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Argument estimates of certain meromorphically multivalent functions associated with generalized hypergeometric function

M.K. Aouf

Faculty of Science, Mansoura University, Mansoura 35516, Egypt

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ABSTRACT

The object of this paper is to obtain some argument properties of meromorphically multivalent functions associated with generalized hypergeometric function. We also derive the integral preserving properties in a sector.

1. Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in N = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the punctured disc $U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$. For functions $f(z) \in \Sigma_p$ given by (1.1), and $g(z) \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (p \in N), \tag{1.2}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p}. \tag{1.3}$$

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s (\beta_j \notin Z_0^- = \{0, -1, -2, \ldots\};\ j = 1, \ldots, s)$, we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q;\ \beta_1, \ldots, \beta_s;\ z)$ by

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k} \cdot \frac{z^k}{k!} \quad (q \leqslant s+1; q, s \in N_0 = N \cup \{0\}; z \in U), \tag{1.4}$$

where $(\theta)v$ is the Pochhammer symbol defined, in terms the Gamma function Γ, by

$$(\theta)v = \frac{\Gamma(\theta + v)}{\Gamma(\theta)} = \begin{cases} 1, & (v = 0; \theta \in C \setminus \{0\}), \\ \theta(\theta+1)\cdots(\theta+v-1), & (v \in N; \theta \in C). \end{cases} \tag{1.5}$$

E-mail address: mkaouf127@yahoo.com

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Corresponding to a function $h_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ defined by

$$h_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) = z^{-p}{}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z),$$

Liu and Srivastava [13] (see, for details, [6,7]) introduced a linear operator

$$H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s) : \Sigma_p \to \Sigma_p,$$

which is defined by the following Hadamard product (or convolution)

$$H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z) = h_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) * f(z). \tag{1.6}$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z) = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_q)_k}{(\beta_1)_k\cdots(\beta_s)_k}\cdot\frac{a_{k-p}}{k!}z^{k-p}. \tag{1.7}$$

If, for convenience, we write

$$H_{p,q,s}(\alpha_1) = H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s), \tag{1.8}$$

then one can easily verify from the definition (1.6) that

$$z(H_{p,q,s}(\alpha_1)f(z))' = \alpha_1 H_{p,q,s}(\alpha_1+1)f(z) - (\alpha_1 - p)H_{p,q,s}(\alpha_1)f(z). \tag{1.9}$$

Some interesting subclasses of analytic functions, associated with the generalized hypergeometric function, were considered recently by (for example) Gangadharan et al. [9], Liu [11] and Aouf [2].

Let $f(z)$ and $g(z)$ be analytic in U. Then we say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic function $w(z)$ in U such that $|w(z)| < 1 (z \in U)$ and $g(z) = f(w(z))$. For this subordination, the symbol $g(z) \prec f(z)$ is used. In case $f(z)$ is univalent in U, the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$.

For a function $f(z) \in \Sigma_p$ and $v > 0$ the integral operator $F_{v,p}(f)(z) : \Sigma_p \to \Sigma_p$ is defined by

$$F_{v,p}(f)(z) = \frac{v}{z^{v+p}}\int_0^z t^{v+p-1}f(t)dt = \left(z^{-p} + \sum_{k=1}^{\infty}\left(\frac{v}{v+k}\right)z^{p-k}\right) * f(z) = z^{-p}{}_2F_1(v,1;v+1;z) * f(z) \quad (v > 0; z \in U). \tag{1.10}$$

It follows from (1.10) that

$$z(H_{p,q,s}(H_{p,q,s}(\alpha_1)F_{v,p}(f)(z)))' = vH_{p,q,s}(\alpha_1)f(z) - (v+p)H_{p,q,s}(\alpha_1)F_{v,p}(f)(z). \tag{1.11}$$

The operator $F_{v,p}(f)(z)$ was investigated by many authors (see for example [1,17,18]).

We note that:

(i) For $q = 2, s = 1$ and $\alpha_2 = 1$, we obtain the linear operator

$$H_{p,2,1}(\alpha_1,1;\beta_1)f(z) = \ell_p(\alpha_1,\beta_1)f(z) \quad (f \in \Sigma_p),$$

which was introduced and studied by Liu and Srivastava [12].

