Computational Methods and Function Theory
Volume 11 (2011), No. 1, 263–282

On the Distribution of Zeros of Faber Polynomials

Vladimir V. Andrievskii and Hans-Peter Blatt

(Communicated by Alexander Aptekarev)

Abstract. We prove a local version of the potential theoretic interpretation of the Erdős-Turán result on the distribution of zeros of complex algebraic polynomials with a given uniform norm on a closed subdisk of the open unit disk. As a consequence, we derive the exact rate of convergence of the unit counting measure for zeros of Faber polynomials with respect to a Dini-smooth Jordan arc to the equilibrium measure of the arc.

Keywords. Faber polynomial, conformal mapping, discrepancy theorem, equilibrium measure.

2000 MSC. 30C10, 30E10.

1. Introduction

Let $K \subset \mathbb{C}$ be a compact set containing more than a single point whose complement $\Omega := \overline{\mathbb{C}} \setminus K$ with respect to the extended complex plane $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is a simply connected domain. Denote by $\Psi : \mathbb{D}^* \to \Omega$, the Riemann conformal mapping of $\mathbb{D}^* := \{w \in \overline{\mathbb{C}} \colon |w| > 1\}$ onto Ω such that $\Psi(\infty) = \infty$ and $\Psi'(\infty) > 0$. Then, the nth *Faber polynomial* $F_n = F_{n,K}$, $n = 0, 1, 2, \dots$ with respect to K is defined to be an appropriate coefficient of a Laurent series

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \qquad z \in K.$$

The monographs [22, 5, 9, 24] and the papers [4, 28] are reference sources for the numerous applications that Faber polynomials have found in approximation theory, orthogonal polynomials, and geometric function theory. We are interested in the asymptotic behavior of the distribution of zeros

$$Z_n = Z_{n,K} := \{z \in \mathbb{C} \colon F_n(z) = 0\}$$

Received July 14, 2010, in revised form December 8, 2010.
Published online February 12, 2011.
The work was supported in part by DFG-Research Grant (Germany) BL 272/10-1. The work of VA was conducted while visiting the Katholische Universität Eichstätt-Ingolstadt.

ISSN 1617-9447/\$ 2.50

of Faber polynomials as $n \to \infty$. This problem attracted considerable attention, see [25, 13, 26, 11, 6, 12, 15, 14, 17, 18] and the references therein for a comprehensive survey of this subject.

As a starting point of our investigation, we mention a consequence of a general result by Ullman [25, Thm. 1] stating that if K has empty interior then

$$K = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n\geq k} Z_n},$$

i.e. all points of K and only they attract zeros of Faber polynomials.

Let $\nu_n, n \in \mathbb{N} := \{1, 2, \ldots\}$ be the normalized counting measure for Z_n, i.e.

$$\nu_n = \nu_{n,K} := \frac{1}{n} \sum_{z\in Z_n} \delta_z,$$

where δ_z is the unit *Dirac measure* at $z \in \mathbb{C}$ and the zeros are counted according to their multiplicity. Kuijlaars and Saff [15, Thm. 1.3] strengthened the Ullman result showing that if K has empty interior then the sequence $\{\nu_n\}_{n=1}^{\infty}$ converges to the equilibrium measure $\mu = \mu_K$ for K in the weak-star topology.

In this paper we prove a local version of the Erdős-Turán result [7] on the distribution of zeros of complex polynomials. More precisely, we establish its modern potential theoretic interpretation (see [3, pp. 152–172]) which, in particular, allows us to obtain the following sharp quantitative extension of the Kuijlaars-Saff Theorem for smooth Jordan arcs.

Let $K = L$ be a smooth Jordan arc with endpoints ζ_1 and $\zeta_2 \neq \zeta_1$. We extend Ψ continuously to $\overline{\mathbb{D}^*}$. For $\xi \in \Omega \setminus \{\infty\}$, let

$$\xi_L := \Psi\left(\frac{\Phi(\xi)}{|\Phi(\xi)|}\right),$$

where $\Phi := \Psi^{-1}$, and let

$$S(J) := \{\xi \in \Omega \setminus \{\infty\} \colon \xi_L \in J\}, \qquad J \subset L.$$

Let $|L|$ be the length of L and let $\beta(s)$, $0 \leq s \leq |L|$ be the angle between the tangent to L and the fixed ray in $\mathbb{C}$ as a continuous function of the *arc length parameter* s. We say that L is *Dini-smooth* if

$$|\beta(s_2) - \beta(s_1)| \leq h(s_2 - s_1), \qquad s_1 < s_2,$$

where h is an increasing function for which

$$\int_0^1 \frac{h(x)}{x}\, dx < \infty.$$

Theorem 1. *Let L be a Dini-smooth arc. Then, for any arc $J \subset L$ and $n \in \mathbb{N}$, the inequality*

$$\left|\nu_n(\overline{S(J)}) - \mu_L(J)\right| \leq \frac{c}{n} \tag{1}$$

holds with a constant $c = c(L) > 0$ *that depends only on* L.

