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Affine configurations of 4 lines in $\mathbb{R}^3$

By

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Abstract. We prove that affine configurations of 4 lines in $\mathbb{R}^3$ are topologically and combinatorially homeomorphic to affine configurations of 6 points in $\mathbb{R}^4$.

1. Introduction. Consider four lines $\ell_1, \ell_2, \ell_3, \ell_4$ in 3-dimensional space $\mathbb{R}^3$; their *affine configuration* is their equivalence class under the natural (diagonal) action of the affine group Aff(3). We say that their directions are in *general position* if their corresponding four points at infinity are in general position in the projective plane $\mathbb{P}^2$. We say that they *fix* $\mathbb{R}^3$ (or that their affine configuration *fixes* $\mathbb{R}^3$) if the only affine isomorphism that fixes them, as sets, is the identity (i.e., they lie on a free orbit of the action) –it is easy to see that four lines with directions in general position fix $\mathbb{R}^3$ if and only if they are not concurrent.

The purpose of this paper is to describe the space, which we denote $\mathbb{A}^3_{4,1}$, of affine configurations of four lines in $\mathbb{R}^3$ that fix and have directions in general position. Topologically, it is the 4 dimensional projective space $\mathbb{P}^4$, but furthermore, it has the polyhedral structure of the space of affine configurations of 6 points in $\mathbb{R}^4$. The combinatorial structure, or decomposition, of $\mathbb{A}^3_{4,1}$ arises naturally from the fact that there is a clear notion of degeneracy: configurations where some of the lines meet. So that we can say that two affine configurations of lines are *equivalent* if they have the same meeting pattern (a set of pairs of meeting lines) and one representative may be continuously moved to the other without ever changing that pattern. We will prove that these equivalence classes are cells (in fact, the interior of products of simplices) corresponding to the Radon partitions of the six possible pairs of the (four) indices.

An affine configuration of points is an affine equivalence class of k points in $\mathbb{R}^n$ that affinely generate it; the space of such is denoted $\mathbb{A}^n_{k,0}$. Following the ideas of [3] for vector

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configurations, these spaces are seen to be grassmannians –namely, $\mathbb{A}^n_{k,0} = G(k-1-n, n)$: the grassmannian of $(k-n-1)$ dimensional subspaces of $\mathbb{R}^{k-1}$. They come with a natural stratification given, again, by the notion of degeneracy, with "cells" corresponding to oriented matroids. In this context, the present paper studies one of the first examples of their natural generalization to spaces of configurations of flats of dimensions other than 0. It is remarkable that $\mathbb{A}^3_{4,1}$ is again a grassmannian because the space of 4 different lines that fix the plane $\mathbb{R}^2$ (modulo the affine group Aff(2), of course) which we call $\mathbb{A}^2_{4,1}$ is the surface of non-oriented genus 5 with combinatorial symmetry group S_5 [1].

2. The homeomorphism $\mathbb{A}^3_{4,1} \simeq \mathbb{A}^1_{6,0}$. Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four lines in $\mathbb{R}^3$ with respective directional vectors $d_1, \dots, d_4$ in general position. We then have a non trivial linear relation $\sum \mu_i d_i = 0$ with $\mu_i \neq 0$ for all i (if otherwise, three of the points at infinity would be colinear), which is unique up to a constant non-zero factor. Then, rescaling the directions ($d_i := \mu_i d_i$), we may assume that

$$\sum_{i=1}^{4} d_i = 0$$

in which case we say that they are *normalized*. Now, we associate to each pair of lines ℓ_i, ℓ_j a number λ_{ij} which, in a sense, measures the *distance* between them.

To fix ideas, consider the lines ℓ_1, ℓ_2. It is easy to see that there are unique segments with directions d_3 and d_4 and endpoints in ℓ_1 and ℓ_2; call them σ_3^{12} and σ_4^{12} accordingly (Figure 1). These segments together with the segments within ℓ_1 and ℓ_2 between their endpoints form a quadrilateral, which, walked around, clearly gives a relation $\sum_{i=1}^{4} \alpha_i d_i = 0$. Since the directions d_i are normalized, then all the coefficients (α_i) are equal, to λ_{12} say.

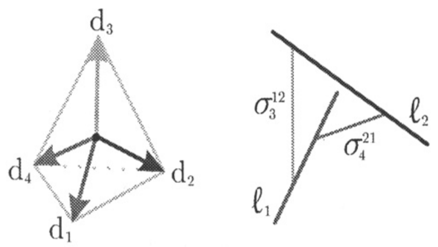

Figure 1.