(ii) For any integer $n > -p$ and $f(z) \in \Sigma_p$, we have

$$H_{p,2,1}(n+p,1;1)f(z) = D^{n+p-1}f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z),$$

where $D^{n+p-1}f(z)$ is the differential operator studied by Uralegaddi and Somanatha [17] and Aouf [1].

(iii) $H_{p,2,1}(v,1;v+1)f(z) = F_{v,p}(f)(z) (v > 0)$. Let

$$\Sigma_p^*[\alpha_1;A,B] = \left\{f \in \Sigma_p : -\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)} \prec p\frac{1+Az}{1+Bz}, -1 \leqslant B < A \leqslant 1, z \in U\right\}. \tag{1.12}$$

We note that:

(i) For $q = 2, s = 1, \alpha_1 = \beta_1 = p, \alpha_2 = 1, A = 1$ and $B = -1$, we note that $\Sigma_p^*[p,1;p;1,-1] = \Sigma_p^*$ is the well-known class of meromorphically starlike functions.

(ii) $q = 2, s = 1, \alpha_1 = \beta_1 = p, \alpha_2 = 1, A = 1 - \frac{2\alpha}{p}, 0 \leqslant \alpha < p$, and $B = -1$, we note that $\Sigma_p^*\left[p,1;p;1-\frac{2\alpha}{p},-1\right] = \Sigma_p^*[\alpha]$ is the well-known class of meromorphically starlike functions of order α (see [3]).

From (1.12) and by using the result of Silverman and Silvia [16], we observe that a function $f(z)$ is in $\Sigma_p^*[\alpha_1;A,B]$ if and only if

$$\left|\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)} + \frac{p(1-AB)}{1+B^2}\right| < \frac{p(A-B)}{1-B^2} \quad (-1 < B < A \leqslant 1; z \in U). \tag{1.13}$$

The object of the present paper is to give some argument properties of meromorphically functions belonging to Σ_p and the integral preserving properties in connection with the operator $H_{p,q,s}(\alpha_1)$ defined by (1.7).

2. Main result

In order to show our main results, we need the following lemmas.

Lemma 1 [8]. *Let h be convex univalent in U with $h(0)=1$ and $\mathrm{Re}(\beta h(z)+\gamma)>0 (\beta,\gamma\in C)$. If q is analytic in U with $q(0)=1$, then*

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\prec h(z) \quad (z\in U)$$

implies

$$q(z)\prec h(z) \quad (z\in U).$$

Lemma 2 [14]. *Let h be convex univalent in U and $\lambda(z)$ be analytic in U $\mathrm{Re}\,\lambda(z)\geqslant 0$. If q is analytic in U and $q(0)=h(0)$, then*

$$q(z)+\lambda(z)zq'(z)\prec h(z) \quad (z\in U)$$

implies

$$q(z)\prec h(z) \quad (z\in U).$$

Lemma 3 [15]. *Let q be analytic in U with $q(0)=1$ and $q(z)\neq 0$ in U. Suppose that there exists a point z_0 in U such that*

$$|\arg q(z)|<\frac{\pi}{2}\alpha \quad \text{for } |z|<|z_0| \tag{2.1}$$

and

$$|\arg q(z_0)|=\frac{\pi}{2}\alpha \quad (0<\alpha\leqslant 1). \tag{2.2}$$

Then we have

$$\frac{z_0q'(z_0)}{q(z_0)}=ik\alpha, \tag{2.3}$$

where

$$k\geqslant\frac{1}{2}\left(a+\frac{1}{a}\right) \quad \text{when } \arg q(z_0)=\frac{\pi}{2}\alpha, \tag{2.4}$$

$$k\geqslant\frac{-1}{2}\left(a+\frac{1}{a}\right) \quad \text{when } \arg q(z_0)=\frac{-\pi}{2}\alpha \tag{2.5}$$

and

$$q(z_0)^{\frac{1}{\alpha}}=\pm ia \quad (a>0). \tag{2.6}$$

At first, with the help of Lemma 1, we obtain the following theorem.

Theorem 1. *Let h be convex univalent in U with $h(0)=1$ and $\mathrm{Re}\,h$ be bounded in U. If $f(z)\in\Sigma_p$ satisfies the condition*

$$-\frac{z(H_{p,q,s}(\alpha_1+1)f(z))'}{pH_{p,q,s}(\alpha_1+1)f(z)}\prec h(z) \quad (z\in U),$$

then

$$-\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{pH_{p,q,s}(\alpha_1)f(z)}\prec h(z) \quad (z\in U)$$

for $\max_{z\in U}\mathrm{Re}\,h(z)<\mathrm{Re}\left(\frac{\alpha_1+p}{p}\right)$ (provided $H_{p,q,s}(\alpha_1)f(z)\neq 0$ in U).

