ and } \sum_{k=p}^{\infty} \frac{k(1+|B|)\Gamma_{k+p}(\alpha_1)}{(A-B)(p-\lambda)} \,||b_k| - |a_k|| \le \delta, \quad (-1 \le B < A \le 1;\ \delta \ge 0;\ 0 \le \lambda < p;\ p \in N) \right\}. \quad \square \tag{4.11}$$

Theorem 9. *Let the function $f(z)$ defined by (1.12) be in the class $\Sigma^+_{p,q,s}(\alpha_1+1; A, B, \lambda)$, $-1 \le B < A \le 1$, $-1 \le B \le 0$, $0 \le \lambda < p$ and $p \in N$, then*

$$N_\delta^+(f) \subset \Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda) \quad \left(\delta = \frac{2p}{\alpha_1 + 2p}\right). \tag{4.12}$$

The result is sharp in the sense that δ cannot be increased.

Proof. Making use of the same method as in the proof of Theorem 8, we can show that [cf. Eq. (4.6)]

$$h(z) = z^{-p} + \sum_{k=p}^{\infty} c_k z^k = z^{-p} + \sum_{k=p}^{\infty} \frac{k(1-B\sigma)\Gamma_{k+p}(\alpha_1)}{(B-A)(p-\lambda)\sigma} z^k. \tag{4.13}$$

Thus, under the hypothesis $-1 \le B < A \le 1, -1 \le B \le 0, 0 \le \lambda < p$ and $p \in N$, if $f(z) \in \Sigma^+_{p,q,s}(\alpha_1+1; A, B, \lambda)$ is given by (1.12), we obtain

$$\left|\frac{(f*h)(z)}{z^{-p}}\right| = \left|1 + \sum_{k=p}^{\infty} c_k |a_k| z^{k+p}\right|$$

$$\geq 1 - \frac{\alpha_1}{\alpha_1 + 2p} \sum_{k=p}^{\infty} \frac{k(1-B)\Gamma_{k+p}(\alpha_1+1)}{(A-B)(p-\lambda)} |a_k|.$$

Also, from Theorem 4, we obtain

$$\left| \frac{(f*h)(z)}{z^{-p}} \right| \geq 1 - \frac{\alpha_1}{\alpha_1 + 2p} = \frac{2p}{\alpha_1 + 2p} = \delta.$$

The remaining part of the proof of Theorem 9 is similar to that of Theorem 8, and we skip the details involved.

To show the sharpness, we consider the functions $f(z)$ and $g(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(p-\lambda)}{p(1-B)\Gamma_{2p}(\alpha_1+1)} z^p \in \Sigma^+_{p,q,s}(\alpha_1+1; A, B, \lambda) \tag{4.14}$$

and

$$g(z) = z^{-p} + \left[\frac{(A-B)(p-\lambda)}{p(1-B)\Gamma_{2p}(\alpha_1+1)} + \frac{(A-B)(p-\lambda)\delta'}{p(1-B)\Gamma_{2p}(\alpha_1)} \right] z^p, \tag{4.15}$$

where $\delta' > \delta = \frac{2p}{\alpha_1+2p}$. Clearly, the function $g(z)$ belongs to $N^+_{\delta'}(f)$. On the other hand, we find from Theorem 4 that $g(z)$ is not in the class $\Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda)$. Thus the proof of Theorem 9 is completed. □

Next we prove the following result.

Theorem 10. *Let $-1 \leq B \leq 0$. Suppose also that $f(z) \in \Sigma_p$ is given by (1.1) and define the partial sums $s_1(z)$ and $s_n(z)$ as follows:*

$$s_1(z) = z^{-p} \quad \textit{and} \quad s_n(z) = z^{-p} + \sum_{k=1}^{n-1} a_{k-p} z^{k-p} \quad (n \in N \setminus \{1\}). \tag{4.16}$$

Suppose also that

$$\sum_{k=1}^{\infty} d_k |a_{k-p}| \leq 1 \quad \left(d_k = \frac{(k+p)(1+|B|)\Gamma_k(\alpha_1)}{(A-B)(p-\lambda)} \right). \tag{4.17}$$

Then

(i) $f(z) \in \Sigma_{p,q,s}(\alpha_1; A, B, \lambda)$.
(ii) *If $\{\Gamma_k(\alpha_1)\}(k \in N)$ is nondecreasing and*

$$\Gamma_1(\alpha_1) > \frac{(A-B)(p-\lambda)}{(1+p)(1+|B|)}, \tag{4.18}$$

then

$$\operatorname{Re}\left\{ \frac{f(z)}{s_n(z)} \right\} > 1 - \frac{1}{d_n} \quad (z \in U;\ n \in N), \tag{4.19}$$

and

$$\operatorname{Re}\left\{ \frac{s_n(z)}{f(z)} \right\} > \frac{d_n}{1+d_n} \quad (z \in U;\ n \in N). \tag{4.20}$$

Each of the bounds in (4.19) and (4.20) is the best possible for each $n \in N$.