The trivial estimate

$$\sup_{\zeta\in L}\left|\nu_n(\overline{S(\{\zeta\})}) - \mu_L(\{\zeta\})\right| = \sup_{\zeta\in L}\left|\nu_n(\overline{S(\{\zeta\})})\right| \geq \frac{1}{n}$$

shows the sharpness (up to a constant) of the inequality (1).

In order to prove Theorem 1 we are going to use the potential theoretical approach. The starting point is a theorem of Erdős and Turán [7] which describes discrepancy estimates between the normalized zero counting measure of a monic polynomial p_n and the equilibrium distribution of the unit disc if the norm of the polynomial is given on a smaller (resp. larger) circle $K_\theta := \{z\colon |z| = \theta\}$, $0 < \theta < 1$ (resp. $1 < \theta < \infty$), and all zeros of p_n lie outside the open unit disk (resp. on the unit disk).

In [3, Ch. 4], this result was generalized to Dini-smooth Jordan arcs L where a probability measure ν is given on L together with one-sided bound of the difference of the logarithmic potentials $U^\nu - U^{\mu_L}$ on some level curve of Green's function of $\Omega = \overline{\mathbb{C}} \setminus L$ with pole at infinity, where μ_L is the equilibrium measure of L. But this generalization [3, Thm. 4.1]) cannot be applied since the normalized zero counting measures ν_n of the Faber polynomials are not supported on L in general (cf. [11, 17]).

The outline of the paper is as follows. In Section 2 the generalization of the Erdős-Turán Theorem is formulated to prove Theorem 1 in Section 3. Section 4 is devoted to the construction of auxiliary functions that are approximated by real parts of complex polynomials (see Section 5). Finally, in Section 6 the proof of the Erdős-Turán-type Theorem is given.

For the basic concepts of potential theory in $\mathbb{C}$ see [20, 21].

2. An Erdős-Turán-type Theorem

Let $L \subset \mathbb{C}$ be a smooth arc with endpoints at ζ_1 and $\zeta_2 \neq \zeta_1$ and let $K_1 \subset \mathbb{C}$ and $K_2 \subset \mathbb{C}$ be such that

$$K_1 \cap K_2 = \emptyset, \quad K_j \cap L = \{\zeta_j\}, \qquad j = 1, 2,$$

and $K := K_1 \cup L \cup K_2$ is a continuum with the simply connected complement $\Omega := \overline{\mathbb{C}} \setminus K$. We do not exclude the case where $K_1 = \{\zeta_1\}$ or $K_2 = \{\zeta_2\}$.

Consider the Riemann mapping function $\Phi\colon \Omega \to \mathbb{D}^*$ normalized by $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$. Denote by $L(\zeta', \zeta'')$ the subarc of L between $\zeta', \zeta'' \in L$.

Next, we fix points $\zeta_3 \neq \zeta_4$ on L such that $\zeta_3 \in L(\zeta_1, \zeta_4)$. We always assume that $\zeta_3 = \zeta_1$ if and only if $K_1 = \{\zeta_1\}$; and $\zeta_4 = \zeta_2$ if and only if $K_2 = \{\zeta_2\}$. For any subarc $J \subset L$ let

$$S(J) := \{\xi \in \Omega \setminus \{\infty\} \colon \xi_K \in J\},$$

where $\xi_K := \Psi(\Phi(\xi)/|\Phi(\xi)|)$ is well defined (by continuity) for any $\xi \in \Omega \setminus \{\infty\}$ such that the inverse mapping function $\Phi^{-1} =: \Psi\colon \mathbb{D}^* \to \Omega$ has the continuous extension to the point $e^{i\theta}$, $\theta = \arg \Phi(\xi)$. For $\xi \in B := \partial K$, we set $\xi_K := \xi$. Let for $t > 0$,

$$B_t := \{\xi \in \Omega\colon |\Phi(\xi)| = 1 + t\},$$
$$K_t^* := \{\xi \in \Omega\colon |\Phi(\xi)| \le 1 + t\} \cup K.$$

Let E be a bounded subset of $\mathbb{C}$, denote by $\mathcal{M}(E)$ the set of positive unit Borel measures supported on E. Let $\nu \in \mathcal{M}(E)$ and

$$U^\nu(\xi) := -\int \log|z-\xi|\, d\nu(z), \qquad \xi \in \mathbb{C},$$

be the (logarithmic) *potential* of ν. Let $\mu = \mu_K$ be the *equilibrium measure* for K.

Our main result is formulated in terms of the one-sided bounds on the potential of a signed measure $\mu - \nu$ on the level line B_t, $t > 0$, i.e. the quantities

$$a_{\sup}(t) := \sup_{\xi \in B_t} U^{\mu-\nu}(\xi),$$
$$a_{\inf}(t) := -\inf_{\xi \in B_t} U^{\mu-\nu}(\xi),$$
$$a(t) := \min(a_{\sup}(t), a_{\inf}(t)),$$

where for $\xi \in \Omega \setminus \{\infty\}$,

$$U^{\mu-\nu}(\xi) := U^\mu(\xi) - U^\nu(\xi) = -\log|\Phi(\xi)| - \log \operatorname{cap} K - U^\nu(\xi)$$

and $\operatorname{cap} K$ is the (logarithmic) *capacity* of K.