Proof. Let

$$q(z)=-\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)} \quad (z\in U).$$

By using (1.9), we have

$$q(z) - \left(\frac{\alpha_1 + p}{p}\right) = -\frac{\alpha_1 H_{p,q,s}(\alpha_1 + 1)f(z)}{pH_{p,q,s}(\alpha_1)f(z)}. \tag{2.7}$$

Taking logarithmic derivatives in both sides of (2.7) and multiplying by z, we get

$$\frac{zq'(z)}{-pq(z) + (\alpha_1 + p)} + q(z) = -\frac{z(H_{p,q,s}(\alpha_1 + 1)f(z))'}{pH_{p,q,s}(\alpha_1 + 1)f(z)} \prec h(z) \quad (z \in U).$$

From Lemma 1, it follows that $q(z) \prec h(z)$ for $\mathrm{Re}\left\{-h(z) + \left(\frac{\alpha_1+p}{p}\right)\right\} > 0 (z \in U)$, which means

$$-\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{pH_{p,q,s}(\alpha_1)f(z)} \prec h(z) \quad (z \in U)$$

for $\max_{z\in U} \mathrm{Re}\, h(z) < \mathrm{Re}\left(\frac{\alpha_1+p}{p}\right)$. □

Taking $q = 2, s = 1, \alpha_1 = n + p (n > -p), \alpha_2 = \beta_1 = 1$ and $h(z) = \frac{1+z}{1-z}$ in Theorem 1, we obtain the result obtained by Aouf [1, Theorem 1].

Using Lemmas 1 and 2 and Theorem 1, we now derive

Theorem 2. *Let $f(z) \in \Sigma_p$. Choose $\alpha_1 \in R$ such that*

$$\alpha_1 \geqslant \frac{p(A - B)}{1 + B},$$

where $-1 < B < A \leqslant 1$ and $p \in N$. If

$$\left|\arg\left(-\frac{z(H_{p,q,s}(\alpha_1 + 1)f(z))'}{H_{p,q,s}(\alpha_1 + 1)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \quad (0 \leqslant \gamma < p; 0 < \delta \leqslant 1)$$

for some $g \in \Sigma_p^*(\alpha_1 + 1; A, B)$, then

$$\left|\arg\left(-\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where $\alpha (0 < \alpha \leqslant 1)$ is the solution of the equation

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left\{\frac{\alpha \sin\frac{\pi}{2}(1 - t(A, B))}{\frac{\alpha_1(1-B)+p(A-B)}{1-B} + \alpha\cos\frac{\pi}{2}(1 - t(A, B))}\right\} \tag{2.8}$$

when

$$t(A, B) = \frac{2}{\pi}\sin^{-1}\left[\frac{p(A - B)}{(\alpha_1 + p)(1 - B^2) - p(1 - AB)}\right]. \tag{2.9}$$

Proof. Let

$$q(z) = -\frac{1}{p - \gamma}\left(\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)g(z)} + \gamma\right) \quad (z \in U).$$

By using the identity (1.9), we have

$$\begin{aligned}&(p - \gamma)zq'(z)H_{p,q,s}(\alpha_1)g(z) + (p - \gamma)q(z)z(H_{p,q,s}(\alpha_1)f(z))' + \gamma z(H_{p,q,s}(\alpha_1)g(z))' \\ &\quad = (\alpha_1 + p)z(H_{p,q,s}(\alpha_1)f(z))' - \alpha_1 z(H_{p,q,s}(\alpha_1 + 1)f(z))'.\end{aligned} \tag{2.10}$$