Proof. (i) It is not difficult to see that

$$z^{-p} \in \Sigma_{p,q,s}(\alpha_1; A, B, \lambda) \quad (p \in N). \tag{4.21}$$

Thus, from Theorem 8 and the hypothesis (4.17) of Theorem 10 we have $N_1(z^{-p}) \subset \Sigma_{p,q,s}(\alpha_1; A, B, \lambda)$ as asserted by Theorem 10.

(ii) Under the hypothesis in Part (ii) of Theorem 10, we can see from (4.17) that

$$d_{k+1} > d_k > 1 \quad (k \in N). \tag{4.22}$$

Therefore, we have

$$\sum_{k=1}^{n-1} |a_{k-p}| + d_n \sum_{k=n}^{\infty} |a_{k-p}| \le \sum_{k=1}^{\infty} d_k |a_{k-p}| \le 1, \tag{4.23}$$

by using hypothesis (4.17) of Theorem 10 again.

By setting

$$g_1(z) = d_n \left[\frac{f(z)}{s_n(z)} - \left(1 - \frac{1}{d_n}\right) \right] = 1 + \frac{d_n \sum_{k=n}^{\infty} a_{k-p} z^k}{1 + \sum_{k=1}^{n-1} a_{k-p} z^k}, \tag{4.24}$$

and applying (4.23), we find that

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \le \frac{d_n \sum_{k=n}^{\infty} |a_{k-p}|}{2 - 2\sum_{k=1}^{n-1} |a_{k-p}| - d_n \sum_{k=n}^{\infty} |a_{k-p}|} \le 1 \quad (z \in U), \tag{4.25}$$

which readily yields the assertion (4.19) of Theorem 10. If we take

$$f(z) = z^{-p} - \frac{z^{n-p}}{d_n}, \tag{4.26}$$

then

$$\frac{f(z)}{s_n(z)} = 1 - \frac{z^n}{d_n} \to 1 - \frac{1}{d_n} \quad (z \to 1^-),$$

which shows that the bound in (4.19) is the best possible for each $n \in N$.

Similarly, if we put

$$\begin{aligned} g_2(z) &= (1 + d_n) \left(\frac{s_n(z)}{f(z)} - \frac{d_n}{1 + d_n} \right) \\ &= 1 - \frac{(1 + d_n) \sum_{k=n}^{\infty} a_{k-p} z^k}{1 + \sum_{k=1}^{\infty} a_{k-p} z^k} \end{aligned} \tag{4.27}$$

and make use of (4.23), we can deduce that

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \le \frac{(1 + d_n) \sum_{k=n}^{\infty} |a_{k-p}|}{2 - 2\sum_{k=1}^{n-1} |a_{k-p}| + (1 - d_n) \sum_{k=n}^{\infty} |a_{k-p}|} \le 1 \quad (z \in U), \tag{4.28}$$

which leads us immediately to the assertion (4.20) of Theorem 10.

The bound in (4.20) is sharp for each $n \in N$, with the extremal function $f(z)$ given by (4.26). The proof of Theorem 10 is thus completed. □

5. Convolution properties

For the functions $f_j(z)(j = 1, 2)$ defined by (3.8) we denote by $(f_1 * f_2)(z)$ the Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, which is,

$$(f_1 * f_2)(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,1}||a_{k,2}|z^k. \tag{5.1}$$

Throughout this section, we assume further that the sequence $\{k\Gamma_{k+p}(\alpha_1)\}(k \geq p;\ p \in N)$ is nondecreasing.