We then have that, as a vector, $\sigma_3^{12} = \lambda_{12} d_3$, and similarly $\sigma_4^{12} = \lambda_{12} d_4$. Observe that λ_{12} is well defined up to sign, because walking around the quadrilateral in the opposite direction simply changes its sign. The two directions correspond to choosing one of the cyclic orders 1324 or 1423 indicating the order of (the directions of) the segments in the quadrilateral. Observe also that this procedure analogously gives λ_{ij} for the six pairs of indices in the set

$\Delta_4 := \{1, 2, 3, 4\}$. So that we are left to give a rule for choosing orientations to eliminate the ambiguity in the signs of λ_{ij}.

The natural rule follows from fixing an orientation of the tetrahedron Δ with vertices Δ_4 (let us establish 432 as the positive cyclic orientation around vertex 1 as in Figure 2). Then, for the edge ij choose the positive orientation of the quadrilateral of which it is a diagonal (e.g., for the edge 12 we must choose the cyclic order 1324).

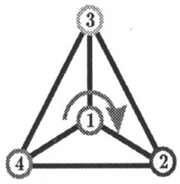

Figure 2.

Before proceeding, let us precise our notation for later use. The segments σ_i^{jk} are now oriented. They go from ℓ_j to ℓ_k having direction d_i and, abusing notation that forgets the starting point in ℓ_j, we may write

$$\sigma_i^{jk} = \lambda_{jk}\, d_i$$

where, moreover, the triangle ijk of Δ has orientation jik. Thus, according to our conventions, we should rewrite σ_4^{21} instead of the previously used σ_4^{12}. Observe also that $\lambda_{ij} = \lambda_{ji}$ because our subindices for λ are understood unordered, but not the indices for σ.

We have associated six numbers $\lambda_{ij} \in \mathbb{R}$ to a configuration $\ell_1, \ell_2, \ell_3, \ell_4$ depending on the choice of a normalized set of directions. Since these are defined up to a non-zero constant factor, so are the λ_{ij}. Observe that

$$\lambda_{ij} = 0 \quad \Leftrightarrow \quad \ell_i \cap \ell_j \neq \emptyset$$

so that all the λ_{ij} are zero if and only if the four lines are concurrent (since their directions are in general position, if three of them meet by pairs then they are concurrent). Therefore, if $\ell_1, \ell_2, \ell_3, \ell_4$ fix $\mathbb{R}^3$ we have a well defined point $[\lambda_{ij}] \in \mathbb{P}^5$, which, moreover, only depends on the affine configuration. To see this, observe that a translation does not change the λ_{ij} and that a linear map sends a normalized set of directions to a corresponding normalized set of directions so that it does not change them either. Summarizing, we have defined a map

$$\mathbb{A}^3_{4,1} \to \mathbb{P}^5$$
$$[\ell_1, \dots, \ell_4] \mapsto [\lambda_{ij}]$$

which, as we will now see, is really a map to a hyperplane ($\mathbb{P}^4$).

Lemma 1. *Let λ_{ij} correspond to the lines $\ell_1, \dots, \ell_4$ with normalized directions $d_1, \dots, d_4$ as above. Then*

$$\sum \lambda_{ij} = 0$$

where the sum is taken over the six pairs of $\Delta_4 = \{1, 2, 3, 4\}$.

Proof. According to our orientation convention, we have defined three oriented segments with direction d_1, namely $\sigma_1^{23}, \sigma_1^{34}, \sigma_1^{42}$, which join the lines ℓ_2, ℓ_3, ℓ_4 in that cyclic order. Therefore, intercalating segments on the lines ℓ_3, ℓ_4, ℓ_2 we get an oriented hexagon, which traversed orientedly yields, for some $\beta_2, \beta_3, \beta_4 \in \mathbb{R}$, a relation

$$\lambda_{23}\, d_1 + \beta_3\, d_3 + \lambda_{34}\, d_1 + \beta_4\, d_4 + \lambda_{42}\, d_1 + \beta_2\, d_2 = 0 \tag{1}$$

Because the directions are normalized, this implies that

$$\beta_2 = \beta_3 = \beta_4 = \lambda_{23} + \lambda_{34} + \lambda_{42} =: \gamma_1$$

This can clearly be done for any $i \in \Delta_4$, yielding that the segment in ℓ_j from the endpoint of σ_i^{kj} to the starting point of σ_i^{jr} is precisely $\gamma_i d_j$, where

$$\gamma_i := \lambda_{jk} + \lambda_{kr} + \lambda_{rj} \text{ with } \{i, j, k, r\} = \Delta_4.$$