Dividing (2.10) by $H_{p,q,s}(\alpha_1)g(z)$ and simplifying, we obtain

$$q(z) + \frac{zq'(z)}{-r(z) + \alpha_1 + p} = -\frac{1}{p - \gamma}\left(\frac{z(H_{p,q,s}(\alpha_1 + 1)f(z))'}{H_{p,q,s}(\alpha_1 + 1)g(z)} + \gamma\right), \tag{2.11}$$

where

$$r(z) = -\frac{z(H_{p,q,s}(\alpha_1)g(z))'}{H_{p,q,s}(\alpha_1)g(z)}.$$

Since $g(z) \in \Sigma_p^*(\alpha_1 + 1; A, B)$, from Theorem 1, we have

$$r(z) \prec \frac{1 + Az}{1 + Bz},$$

using (1.13), we have

$$-r(z)+\alpha_1+p=\rho e^{i\frac{\pi}{2}\varphi},$$

where

$$\frac{\alpha_1(1+B)-p(A-B)}{1+B}<\rho<\frac{\alpha_1(1-B)+p(A-B)}{1-B},\quad -t(A,B)<\varphi<t(A,B),$$

where $t(A,B)$ is given by (2.9).

Let h be a function which maps onto the angular domain $\{w:|\arg w|<\frac{\pi}{2}\delta\}$ with $h(0)=1$. Applying Lemma 2 for this h with $\lambda(z)=\frac{1}{-r(z)+\alpha_1+p}$, we see that $\mathrm{Re}\,q(z)>0$ in U and hence $q(z)\neq 0$ in U.

If there exists a point $z_0\in U$ such that the conditions (2.1) and (2.2) are satisfied, then by Lemma 3, we have (2.3) under the restrictions (2.4)–(2.6).

At first, suppose that $q(z_0)^{\frac{1}{\alpha}}=ia\,(a>0)$. Then we obtain

$$\begin{aligned}\arg\left[-\frac{1}{p-\gamma}\left(\frac{z_0(H_{p,q,s}(\alpha_1+1)f(z_0))'}{H_{p,q,s}(\alpha_1+1)g(z_0)}+\gamma\right)\right]&=\arg\left[q(z_0)+\frac{z_0q'(z_0)}{-r(z_0)+\alpha_1+p}\right]=\frac{\pi}{2}\alpha+\arg\left(1+i\alpha k(\rho e^{i\frac{\pi}{2}\varphi})^{-1}\right)\\&=\frac{\pi}{2}\alpha+\tan^{-1}\left(\frac{\alpha k\sin\frac{\pi}{2}(1-\varphi)}{\rho+\alpha k\cos\frac{\pi}{2}(1-\varphi)}\right)\\&\geqslant\frac{\pi}{2}\alpha+\tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}(1-t(A,B))}{\frac{\alpha_1(1-B)+p(A-B)}{1-B}+\alpha\cos\frac{\pi}{2}(1-t(A,B))}\right)=\frac{\pi}{2}\delta,\end{aligned}$$

where δ and $t(A,B)$ are given by (2.8) and (2.9), respectively. This is a contradiction to the assumption of our theorem.

Next, suppose that $p(z_0)^{\frac{1}{\alpha}}=-ia\,(a>0)$. Applying the same method as the above, we have

$$\arg\left[-\frac{1}{p-\gamma}\left(\frac{z_0(H_{p,q,s}(\alpha_1+1)f(z_0))'}{H_{p,q,s}(\alpha_1+1)g(z_0)}+\gamma\right)\right]\leqslant-\frac{\pi}{2}\alpha-\tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}(1-t(A,B))}{\frac{\alpha_1(1-B)+p(A-B)}{1-B}+\alpha\cos\frac{\pi}{2}(1-t(A,B))}\right)=\frac{-\pi}{2}\delta,$$

where δ and $t(A,B)$ are given by (2.8) and (2.9), respectively, which contradicts the assumption. Therefore we complete the proof of Theorem 2. □

Taking $A=1, B=0$ and $\delta=1$ in Theorem 2, we have the following corollary.

Corollary 1. *Let $f(z)\in\Sigma_p$. If*

$$-\mathrm{Re}\left\{\frac{z(H_{p,q,s}(\alpha_1+1)f(z))'}{H_{p,q,s}(\alpha_1+1)g(z)}\right\}>\gamma\quad(0\leqslant\gamma<p)$$

for some $g\in\Sigma_p$ satisfying the condition

$$\left|\frac{z(H_{p,q,s}(\alpha_1+1)g(z))'}{H_{p,q,s}(\alpha_1+1)g(z)}+p\right|<p,$$

then

$$-\mathrm{Re}\left\{\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)g(z)}\right\}>\gamma\quad(0\leqslant\gamma<p).$$

Remark 1. Taking $q=s+1, \alpha_1=\beta_1=p, \alpha_j=1\ (j=2,3,\ldots,s+1)$ and $\beta_j=1\ (j=2,\ldots,s)$ in Corollary 1 we obtain the result obtained by Lashin [10, Corollary 2.4].