Theorem 11. *Let the functions $f_j(z)(j = 1, 2)$ defined by (3.8) be in the class $\Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda)$, then $(f_1 * f_2)(z) \in \Sigma^+_{p,q,s}(\alpha_1; A, B, \zeta)$, where*

$$\zeta = p - \frac{(A - B)(p - \lambda)^2}{p(1 - B)\Gamma_{2p}(\alpha_1)}. \tag{5.2}$$

The result is sharp for the functions $f_j(z)(j = 1, 2)$ given by

$$f_j(z) = z^{-p} + \frac{(A - B)(p - \lambda)}{p(1 - B)\Gamma_{2p}(\alpha_1)} z^p \quad (j = 1, 2;\ p \in N). \tag{5.3}$$

Proof. Employing the techniques used earlier by Schild and Silverman [28], we need to find the largest ζ such that

$$\sum_{k=p}^{\infty} \frac{k(1 - B)\Gamma_{k+p}(\alpha_1)}{(A - B)(p - \zeta)} |a_{k,1}||a_{k,2}| \leq 1 \tag{5.4}$$

for $f_j(z) \in \Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda)(j = 1, 2)$. Since $f_j(z) \in \Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda)(j = 1, 2)$, we readily see that

$$\sum_{k=p}^{\infty} \frac{k(1 - B)\Gamma_{k+p}(\alpha_1)}{(A - B)(p - \lambda)} |a_{k,j}| \leq 1 \quad (j = 1, 2). \tag{5.5}$$

Therefore, by the Cauchy–Schwarz inequality, we obtain

$$\sum_{k=p}^{\infty} \frac{k(1 - B)\Gamma_{k+p}(\alpha_1)}{(A - B)(p - \lambda)} \sqrt{|a_{k,1}||a_{k,2}|} \leq 1. \tag{5.6}$$

This implies that we only need to show that

$$\frac{1}{(p - \zeta)} \cdot |a_{k,1}||a_{k,2}| \leq \frac{1}{(p - \lambda)} \cdot \sqrt{|a_{k,1}||a_{k,2}|} \quad (k \geq p) \tag{5.7}$$

or, equivalently that

$$\sqrt{|a_{k,1}||a_{k,2}|} \leq \frac{(p - \zeta)}{(p - \lambda)} \quad (k \geq p). \tag{5.8}$$

Hence, by the inequality (5.6), it is sufficient to prove that

$$\frac{(A - B)(p - \lambda)}{k(1 - B)\Gamma_{k+p}(\alpha_1)} \leq \frac{(p - \zeta)}{(p - \lambda)} \quad (k \geq p). \tag{5.9}$$

It follows from (5.9) that

$$\zeta \leq p - \frac{(A - B)(p - \lambda)^2}{k(1 - B)\Gamma_{k+p}(\alpha_1)} \quad (k \geq p). \tag{5.10}$$

Now, defining the function $\varphi(k)$ by

$$\varphi(k) = p - \frac{(A-B)(p-\lambda)^2}{k(1-B)\Gamma_{k+p}(\alpha_1)} \quad (k \geq p).$$

We see that $\varphi(k)$ is an increasing function of k. Therefore, we conclude that

$$\zeta \leq \varphi(p) = p - \frac{(A-B)(p-\lambda)^2}{p(1-B)\Gamma_{2p}(\alpha_1)}, \tag{5.11}$$

which evidently completes the proof of Theorem 11. □

Putting (i) $A = \beta$ and $B = -\beta(0 < \beta \leq 1)$ (ii) $A = \beta$ and $B = -(2\gamma - 1)\beta\,(\frac{1}{2} \leq \gamma \leq 1$ and $0 < \beta \leq 1)$ in Theorem 11, we obtain:

Corollary 2. *Let the functions $f_j(z)(j = 1, 2)$ defined by (3.8) be in the class $\Sigma^+_{p,q,s}(\alpha_1; \lambda, \beta)$, then $(f_1 * f_2)(z) \in \Sigma^+_{p,q,s}(\alpha_1; \zeta, \beta)$, where*

$$\zeta = p - \frac{2\beta(p-\lambda)^2}{p(1+\beta)\Gamma_{2p}(\alpha_1)}. \tag{5.12}$$

The result is sharp for the functions $f_j(z)(j = 1, 2)$ given by

$$f_j(z) = z^{-p} + \frac{2\beta(p-\lambda)}{p(1+\beta)\Gamma_{2p}(\alpha_1)} z^p \quad (j = 1, 2;\ p \in N). \tag{5.13}$$

Corollary 3. *Let the functions $f_j(z)(j = 1, 2)$ defined by (3.8) be in the class $\Sigma^+_{p,q,s}(\alpha_1; \lambda, \beta, \gamma)$, then $(f_1 * f_2)(z) \in \Sigma^+_{p,q,s}(\alpha_1; \zeta, \beta, \gamma)$, where*

$$\zeta = p - \frac{2\beta\gamma(p-\lambda)^2}{p(1+2\beta\gamma-\beta)\Gamma_{2p}(\alpha_1)}. \tag{5.14}$$