Theorem 2. *Let L be Dini-smooth and let $t > 0$ and $\zeta_3, \zeta_4 \in L$. Then there exist $c_k = c_k(K, t, \zeta_3, \zeta_4) > 1$, $k = 1, 2$ such that for any u with $0 < 2u < t$, any $m \in \mathbb{N}$ satisfying $mu > c_1$, any subarc $J \subset L(\zeta_3, \zeta_4)$, and any measure $\nu \in \mathcal{M}(K_u^*)$ the following inequality holds:*

$$|\nu(\overline{S(J)}) - \mu_K(J)| \le c_2 \left(\frac{1}{m} + m^6 (1+2t)^m a(t) \right). \tag{2}$$

Let the parameter m in Theorem 2 be chosen as follows

$$m = \left[\varepsilon \log \frac{1}{a(t)} \right] + 1,$$

where $\varepsilon > 0$ is sufficiently small and $[b]$ denotes the integral part of $b \ge 0$. Then the following simplified version of (2) can be stated.

Theorem 3. *Let L be Dini-smooth and let $t > 0$ and $\zeta_3, \zeta_4 \in L$. Let $\nu \in \mathcal{M}(K)$ satisfy $0 < a(t) \le 1/2$. Then there exists $c_3 = c_3(K, t, \zeta_3, \zeta_4) > 0$ such that for* *any subarc $J \subset L(\zeta_3, \zeta_4)$ the following holds:*

$$|(\nu - \mu_K)(J)| \le c_3 \left(\log \frac{1}{a(t)}\right)^{-1}.$$

Theorem 3 is a local version of the potential theoretical interpretation (see [3, Sec. 4.3 and 4.4]) of the Erdős-Turán result [7] on the distribution of zeros of polynomials with a given uniform norm on a closed subdisk of the open unit disk.

Below we continue to use $c, c_1, \dots$ to denote positive constants (possibly different in different sections) that are either absolute or depend on parameters not essential for the argument (such as , for example, $L, K, \zeta_3, \zeta_4, t$). For $a \ge 0$ and $b \ge 0$ we use the notation $a \preceq b$ (the *order inequality*) if $a \le cb$. The expression $a \asymp b$ denotes that $a \preceq b$ and $b \preceq a$ simultaneously.

Proof of Theorem 1. We are going to apply Theorem 2 for the case $K = L$, i.e. $K_j = \{\zeta_j\}$, $j = 1, 2$. Since L is Dini-smooth, we have

$$|F_n(\zeta)| \preceq 1, \qquad \zeta \in L. \tag{3}$$

This fact is known for the case of a Jordan domain bounded by a piecewise Dini-smooth curve (see [10, Thm. 3 and 4]). The validity of (3) for more general continua, which is indicated in [8], relies on two facts. The first fact is the following representation of the Faber polynomials, the proof of which follows the same lines as that of the similar formula derived first by Pommerenke [19] (see also [1, p. 4], [24, Ch. IX, Thm. 11]):

$$F_n(\zeta) = \frac{1}{\pi} \int_0^{2\pi} e^{in\theta}\, d_\theta \arg(\Psi(e^{i\theta}) - \zeta), \qquad \zeta \in L \setminus \{\zeta_1, \zeta_2\},$$

implying that for $\zeta \in L \setminus \{\zeta_1, \zeta_2\}$,

$$|F_n(\zeta)| \le \frac{1}{\pi} \int_0^{2\pi} |\, d_\theta \arg(\Psi(e^{i\theta}) - \zeta)|. \tag{4}$$

The second fact is the result by Gaier [10, Thm. 4] according to which the right-hand side of (4) is uniformly bounded for any Dini-smooth arc.

Since

$$F_n(\xi) = \Phi(\xi)^n + \kappa_n(\xi), \qquad \xi \in \Omega,$$

where the function κ_n is analytic in Ω and $\kappa_n(\infty) = 0$, according to (3) and the Maximum Modulus Principle for κ in Ω, we have

$$|\kappa_n(\xi)| \preceq 1, \qquad \xi \in \Omega,$$

which yields that $\nu_n \in \mathcal{M}(L_\tau^*) = \mathcal{M}(K_\tau^*)$ for $\tau = c_4/n$ with some $c_4 = c_4(L)$.

Moreover, for $\xi \in \Omega \setminus \{\infty\}$,

$$\begin{aligned} U^{\mu-\nu_n}(\xi) &= -\log|\Phi(\xi)| - \log\operatorname{cap} L + \frac{1}{n}\log|F_n(\xi)|(\operatorname{cap} L)^n \\ &= \frac{1}{n}\log\frac{|\Phi(\xi)^n + \kappa(\xi)|}{|\Phi(\xi)|^n} \le \frac{c_5}{n|\Phi(\xi)|^n}, \end{aligned}$$

i.e. for sufficiently large $n \ge n_0$, which we can assume without loss of generality, we obtain

$$a(1) \le a_{\sup}(1) \le 2^{-n}.$$

Since for

$$u = \frac{2c_1 + c_4}{n} \qquad \text{and} \qquad m = \left[\frac{n}{2}\right] + 1$$

we have

$$u > \frac{c_4}{n} \qquad \text{and} \qquad mu > c_1,$$

applying (2) we obtain

$$\left|\nu_n(\overline{S(J)}) - \mu(J)\right| \preceq \frac{1}{n} + n^6 3^{n/2} 2^{-n} \preceq \frac{1}{n},$$

which completes the proof of Theorem 1. ∎

As we already mentioned, Theorem 2 is of local nature. An insignificant modification of the proof of Theorem 1 would allow us to establish its local analogue. We do not dwell on this purely technical issue.