Taking $A=1, B=0$ and $g(z)=\frac{1}{z^p}$ in Theorem 2, we have the following corollary.

Corollary 2. *Let $f(z)\in\Sigma_p$ and $\alpha_1\geqslant p$. If*

$$|\arg[-z^{p+1}(H_{p,q,s}(\alpha_1+1)f(z))'-\gamma]|<\frac{\pi}{2}\delta\quad(0\leqslant\gamma<p;0<\gamma\leqslant 1),$$

then

$$|\arg[-z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'-\gamma]|<\frac{\pi}{2}\delta.$$

Taking $q=s+1, \alpha_1=\beta_1=p, \alpha_j=1\ (j=2,3,\ldots,s+1), \beta_j=1\ (j=2,3,\ldots,s)$ and $\delta=1$ in Corollary 2, we have the following corollary.

Corollary 3. *Let $f(z)\in\Sigma_p$. If*

$$-\mathrm{Re}\{z^{p+1}[zf''(z)+(2p+1)f'(z)]\}>\gamma\quad(0\leqslant\gamma<p),$$

then

$$-\text{Re}\{z^{p+1}f'(z)\} > \gamma.$$

Remark 2. Taking $q = 2, s = 1, \alpha_1 = \nu, \alpha_2 = 1$ and $\beta_1 = \nu + 1, \nu > -1$, in Theorem 2, we obtain the result obtained by Lashin [10, Corollary 2.3].

Remark 3

(i) Putting $q = 2, s = 1, \alpha_1 = n + p (n > -p), \alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 2, we obtain the result obtained by Cho and Owa [5, Theorem 2.1].
(ii) Putting $p = 1, q = 2, s = 1, \alpha_1 = n + 1 (n > -1), \alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 2, we obtain the result obtained by Cho [4, Theorem 2.1].
(iii) Putting $q = 2, s = 1, \alpha_1 = a > 0, \alpha_2 = 1$ and $\beta_1 = c > 0$ in Theorem 2, we obtain the result obtained by Lashin [10, Theorem 2.2].

By the same techniques as in the proof of Theorem 2, we obtain

Theorem 3. *Let $f(z) \in \Sigma_p$. Choose $\alpha_1 \in R$ such that*

$$\alpha_1 \geqslant \frac{p(A-B)}{1+B},$$

where $-1 < B < A \leqslant 1$ and $p \in N$. If

$$\left|\arg\left(\frac{z(H_{p,q,s}(\alpha_1+1)f(z))'}{H_{p,q,s}(\alpha_1+1)g(z)} + \gamma\right)\right| < \frac{\pi}{2}\delta \quad (\gamma > p; 0 < \delta \leqslant 1)$$

for some $g \in \Sigma_p^*[\alpha_1 + 1; A, B]$, then

$$\left|\arg\left(\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)g(z)} + \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where $\alpha (0 < \alpha \leqslant 1)$ is the solution of the equation given by (2.8).

Theorem 4. *Let h be convex univalent in U with $h(0) = 1$ and $\text{Re}\, h$ be bounded in U. Let $F_{\nu,p}(f)(z)$ be the integral operator defined by (1.10). If $f \in \Sigma_p$ satisfies the condition*

$$-\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{pH_{p,q,s}(\alpha_1)f(z)} \prec h(z) \quad (z \in U),$$

then

$$-\frac{z(H_{p,q,s}(\alpha_1)F_{\nu,p}(f)(z))'}{pH_{p,q,s}(\alpha_1)F_{\nu,p}(f)(z)} \prec h(z) \quad (z \in U)$$

for $\max_{z\in U}\text{Re}\, h(z) < \frac{\nu+p}{p}$ (provided $H_{p,q,s}F_{\nu,p}(f)(z) \neq 0$ in U).