The result is sharp for the functions

$$f_j(z) = z^{-p} + \frac{2\beta\gamma(p-\lambda)}{p(1+2\beta\gamma-\beta)\Gamma_{2p}(\alpha_1)} z^p \quad (j = 1, 2;\ p \in N). \tag{5.15}$$

Using arguments similar to those in the proof of Theorem 11, we obtain the following result:

Theorem 12. *Let the function $f_1(z)$ defined by (3.8) be in the class $\Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda)$, and the function $f_2(z)$ defined by (3.8) be in the class $\Sigma^+_{p,q,s}(\alpha_1; A, B, \varphi)$. Then $(f_1 * f_2)(z) \in \Sigma^+_{p,q,s}(\alpha_1; A, B, \xi)$, where*

$$\xi = p - \frac{(A-B)(p-\lambda)(p-\varphi)}{p(1-B)\Gamma_{2p}(\alpha_1)}. \tag{5.16}$$

The result is sharp for the functions $f_j(z)(j = 1, 2)$ given by

$$f_1(z) = z^{-p} + \frac{(A-B)(p-\lambda)}{p(1-B)\Gamma_{2p}(\alpha_1)} z^p \quad (p \in N) \tag{5.17}$$

and

$$f_2(z) = z^{-p} + \frac{(A-B)(p-\varphi)}{p(1-B)\Gamma_{2p}(\alpha_1)} z^p \quad (p \in N). \tag{5.18}$$

Putting (i) $A = \beta$ and $B = -\beta\,(0 < \beta \leq 1)$ (ii) $A = \beta$ and $B = -(2\gamma - 1)\beta\,(\frac{1}{2} \leq \gamma \leq 1$ and $0 < \beta \leq 1)$ in Theorem 12, we obtain:

Corollary 4. *Let the function $f_1(z)$ defined by (3.8) be in the class $\Sigma^+_{p,q,s}(\alpha_1; \lambda, \beta)$, and the function $f_2(z)$ defined by (3.8) be in the class $\Sigma^+_{p,q,s}(\alpha_1; \varphi, \beta)$. Then $(f_1 * f_2)(z) \in \Sigma^+_{p,q,s}(\alpha_1; \Im, \beta)$, where*

$$\Im = p - \frac{2\beta(p-\lambda)(p-\varphi)}{p(1+\beta)\Gamma_{2p}(\alpha_1)}. \tag{5.19}$$

The result is the best possible for the functions $f_j(z)(j = 1, 2)$ given by

$$f_1(z) = z^{-p} + \frac{2\beta(p-\lambda)}{p(1+\beta)\Gamma_{2p}(\alpha_1)} z^p \quad (p \in N) \tag{5.20}$$

and

$$f_2(z) = z^{-p} + \frac{2\beta(p-\varphi)}{p(1+\beta)\Gamma_{2p}(\alpha_1)} z^p \quad (p \in N). \tag{5.21}$$

Corollary 5. *Let the function $f_1(z)$ defined by (3.8) be in the class $\Sigma^+_{p,q,s}(\alpha_1; \lambda, \beta, \gamma)$, and the function $f_2(z)$ defined by (3.8) be in the class $\Sigma^+_{p,q,s}(\alpha_1; \varphi, \beta, \gamma)$. Then $(f_1 * f_2)(z) \in \Sigma^+_{p,q,s}(\alpha_1; \xi, \beta, \gamma)$, where*

$$\xi = p - \frac{2\beta\gamma(p-\lambda)(p-\varphi)}{p(1+2\beta\gamma-\beta)\Gamma_{2p}(\alpha_1)}. \tag{5.22}$$

The result is the best possible for the functions $f_j(z)(j = 1, 2)$ given by

$$f_1(z) = z^{-p} + \frac{2\beta\gamma(p-\lambda)}{p(1+2\beta\gamma-\beta)\Gamma_{2p}(\alpha_1)} z^p \quad (p \in N) \tag{5.23}$$

and

$$f_2(z) = z^{-p} + \frac{2\beta\gamma(p-\varphi)}{p(1+2\beta\gamma-\beta)\Gamma_{2p}(\alpha_1)} z^p \quad (p \in N). \tag{5.24}$$