3. Local properties of Φ and Ψ

For the sake of later reference, we explicitly state certain known facts concerning the conformal mappings Φ and Ψ. For details, see [2, Ch. 5] and [3, Sec. 1.2.6].

Let K, ζ_j, $j = 1, 2, 3, 4$ be as in Theorem 2 and let $L(\zeta_3, \zeta_4) \subset J := L(\zeta_5, \zeta_6) \subset L$, where points $\zeta_5 \in L(\zeta_1, \zeta_3)$ and $\zeta_6 \in L(\zeta_2, \zeta_4)$ satisfy

$$(5) \qquad |\zeta_1 - \zeta_5| = |\zeta_3 - \zeta_5| \qquad \text{and} \qquad |\zeta_2 - \zeta_6| = |\zeta_4 - \zeta_6|.$$

The set $S(J)$ consists of two unbounded Jordan domains which we denote by $S^{\pm}(J)$.

Our first remark concerns the geometry of an arc $S^{\pm}_{\zeta} := S(\{\zeta\}) \cap S^{\pm}(J)$ for any $\zeta \in J$. Let $z_1, z_2 \in S^{\pm}_{\zeta}$ and $w_j = \Phi(z_j)$, $j = 1, 2$ be such that $|w_1| < |w_2|$. Then

$$(6) \qquad |z_1 - \zeta| \preceq |z_2 - \zeta| \asymp d(z_2, K),$$

where

$$d(\xi, E) = \operatorname{dist}(\xi, E) := \inf_{z \in E} |\xi - z|, \qquad \xi \in \mathbb{C}, E \subset \mathbb{C}.$$

Moreover, the length of $S^{\pm}_{\zeta}(z_1, z_2)$ satisfies

$$(7) \qquad |S^{\pm}_{\zeta}(z_1, z_2)| \asymp |z_1 - z_2|.$$

In addition, if $|w_1| - 1 \asymp |w_2| - 1$, then

$$|z_1 - \zeta| \asymp |z_2 - \zeta|. \tag{8}$$

Denote by $\Phi_\pm$ the continuous extension of Φ from $S^\pm(J)$ to its closure $\overline{S^\pm(J)}$. For $t > 0$ and $z \in \overline{S^\pm(J)}$, let

$$\tilde{z}_t = \tilde{z}_t^\pm := \Psi((1+t)\Phi_\pm(z)) .$$

Since L is Dini-smooth, we have

$$|\zeta - \tilde{\zeta}_t^+| \asymp |\zeta - \tilde{\zeta}_t^-|, \qquad \zeta \in J. \tag{9}$$

Next, for $\zeta \in J$ and $z \in \overline{S^\pm(J)}$ with $|z| \preceq 1$ we obtain

$$\left| \frac{\tilde{z}_t^\pm - z}{\tilde{z}_t^\pm - \zeta} \right| \preceq \left| \frac{\tilde{\zeta}_t^\pm - \zeta}{\tilde{z}_t^\pm - \zeta} \right|^{1/2} . \tag{10}$$

Moreover, let $\zeta \in J$, $z \in \overline{S^\pm(J)}$, and $t > 0$ satisfy $|\zeta - \tilde{\zeta}_t^\pm| \preceq |\zeta - z|$. Then

$$\left| \frac{\zeta - \tilde{\zeta}_t^\pm}{\zeta - z} \right| \preceq \frac{t}{|\Phi_\pm(\zeta) - \Phi_\pm(z)|}. \tag{11}$$

The proof of (10) and (11) is based on the investigation of the module of different families of crosscuts of Ω. We also use this notion in the next section to derive the properties of the harmonic measure. For the definition and basic properties of the module of a family of curves see [16, 3].

For example, in order to prove (11) we let $\tau^\pm := \Phi_\pm(\zeta)$ and $w^\pm := \Phi_\pm(z)$. Denote by Γ the family of all crosscuts of Ω that separate ζ and $\tilde{\zeta}_t^\pm$ from z and ∞. Since L is Dini-smooth we have

$$m(\Gamma) \leq \frac{1}{\pi} \log \left| \frac{\zeta - z}{\zeta - \tilde{\zeta}_t^\pm} \right| + c_1. \tag{12}$$

Meanwhile, for the family $\Gamma' := \Phi(\Gamma)$ we obtain

$$m(\Gamma') \geq \frac{1}{\pi} \log \frac{|\tau^\pm - w^\pm|}{t} - c_2. \tag{13}$$

Comparing (12) and (13) we have (11).

Since L is Dini-smooth we also have

$$t^2 \preceq |z - \tilde{z}_t^\pm| \preceq t, \qquad z \in S^\pm(J). \tag{14}$$

4. Auxiliary functions

Let K be as in Theorem 2. Our first objective is to construct a neighborhood $G = G_{\delta,\zeta}$ of K with some special properties for a sufficiently small $\delta > 0$ and an arbitrary point $\zeta \in L(\zeta_3, \zeta_4)$.