Now we have enough measures between the six points defined as endpoints of the segments σ_i^{jk} in any given line. In ℓ_1 for example, we know the distances between consecutive ℓ_1-endpoints of the segments $\sigma_2^{31}, \sigma_2^{14}, \sigma_3^{41}, \sigma_3^{12}, \sigma_4^{21}, \sigma_4^{13}$ which, preserving that cyclic order, yields the relation

$$(\gamma_2 - \lambda_{14} + \gamma_3 - \lambda_{12} + \gamma_4 - \lambda_{13})\, d_1 = 0 \tag{2}$$

from which the Lemma follows directly by the definition of γ_i. $\square$

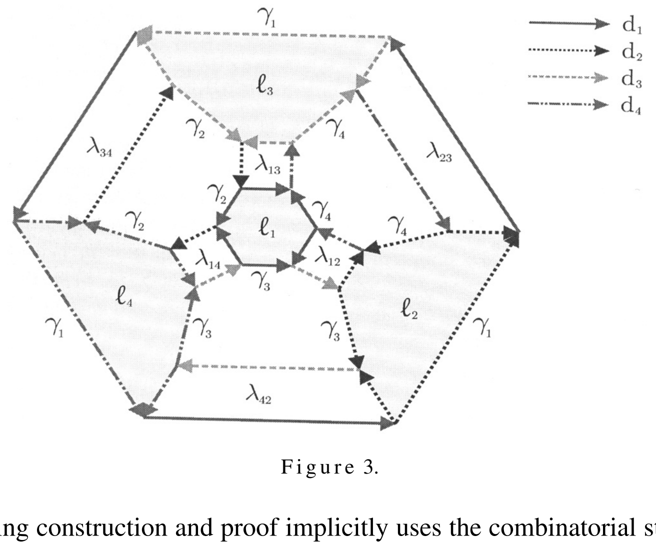

Figure 3.

The preceding construction and proof implicitly uses the combinatorial structure of the truncated octahedron. In Figure 3, the different styles of directed edges correspond to the four directions; the coefficients of the vectors used in the proof appear respectively as labels of the quadrilaterals or the edges. The hexagons with only one type of edge lie within the lines; they give equations of type (2) at the end of the proof. The edges between these hexagons correspond to the segments σ_i^{kj} and they group in the quadrilaterals that defined the λ's. The other type of hexagons were used to define the γ's and give equations of type (1).

We must finally remark that the map $\mathbb{A}^3_{4,1} \to \mathbb{P}^4$ we have defined is a homeomorphism. To see this, observe that if the six λ_{ij}, adding zero, are given, one can construct the lines. Fix one of them with an arbitrary base point. Then, enough geometric information is given by the coefficients to know where on the line precisely defined segments should go to the other three lines. The linear condition, and Figure 3 imply that the result is independent of the choices.

3. Duality of affine configurations of points. Two affine configurations of points are *dual* if they are the orthogonal projections of (the vertices of) the standard regular simplex to a pair of orthogonal complementary subspaces; where the standard regular simplex has all vertices equidistant. We will see that duality gives a homeomorphism

$$\mathbb{A}^n_{k,0} \longleftrightarrow \mathbb{A}^{k-n-1}_{k,0}$$

and characterize it completely for the case $n = 1$. The basic ideas come from the classic duality of matroids and vector configurations [4], see also [3].

Consider an affine configuration of k points in dimension n, $\mathbf{p} \in \mathbb{A}^n_{k,0}$. It is represented by points $p_1, \dots, p_k \in \mathbb{R}^n$ (written, $\mathbf{p} = [p_1, \dots, p_k]$) such that they affinely generate $\mathbb{R}^n$. Since their barycenter $\sum (1/k) p_i$ is a well defined affine invariant, we may translate it to the origin and assume that $p_1, \dots, p_k$ is *centered*, that is, that

$$\sum_{i=1}^{k} p_i = 0.$$

Observe that choosing centered configurations leaves our ambiguity in the general linear group $\mathrm{Gl}(n)$, that is,

$$\mathbb{A}^n_{k,0} = \left\{ p_1, \dots, p_k \in \mathbb{R}^n \;\middle|\; \begin{array}{c} p_1, \dots, p_k \text{linearly generate} \mathbb{R}^n, \\ \sum p_i = 0 \end{array} \right\} / \,\mathrm{Gl}(n).$$