Theorem 13. *Let the functions $f_j(z)(j = 1, 2)$ defined by (3.8) be in the class $\Sigma^+_{p,q,s}(\alpha_1 A, B, \lambda)$. Then the function $h(z)$ defined by*

$$h(z) = z^{-p} + \sum_{k=p}^{\infty} (|a_{k,1}|^2 + |a_{k,2}|^2) z^k; \tag{5.25}$$

belongs to the class $\Sigma^+_{p,q,s}(\alpha_1; A, B, \zeta)$, where

$$\zeta = p - \frac{2(A-B)(p-\lambda)^2}{p(1-B)\Gamma_{2p}(\alpha_1)}. \tag{5.26}$$

This result is sharp for the functions $f_j(z)(j = 1, 2)$ given already by (5.3).

Proof. Noting that

$$\sum_{k=p}^{\infty} \left[\frac{k(1-B)\Gamma_{k+p}(\alpha_1)}{(A-B)(p-\lambda)}\right]^2 |a_{k,j}|^2 \le \left[\sum_{k=p}^{\infty} \frac{k(1-B)\Gamma_{k+p}(\alpha)_1}{(A-B)(p-\lambda)} |a_{k,j}|\right]^2 \le 1 \quad (j = 1, 2), \tag{5.27}$$

for $f_j(z) \in \Sigma^+_{p,q,s}(\alpha_1; A, B, \lambda)(j = 1, 2)$, we have

$$\sum_{k=p}^{\infty} \frac{1}{2} \left[\frac{k(1-B)\Gamma_{k+p}(\alpha_1)}{(A-B)(p-\lambda)}\right]^2 (|a_{k,1}|^2 + |a_{k,2}|^2) \le 1. \tag{5.28}$$

Therefore, we have to find the largest ζ such that

$$\frac{1}{(p-\zeta)} \leq \frac{k(1-B)\Gamma_{k+p}(\alpha_1)}{2(A-B)(p-\lambda)^2} \quad (k \geq p), \tag{5.29}$$

so that

$$\zeta \leq p - \frac{2(A-B)(p-\lambda)^2}{p(1-B)\Gamma_{k+p}(\alpha_1)} \quad (k \geq p). \tag{5.30}$$

Now, we define a function $\Psi(k)$ by

$$\Psi(k) = p - \frac{2(A-B)(p-\lambda)^2}{k(1-B)\Gamma_{k+p}(\alpha_1)} \quad (k \geq p). \tag{5.31}$$

We observe that $\Psi(k)$ is an increasing function of k $(k \geq p)$. Thus, we conclude that

$$\zeta \leq \Psi(p) = p - \frac{2(A-B)(p-\lambda)^2}{p(1-B)\Gamma_{2p}(\alpha_1)}, \tag{5.32}$$

which completes the proof of Theorem 13. □

Putting (i) $A = \beta$ and $B = -\beta$ $(0 < \beta \leq 1)$ (ii) $A = \beta$ and $B = -(2\gamma - 1)\beta$ $(\frac{1}{2} \leq \gamma \leq 1$ and $0 < \beta \leq 1)$ in Theorem 13, we obtain:

Corollary 6. *Let the functions $f_j(z)(j = 1, 2)$ defined by (3.8) be in the class $\Sigma^{+}_{p,q,s}(\alpha_1; \lambda, \beta)$, then the function $h(z)$ defined by (5.25) belongs to the class $\Sigma^{+}_{p,q,s}(\alpha_1; \varphi, \beta)$, where*

$$\varphi = p - \frac{4\beta(p-\lambda)^2}{p(1+\beta)\Gamma_{2p}(\alpha_1)}. \tag{5.33}$$

The result is sharp for the functions $f_j(z)(j = 1, 2)$ defined by (5.13).

Corollary 7. *Let the functions $f_j(z)(j = 1, 2)$ defined by (3.8) be in the class $\Sigma^{+}_{p,q,s}(\alpha_1; \lambda, \beta, \gamma)$, then the function $h(z)$ defined by (5.25) belongs to the class $\Sigma^{+}_{p,q,s}(\alpha_1; \varphi, \beta, \gamma)$, where*

$$\varphi = p - \frac{4\beta\gamma(p-\lambda)^2}{p(1+2\beta\gamma-\beta)\Gamma_{2p}(\alpha_1)}. \tag{5.34}$$

The result is sharp for the functions $f_j(z)(j = 1, 2)$ defined by (5.15).

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