Denote by $N = N_\zeta$ the straight line passing through ζ which is perpendicular to L at ζ. Let $\zeta^\pm \in N$ be two distinct points satisfying $|\zeta^\pm - \zeta| = \delta$. Let for $z, \zeta \in \mathbb{C}$,

$$[z,\zeta] := \{\xi = z + t(\zeta - z)\colon 0 \le t \le 1\},$$
$$(z,\zeta) := \{\xi = z + t(\zeta - z) : 0 < t < 1\}.$$

Since L is smooth, we can choose $\zeta^\pm$ such that for a sufficiently small fixed number $\varepsilon > 0$ there exist four distinct points $\zeta_j^\pm$, $j = 1,2$ on B_ε, i.e.

$$\Phi(\zeta_j^\pm) = \varepsilon \exp(i\theta_j^\pm), \qquad \theta_1^+ < \theta_2^+ < \theta_1^- < \theta_2^- < \theta_1^+ + 2\pi$$

with the following properties:

$$\arg \frac{\zeta^\pm - \zeta}{\zeta_j^\pm - \zeta^\pm} = (-1)^{j+1}\frac{3\pi}{8}$$

and for the intervals $l_j^\pm := [\zeta^\pm, \zeta_j^\pm]$,

$$d(\xi, K) \asymp |\xi - \zeta| \asymp \delta + |\xi - \zeta^\pm|, \qquad \xi \in l_j^\pm.$$

Consider three finite simply connected domains G_1, G_2 and G in $\mathbb{C}$ bounded by the following Jordan curves

$$\partial G_1 := [\zeta^-, \zeta^+] \cup l_1^+ \cup l_2^- \cup \{\xi \in B_\varepsilon \colon \theta_2^- - 2\pi \le \arg \Phi(\xi) \le \theta_1^+\},$$
$$\partial G_2 := [\zeta^-, \zeta^+] \cup l_2^+ \cup l_1^- \cup \{\xi \in B_\varepsilon : \theta_2^+ \le \arg \Phi(\xi) \le \theta_1^-\},$$
$$\partial G := (\partial G_1 \cup \partial G_2) \setminus (\zeta^-, \zeta^+).$$

Without loss of generality we assume that $\zeta_j \in G_j$, $j = 1,2$.

Let $\omega(z, G, S)$ be the *harmonic measure* of a (Borel) set $S \subset \partial G$ at a point $z \in G$ with respect to G. Consider the function

$$f_1(z) = f_{1,\delta,\zeta}(z) := \omega(z, G, \partial G_1 \cap \partial G), \qquad z \in G.$$

Lemma 1. *Let $z \in G$ satisfy*

$$d(z, L) \le c_1 d(z, \partial G) \qquad \text{and} \qquad |z - \zeta| \le c_2\delta.$$

Then

$$0 < c_3 \le f_1(z) \le c_4 < 1 \tag{15}$$

holds with $c_k = c_k(c_1, c_2, K, \zeta_3, \zeta_4, \varepsilon)$, $k = 3,4$.

Moreover, if $z \in G$ satisfies

$$|z - \zeta| \le c_5 d(z, \partial G), \tag{16}$$

then the inequalities

$$c_6 \le f_1(z)\left(\frac{|z-\zeta|+\delta}{\delta}\right)^4 \le c_7, \qquad z \in G_2, \tag{17}$$

$$c_8 \le (1-f_1(z))\left(\frac{|z-\zeta|+\delta}{\delta}\right)^4 \le c_9, \qquad z \in G_1, \tag{18}$$

hold with $c_k = c_k(c_5, K, \zeta_3, \zeta_4, \varepsilon)$, $k = 6, 7, 8, 9$.

Proof. By [3, p. 27] for $z \in G$,

$$f_1(z) \asymp \exp(-\pi m(\Gamma_1)), \tag{19}$$

$$1 - f_1(z) \asymp \exp(-\pi m(\Gamma_2)), \tag{20}$$

where Γ_j, $j = 1, 2$ is the family of all crosscuts of G that separate (in G) the point z from $\partial G \cap \partial G_j$.

Considering the metric

$$\rho(\xi) = \begin{cases} 1 & \text{if } \xi \in G, |\xi - \zeta| \le (c_2+1)\delta, \\ 0 & \text{otherwise in } \mathbb{C}, \end{cases}$$

in the definition of the module of a family of curves (see [16, p. 133] or [3, p. 342]) we have

$$m(\Gamma_j) \le \frac{\int_G \rho^2(\xi)\, dm(\xi)}{\left(\inf_{\gamma\in\Gamma_j} \int_\gamma \rho(\xi)\, |d\xi|\right)^2} \preceq 1, \tag{21}$$

where dm means integration with respect to the area in $\mathbb{C}$. Comparing (19)–(21) we have (15).

Next, we establish (17) (the proof of (18) is similar). According to (15) we can assume that $|z-\zeta| \ge 2\delta$. We claim that in this case the inequality

$$-c_{10} \le m(\Gamma_1) - \frac{4}{\pi}\log\frac{|z-\zeta|}{\delta} \le c_{10} \tag{22}$$

holds with some $c_{10} = c_{10}(c_5, K, \zeta_3, \zeta_4, \varepsilon)$.