Given $\mathbf{p} = [p_1, \dots, p_k] \in \mathbb{A}^n_{k,0}$ as above, we have a linear map $\varphi_{\mathbf{p}} : \mathbb{R}^k \to \mathbb{R}^n$ defined by $\varphi(e_i) = p_i$, where $e_1, \dots, e_k$ is the canonical basis of $\mathbb{R}^k$. It is unto by hypothesis, so that $\xi'_{\mathbf{p}} := \mathrm{Ker}(\varphi_{\mathbf{p}})$ is a subspace of dimension $k - n$ (this is the subspace associated to the vector configuration). Observe that $\xi'_{\mathbf{p}}$ does not depend on our choice of the centered representative $p_1, \dots, p_k$ of $\mathbf{p}$, because $\varphi_{\mathbf{p}}$ followed by a linear isomorphism has the same kernel. Observe also that from $\xi'_{\mathbf{p}}$ one can obtain $\mathbf{p}$, because the image of the canonical basis in $\mathbb{R}^k/\xi'_{\mathbf{p}}$ (isomorphic via $\varphi_{\mathbf{p}}$ to $\mathbb{R}^n$) is linearly equivalent to $p_1, \dots, p_k$.

Let $\mathbf{1} = (1, \dots, 1) = \sum e_i$, and let Π be its normal hyperplane defined by $\mathbf{1} \cdot x = 0$. Let $v_1, \dots, v_k$ be the orthogonal projection of $e_1, \dots, e_k$ to Π (namely, $v_i = e_i - (1/k)\mathbf{1}$), so that $v_1, \dots, v_k$ are the vertices of a standard regular simplex in Π. Because $p_1, \dots, p_k$ is centered, then $\mathbf{1} \in \xi'_{\mathbf{p}}$, so that $\xi_{\mathbf{p}} := \Pi \cap \xi'_{\mathbf{p}}$, which is a subspace of dimension $k - n - 1$ of the $(k-1)$-dimensional space Π, has all the information to recover $\mathbf{p}$. Indeed, $\mathbf{p}$ is equivalent to the image of $v_1, \dots, v_k$ in $\Pi/\xi_{\mathbf{p}} \simeq \mathbb{R}^n$.

Let $q_1, \dots, q_k$ be, respectively, the (orthogonal) projections of $v_1, \dots, v_k$ (or $e_1, \dots, e_k$) to $\xi_{\mathbf{p}}$, and let $\mathbf{q} \in \mathbb{A}^{k-n-1}_{k,0}$ be the corresponding affine configuration. By construction, $\xi_{\mathbf{p}}$ and $\xi_{\mathbf{q}}$ (defined analogously for $\mathbf{q}$) are orthogonal complementary subspaces of Π ($\simeq \mathbb{R}^{k-1}$) and the projection of the standard regular simplex $v_1, \dots, v_k$ to them gives, respectively, the affine configurations $\mathbf{q}$ and $\mathbf{p}$ (because we can identify $\Pi/\xi_{\mathbf{p}} = \xi_{\mathbf{q}}$). So they are dual. We can summarize by saying that both $\mathbb{A}^n_{k,0}$ and $\mathbb{A}^{k-n-1}_{k,0}$ are naturally homeomorphic to the grassmannians $G(k-n-1, n)$ and $G(n, k-n-1)$ with duality corresponding to orthogonal complementation. In the case that interests us ($n = 1$), duality can be characterized more explicitly.

Theorem 2. *Let $\lambda_1, \dots, \lambda_k$ be a centered affine configuration in $\mathbb{R}^1$ and $\mathbf{p} = [p_1, \dots, p_k]$ an affine configuration in $\mathbb{R}^{k-2}$. Then they are dual if and only if*

$$\sum_{i=1}^{k} \lambda_i p_i = 0. \tag{3}$$

P r o o f. Because multiplication by non zero constant factors does not affect the affine configuration or the equation, we may assume that $\lambda_1, \dots, \lambda_k$ is *normalized*, i.e., that $\sum \lambda_i^2 = 1$; so that $\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ is well defined up to sign.

Let $\xi_{[\lambda]}$ be the $(k-2)$-dimensional subspace of Π defined as above, and $q_1, \dots, q_k$ the (orthogonal) projections to $\xi_{[\boldsymbol{\lambda}]}$ of $v_1, \dots, v_k$ respectively; so that $[\boldsymbol{\lambda}] \in \mathbb{A}^1_{k,0}$ and $\mathbf{q} := [q_1, \dots, q_k] \in \mathbb{A}^{k-2}_{k,0}$ are dual. Observe that $\boldsymbol{\lambda} \in \Pi$ (by the centered hypothesis) and that the defining map for $\xi_{[\boldsymbol{\lambda}]}$ ($e_i \mapsto \lambda_i$) is precisely $x \mapsto x \cdot \lambda$, so that $\xi_{[\lambda]}$ is the orthogonal hyperplane to λ in Π. Then, it is easy to see that