Proof. Let

$$p(z) = -\frac{z(H_{p,q,s}(\alpha_1)F_{\nu,p}(f)(z))'}{pH_{p,q,s}(\alpha_1)F_{\nu,p}(f)(z)} \quad (z \in U).$$

Then, by using (1.11), we have

$$q(z) - (\nu + p) = -c\frac{H_{p,q,s}(\alpha_1)F_{\nu,p}(f)(z)}{H_{p,q,s}(\alpha_1)F_{\nu,p}(f)(z)}. \tag{2.12}$$

Taking logarithmic derivatives in both sides of (2.12) and multiplying by z, we get

$$\frac{zq'(z)}{-pq(z) + (\nu + p)} + q(z) = -\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{pH_{p,q,s}(\alpha_1)f(z)} \prec h(z) \quad (z \in U).$$

Therefore, by using Lemma 1, we have

$$-\frac{z(H_{p,q,s}(\alpha_1)F_{\nu,p}(f)(z))'}{pH_{p,q,s}(\alpha_1)F_{\nu,p}(f)(z)} \prec h(z) \quad (z \in U)$$

for $\max_{z\in U}\text{Re}\, h(z) < \frac{\nu+p}{p}$ (provided $H_{p,q,s}(\alpha_1)F_{\nu,p}(f)(z) \neq 0$ in U). $\square$

Remark 4. Taking $q = 2, s = 1, \alpha_1 = n + p (n > -p), \alpha_2 = \beta_1 = 1$ and $h(z) = \frac{1+z}{1-z}$ in Theorem 4, we obtain the result obtained by Aouf [1, Theorem 3].

Theorem 5. *Let $f(z) \in \Sigma_p$ and choose a positive number v such that*

$$v \geqslant \frac{1+A}{1+B} - p,$$

where $-1 < B < A \leqslant 1$ and $p \in N$. If

$$\left|\arg\left(-\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \quad (0 \leqslant \gamma < p; 0 < \delta \leqslant 1)$$

for some $g \in \Sigma_p^*[\alpha_1; A, B]$, then

$$\left|\arg\left(-\frac{z(H_{p,q,s}(\alpha_1)F_{v,p}(f)(z))'}{H_{p,q,s}(\alpha_1)G_{v,p}(g)(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where $F_{v,p}(f)(z)$ is the integral operator given by (1.10),

$$G_{v,p}(g)(z) = \frac{v}{z^{v+p}} \int_0^z t^{v+p-1} g(t) dt \quad (v > 0) \tag{2.13}$$

and $\alpha (0 < \alpha \leqslant 1)$ is the solution of the equation

$$\delta = \alpha + \frac{2}{\pi} \tan^{-1}\left\{\frac{\alpha \sin\frac{\pi}{2}(1 - t(A, B, v))}{\frac{(v+p)(1-B)+p(A-B)}{1-B} + \alpha \cos\frac{\pi}{2}(1 - t(A, B, v))}\right\}, \tag{2.14}$$

where

$$t(A, B, v) = \frac{2}{\pi} \sin^{-1}\left[\frac{p(A - B)}{(v + p)(1 - B^2) - p(1 - AB)}\right]. \tag{2.15}$$

Proof. Let

$$q(z) = -\frac{1}{p-\gamma}\left(\frac{z(H_{p,q,s}(\alpha_1)F_{v,p}(f)(z))'}{H_{p,q,s}(\alpha_1)G_{v,p}(g)(z)} + \gamma\right) \quad (z \in U).$$

Since $g \in \Sigma_p^*[\alpha_1; A, B]$, from Theorem 4, $G_{v,p}(g)(z) \in \Sigma_p^*[\alpha_1; A, B]$. Using (1.11), we have

$$(p-\gamma)q(z)H_{p,q,s}(\alpha_1)G_{v,p}(g)(z) - (v + p)H_{p,q,s}(\alpha_1)F_{v,p}(f)(z) = -vH_{p,q,s}(\alpha_1)f(z) - \gamma H_{p,q,s}(\alpha_1)G_{v,p}(g)(z).$$

Then, by a simple calculation, we have

$$(p-\gamma)\{zq'(z) + q(z)[-r(z) + v + p]\} + \gamma[-r(z) + v + p] = -\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)G_{v,p}(g)(z)},$$

where

$$r(z) = -\frac{z(H_{p,q,s}(\alpha_1)G_{v,p}(g)(z))'}{H_{p,q,s}(\alpha_1)G_{v,p}(g)(z)}.$$

Hence, we have

$$q(z) + \frac{zq'(z)}{-r(z) + v + p} = -\frac{1}{p-\gamma}\left(\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)g(z)} + \gamma\right).$$

The remaining part of the proof is similar to that of Theorem 2 and so we omit it. □

Putting $q = s + 1, \alpha_1 = \beta_1 = p, \alpha_j = 1 \ (j = 2, 3, \ldots, s + 1), \beta_j = 1 \ (j = 2, 3, \ldots, s), A = 1, B = 0$ and $\delta = 1$ in Theorem 5, we obtain the following result.