Indeed, let

$$b = b_z := \min\left(|z-\zeta|, d(\zeta, \partial G_2 \setminus (l_1^- \cup [\zeta^-, \zeta^+] \cup l_2^+))\right).$$

Denote by $D = D_z$ the finite domain bounded by the Jordan curve

$$\partial D := \left\{\xi \in l_1^- \cup l_2^+ : |\xi - \zeta| \le b\right\} \cup \{\xi \in G_2 : |\xi - \zeta| = b \text{ or } \delta\}.$$

Let $\Gamma_3 = \Gamma_{3,z}$ be the family of all crosscuts of D that separate the boundary circular arcs $\{\xi \in G_2 : |\xi-\zeta| = b\}$ and $\{\xi \in G_2 : |\xi - \zeta| = \delta\}$. According to [3, p. 347] we obtain

$$-c_{11} \le m(\Gamma_3) - \frac{4}{\pi}\log\frac{b}{\delta} \le 0. \tag{23}$$

Since $m(\Gamma_1) \geq m(\Gamma_3)$, the left-hand side of (23) yields the left-hand side of (22).

Next, let

$$\Gamma_4 = \Gamma_{4,z} := \{\gamma \in \Gamma_1 \colon \gamma \cap \{\xi \in G_2 \colon |\xi - \zeta| = \delta\} \neq \emptyset\},$$
$$\Gamma_5 := \Gamma_1 \setminus (\Gamma_3 \cup \Gamma_4).$$

According to the definition of the module, for the metric

$$\rho(\xi) = \begin{cases} 1 & \text{if } \xi \in G, |\xi - \zeta| \leq 2\delta, \\ 0 & \text{otherwise in } \mathbb{C} \end{cases}$$

we have

$$m(\Gamma_4) \leq \frac{\int_G \rho^2(\xi) dm(\xi)}{\left(\inf_{\gamma \in \Gamma_4} \int_\gamma \rho(\xi)\,|d\xi|\right)^2} \preceq 1. \tag{24}$$

Using the same reasoning it can also be shown that

$$m(\Gamma_5) \preceq 1. \tag{25}$$

In order to prove (25), one has to apply the definition of the module and the metric which is defined as follows:

- if $|z - \zeta| \geq b$ then

$$\rho(\xi) = \begin{cases} 1 & \text{if } \xi \in G_2, \\ 0 & \text{otherwise in } \mathbb{C}, \end{cases}$$

- if $|z - \zeta| < b$ then

$$\rho(\xi) = \begin{cases} 1 & \text{if } \xi \in G_2, \dfrac{1}{2} \leq \dfrac{|\xi - \zeta|}{|\xi - z|} \leq 2, \\ 0 & \text{otherwise in } \mathbb{C}. \end{cases}$$

The right-hand side of (22) follows from (23)–(25) and the composition law for the module, i.e. the inequality

$$m(\Gamma_1) \leq m(\Gamma_3) + m(\Gamma_4) + m(\Gamma_5).$$

Comparing (19) and (22) we obtain (17). ∎

Next, consider the function

$$f_2(z) = f_{2,\delta,\zeta}(z) := \omega(z, G, \{\xi \in l_1^+ \cup l_2^+ \colon |\xi - \zeta^+| \leq \delta\}), \qquad z \in G.$$

Using the same ideas as in the proof of Lemma 1 one can establish the following properties of f_2.

Lemma 2. *Let $z \in G$ satisfy* (16). *Then the double inequality*

$$c_{12} \le f_2(z)\left(\frac{|z-\zeta|+\delta}{\delta}\right)^4 \le c_{13}, \qquad z \in G, \tag{26}$$

holds with $c_k = c_k(c_5, K, \zeta_3, \zeta_4, \varepsilon)$, $k = 12, 13$.

Proof. We have to use the formula (19) in which Γ_1 is now defined as the family of all crosscuts of G that separate the point $z \in G$ from $\{\xi \in l_1^+ \cup l_2^+ : |\xi - \zeta^+| \le \delta\}$ and the inequality

$$-c_{14} \le m(\Gamma_1) - \frac{4}{\pi}\log\frac{|z-\zeta|+\delta}{\delta} \le c_{14}$$

which holds in this case. ■

We extend f_j, $j = 1, 2$, to the whole complex plane by setting

$$f_j(z) = 0, \qquad z \in \mathbb{C} \setminus G.$$

Our next objective is to derive an integral representation for f_j (in the neighborhood of K) by averaging it in a particular way. We use some elements of the method described in [2, pp. 175–177].