$$q_i = e_i - (1/k)\mathbf{1} - \lambda_i \boldsymbol{\lambda} \tag{4}$$

where one uses that $\boldsymbol{\lambda}$ is normalized. Therefore,

$$\sum_{i=1}^{k} \lambda_i q_i = \boldsymbol{\lambda} - (1/k)\left(\sum_{i=1}^{k} \lambda_i\right)\mathbf{1} - \left(\sum_{i=1}^{k} \lambda_i^2\right)\boldsymbol{\lambda} = 0$$

because $\lambda_1, \dots, \lambda_k$ is centered and normalized; proving the only if side.

Suppose now that $\mathbf{p} = [p_1, \dots p_k] \in \mathbb{A}^{k-2}_{k,0}$ satisfies the relation (3). Then $\boldsymbol{\lambda} \in \xi_{\mathbf{p}}$ and moreover, $\xi_{\mathbf{p}}$ is the line generated by $\boldsymbol{\lambda}$. By equation (4) $\lambda_i \boldsymbol{\lambda}$ is the orthogonal projection of v_i to $\xi_{\mathbf{p}}$. So that $\lambda_1, \dots, \lambda_k$ represent the dual configuration to $\mathbf{p}$. $\square$

4. The Radon Complex. We have proved that $\mathbb{A}^3_{4,1}$ is homeomorphic to $\mathbb{P}^4$ which is naturally identified with $\mathbb{A}^1_{6,0}$ and then to $\mathbb{A}^4_{6,0}$ by duality. Now we see that its combinatorial structure corresponds to the latter, which is intimately related to the classic Radon's Theorem.

Let $\mathbf{p} = [p_1, \ldots, p_k] \in \mathbb{A}^{k-2}_{k,0}$ and $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_k] \in \mathbb{A}^1_{k,0}$ be dual (related as in Theorem 2 with $\lambda_1, \ldots, \lambda_k$ centered). Then we have a partition of the index set $\Delta_k := \{1, \ldots, k\}$ into three components

$$A = \{i \mid \lambda_i > 0\}; \ B = \{i \mid \lambda_i < 0\}; \ C = \{i \mid \lambda_i = 0\}$$

with A and B non-void, which is called the *Radon partition* of the configuration $\mathbf{p}$. Radon's Theorem states that the interiors of the simplices $\langle p_i \mid i \in A\rangle$ and $\langle p_i \mid i \in B\rangle$ intersect. It is obtained by changing the relation (3) into an equality of barycentric (convex) combinations in the obvious way. All the configurations with the same Radon partition $A; B; C$ can then be parametrized by the product of the interiors of the two abstract simplices $\langle A\rangle$ and $\langle B\rangle$ using the barycentric coordinates of the intersection point; giving the natural cell decomposition of $\mathbb{A}^{k-2}_{k,0}$ which we call the *Radon complex.* Two configurations in the same cell (with the same Radon partition) can be joined by a path of configurations of the same type (the geodesic in $\mathbb{P}^{k-2}$). Configurations in general position are exactly those for which $C = \emptyset$, thus, the Radon complex is obtained by chopping $\mathbb{P}^{k-2}$ by the k hyperplanes $\lambda_i = 0$, in which the configurations $(\mathbf{p})$ have some degeneracy. See [2].

Returning to the affine configurations of four lines in $\mathbb{R}^3$, $\mathbb{A}^3_{4,1}$, we associated to such a configuration $\ell_1, \ldots, \ell_4$ a centered configuration of six points in the line $[\lambda_{ij}] \in \mathbb{A}^1_{6,0}$ in such a way that the degeneracy "ℓ_i meets ℓ_j" corresponds to $\lambda_{ij} = 0$. Thus, the combinatorial structure of $\mathbb{A}^3_{4,1}$ corresponds by duality to that of the Radon complex $\mathbb{A}^4_{6,0}$; with open cells the product of open simplices and with two configurations of lines being combinatorially equivalent if they can be moved from one to the other without changing the intersection pattern of the lines. For each Radon partition of the six pairs ij, consisting of the pairs that have positive, negative and zero distance, there is one cell.

We have proved the following theorem.

Theorem 3. *There is a stratified homeomorphism (preserving degeneracies) between the space* $\mathbb{A}^3_{4,1}$ *of affine configurations of* 4 *lines in* $\mathbb{R}^3$ *and the space* $\mathbb{A}^4_{6,0}$ *of affine configurations of* 6 *points in* $\mathbb{R}^4$.

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