Corollary 4. *Let $v > 0$ and $f(z) \in \Sigma_p$. If*

$$-\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \gamma \quad (0 \leqslant \gamma < p)$$

for some $g(z) \in \Sigma_p$ satisfying the condition

$$\left|\frac{zg'(z)}{g(z)}+p\right|<p,$$

then

$$-\mathrm{Re}\left\{\frac{zF'_{v,p}(f)(z)}{G_{v,p}(g)(z)}\right\}>\gamma \quad (0\leqslant\gamma<p),$$

where $F_{v,p}(f)(z)$ and $G_{v,p}(g)(z)$ are given by (1.10) and (2.13), respectively.

Taking $q=s+1, \alpha_1=\beta_1=p, \alpha_j=1\ (j=2,3,\ldots,s+1), \beta_j=1\ (j=2,3,\ldots,s), B\to A$ and $g(z)=\frac{1}{z^p}$ in Theorem 5, we obtain the following corollary.

Corollary 5. *Let $v>0$ and $f(z)\in\Sigma_p$. If*

$$|\arg(-z^{p+1}f'(z)-\gamma)|<\frac{\pi}{2}\delta \quad (0\leqslant\gamma<p; 0<\delta\leqslant 1)$$

then

$$|\arg(-z^{p+1}F'_{v,p}(f)(z)-\gamma)|<\frac{\pi}{2}\alpha,$$

where $F_{v,p}(f)(z)$ is the integral operator given by (1.10) and $\alpha(0<\alpha\leqslant 1)$ is the solution of the equation

$$\delta=\alpha+\frac{2}{\pi}\tan^{-1}\left(\frac{\alpha}{v+p}\right).$$

By using the same method as in proving Theorem 5, we have

Theorem 6. *Let $f(z)\in\Sigma_p$ and choose a positive number v such that*

$$v\geqslant\frac{1+A}{1+B}-p,$$

where $-1<B<A\leqslant 1$ and $p\in N$. If

$$\left|\arg\left(\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)g(z)}+\gamma\right)\right|<\frac{\pi}{2}\delta \quad (\gamma>p; 0<\delta\leqslant 1)$$

for some $g\in\Sigma_p^*[\alpha_1;A,B]$, then

$$\left|\arg\left(\frac{z(H_{p,q,s}(\alpha_1)F_{v,p}(f)(z))'}{H_{p,q,s}(\alpha_1)G_{v,p}(g)(z)}+\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

where $F_{v,p}(f)(z)$ and $G_{v,p}(g)(z)$ are given by (1.10) and (2.13), respectively, and $\alpha(0<\alpha\leqslant 1)$ is the solution of the equation given by (2.14).

Finally, we derive

Theorem 7. *Let $f(z)\in\Sigma_p$. Choose $\alpha_1\in R$ such that*

$$\alpha_1\geqslant\frac{p(A-B)}{1+B},$$

where $-1<B<A\leqslant 1$ and $p\in N$. If

$$\left|\arg\left(-\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)g(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta \quad (0\leqslant\gamma<p; 0<\delta\leqslant 1)$$

for some $g\in\Sigma_p^*[\alpha_1;A,B]$, then

$$\left|\arg\left(-\frac{z(H_{p,q,s}(\alpha_1+1)F_{v,p}(f)(z))'}{H_{p,q,s}(\alpha_1+1)G_{v,p}(g)(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta,$$

where $F_{v,p}(f)(z)$ and $G_{v,p}(g)(z)$ are given by (1.10) and (2.13), respectively, with $v=\alpha_1$.

Proof. From (1.9) and (1.11) with $v=\alpha_1$, we have $H_{p,q,s}(\alpha_1)f(z)=H_{p,q,s}(\alpha_1+1)F_{v,p}(f)(z)$. Therefore

$$\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)g(z)}=\frac{z(H_{p,q,s}(\alpha_1+1)F_{v,p}(f)(z))'}{H_{p,q,s}(\alpha_1+1)G_{v,p}(g)(z)}$$

and the result follows. □

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