Let $V(z)$, $z \in \mathbb{C}$ be an arbitrary averaging kernel, i.e. V is infinitely partially differentiable function in $\mathbb{C}$ such that

$$\int_{\mathbb{C}} V(z)\,dm(z) = 1,$$
$$V(z) = V(|z|) \ge 0, \qquad z \in \mathbb{C},$$
$$V(z) = 0, \qquad z \in \mathbb{D}^* \setminus \{\infty\}.$$

Denote by $\eta(\xi), \xi \in \Omega$ the regularized distance from ξ to K (see [23, pp. 170–171] or [3, pp. 407–409]), i.e. the infinitely partially differentiable function in Ω that satisfies the inequalities

$$d(\xi, K) \preceq \eta(\xi) \le \frac{1}{2}d(\xi, K), \tag{27}$$

$$\left|\frac{\partial^{j+k}\eta(x+iy)}{\partial x^j \partial y^k}\right| \preceq \eta(x+iy)^{1-j-k}, \qquad j, k = 1, 2. \tag{28}$$

for any $\xi = x + iy \in \Omega \setminus \{\infty\}$.

For $j = 1, 2$ and $\xi \in \Omega \setminus \{\infty\}$, consider the function

$$h_j(\xi) = h_{j,\delta,\zeta}(\xi) := \frac{1}{\eta(\xi)^2}\int_{\mathbb{C}} f_j(z) V\left(\frac{z-\xi}{\eta(\xi)}\right) dm(z)$$
$$= \int_{\mathbb{C}} f_j(\xi + \eta(\xi)z) V(z)\,dm(z)$$

which we extend to K by setting

$$h_j(\xi) := f_j(\xi), \qquad \xi \in K.$$

Due to (27) and (28), the infinitely partially differentiable function h_j in $\mathbb{C}$ satisfies the following properties:

$$h_j(\xi) = f_j(\xi), \qquad \xi \in M, \tag{29}$$

$$h_j(\xi) = 0, \qquad \xi \in T, \tag{30}$$

$$|\Delta h_j(\xi)| := \left| \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) h_j(x+iy) \right| \preceq \frac{1}{d(\xi, K)^2}, \qquad \xi = x + iy \in H, \tag{31}$$

where c is sufficiently large and

$$\begin{aligned} T &:= \{\xi \colon |\xi| \geq c\}, \qquad T \cap G = \emptyset, \\ H = H_{\delta,\zeta} &:= \{\xi \in \Omega \colon d(\xi, K) \geq d(\xi, \partial G), |\xi| < c\}, \\ M = M_{\zeta,\delta} &:= \mathbb{C} \setminus (T \cup H). \end{aligned}$$

According to the Green formula (see [3, p. 7]) and (29)–(31) we have

$$f_j(\xi) = \int_H g_j(z) \log \left| \frac{z - \xi}{z_0 - \xi} \right| dm(z) + \kappa_j(\xi), \qquad \xi \in M, \tag{32}$$

where

$$|g_j(z)| \preceq \frac{1}{|z - \zeta|^2}, \qquad z \in H, \tag{33}$$

and z_0 is a fixed point with $|z_0| = 3c$, and the in $\{\xi \colon |\xi| < 3c\}$ harmonic functions $\kappa_j(\xi)$ satisfy

$$|\kappa_j(\xi)| \preceq \frac{1}{\delta^2}, \qquad |\xi| \leq 2c. \tag{34}$$

5. Harmonic polynomial approximation of f_j

In this section, for the functions f_j, $j = 1, 2$, from Section 4 we construct an approximating harmonic polynomial

$$Q_j(\xi) = Q_{j,m,\zeta}(\xi) = \operatorname{Re} \sum_{k=0}^{m} a_k \xi^k, \qquad a_k \in \mathbb{C},\ m = m_{\delta,\zeta} \in \mathbb{N},$$

with some particular properties.

Bearing in mind the integral representation (32), we first discuss the approximation of the logarithmic kernel

$$\log \left| \frac{z - \xi}{z_0 - \xi} \right|, \qquad z \in H,\ \xi \in K_u^*,$$

where $u = u(\delta, \varepsilon) > 0$ is chosen such that $K_{2u}^* \cap H = \emptyset$ and $d(\zeta, B_u) \asymp \delta$, by the harmonic polynomial kernels

$$q_m(z, \xi) = \operatorname{Re} \sum_{k=0}^{m} b_k(z) \xi^k.$$

According to [2, Thm. 3.3,p. 95] for any $\tau \in \operatorname{ext} B_{2u} := \mathbb{C} \setminus K^*_{2u}$, $|\tau| \le |z_0|$ and $m > 1/u$, there exists the *Dzjadyk polynomial kernel*

$$D_{m-1}(\tau, \xi) = \sum_{k=0}^{m-1} d_k(\tau)\xi^k$$

such that for $\xi \in K^*_u$, the inequalities

$$\left|\frac{1}{\tau - \xi} - D_{m-1}(\tau, \xi)\right| \preceq \frac{|\tilde{\tau} - \tau|^{24}}{|\tau - \xi||\tilde{\tau} - \xi|^{24}} \left(1 + \left|\frac{\tau - \xi}{\tilde{\tau} - \xi}\right|\right)^{24}, \tag{35}$$

$$\left|\frac{\partial}{\partial \xi} D_{m-1}(\tau, \xi)\right| \preceq \frac{1}{|\tilde{\tau} - \xi|^2} \left(1 + \left|\frac{\tau - \xi}{\tilde{\tau} - \xi}\right|\right)^{24} \tag{36}$$

hold, where for $z \in \Omega \setminus \{\infty\}$,

$$\tilde{z} = \tilde{z}_{1/m} := \Psi\left(\left(1 + \frac{1}{m}\right)\Phi(z)\right).$$

For $z \in L$ we let $\tilde{z} := \tilde{z}^+_{1/m}$ (see Section 3).

Next, we join points $z \in H$ and z_0 by an arc $l = l(z, z_0) \subset \operatorname{ext} B_{2u}$ which consists of a subarc of $S(\{z_K\})$ and a subarc of the circle $\{\xi \in \mathbb{C} \colon |\xi| = |z_0|\}$. Hence, by (6) and (7) for $\tau \in l$ we have

$$d(\tau, K^*_u) \asymp |\tau - z| + |z - \zeta|,$$
$$|l(z, \tau)| := \int_{l(z,\tau)} |d\xi| \asymp |z - \tau|,$$

where $l(z, \tau)$ is the subarc of l with the endpoints z and τ.

For $\xi \in K^*_u$ and $z \in H$ consider the polynomials

$$p_m(z, \xi) := \int_l D_{m-1}(\tau, \xi)\, d\tau, \qquad q_m(z, \xi) := \operatorname{Re} p_m(z, \xi),$$

where the orientation of l is chosen such that

$$\log\left|\frac{z - \xi}{z_0 - \xi}\right| = \operatorname{Re} \int_l \frac{d\tau}{\tau - \xi}.$$

Let $\zeta_5 \in L(\zeta_1, \zeta_3)$ and $\zeta_6 \in L(\zeta_2, \zeta_4)$ be auxiliary points satisfying (5). Note that by (6), (8)–(10) and (14) for $z \in H$, $\tau \in l(z, z_0)$, and $\xi \in K^*_u$ with $\xi_K \in L(\zeta_5, \zeta_6)$ we have

$$|\tilde{\tau} - \xi| \asymp |\tau - \xi|, \tag{37}$$

$$\left|\frac{\tilde{\tau} - \tau}{\tilde{\tau} - \xi}\right| \preceq \left|\frac{\tilde{\xi} - \xi}{\tilde{\tau} - \xi}\right|^{1/2}, \tag{38}$$

$$|\tilde{\tau} - \tau| \preceq \frac{1}{m}. \tag{39}$$

By virtue of (35) and (37) we obtain

$$C(\xi,z):=\left|\log\left|\frac{z-\xi}{z_0-\xi}\right|-q_m(z,\xi)\right|\preceq\int_l\left|\frac{\tilde\tau-\tau}{\tilde\tau-\xi}\right|^{24}\frac{|d\tau|}{|\tau-\xi|}.$$

Therefore, if $\xi\in K_u^*$ satisfies $\xi_K\notin J:=L(\zeta_5,\zeta_6)$ then by (39)

$$C(\xi,z)\preceq\frac{1}{m^{24}}. \tag{40}$$

At the same time, if $\xi\in K_u^*$ and $\xi_K\in J$ then by (37) and (38)

$$C(\xi,z)\preceq|\xi-\tilde\xi|^{12}\int_l\frac{|d\tau|}{|\tau-\xi|^{13}}\preceq\left|\frac{\xi-\tilde\xi}{\xi-z}\right|^{12}. \tag{41}$$

Moreover, by (36) and (37) for $\xi\in K_u^*$ and $z\in H$,

$$\begin{aligned}\left|\frac{\partial}{\partial\xi}p_m(z,\xi)\right|&=\left|\int_l\frac{\partial}{\partial\xi}D_{m-1}(\tau,\xi)\,d\tau\right|\preceq\int_l\frac{|d\tau|}{|\tilde\tau-\xi|^2}\\&\asymp\int_l\frac{|d\tau|}{|\tau-\xi|^2}\preceq\frac{1}{|\xi-z|}.\end{aligned} \tag{42}$$

Let f_j, κ_j, and c be as in (32)–(34). As all the functions κ_j are harmonic in $\{\xi\in\mathbb{C}\colon|\xi|<3c\}$, there exist harmonic polynomials t_j of degree at most m which satisfy the inequalities

$$|\kappa_j(\xi)-t_j(\xi)|\preceq\frac{1}{2^m}, \tag{43}$$

$$\begin{aligned}|\operatorname{grad}t_j(\xi)|&:=\left(\left|\frac{\partial t_j(x+iy)}{\partial x}\right|^2+\left|\frac{\partial t_j(x+iy)}{\partial y}\right|^2\right)^{1/2}\\&\le\frac{1}{\pi}\int_{|\eta|=2c}\frac{|t_j(\eta)||d\eta|}{|\eta-\xi|^2}\preceq\frac{1}{\delta^2}\end{aligned} \tag{44}$$

for $|\xi|=|x+iy|<c$. Also, consider the polynomials

$$v_j(\xi):=\int_H g_j(z)q_m(z,\xi)\,dm(z),\qquad Q_j(\xi):=v_j(\xi)+t_j(\xi).$$

We claim that for $\xi\in K_u^*$,

$$|f_j(\xi)-\kappa_j(\xi)-v_j(\xi)|\preceq\begin{cases}\dfrac{1}{m^{24}}\log\dfrac{c_1}{\delta} & \text{if }\xi_K\notin J,\\[2ex]\left(\dfrac{|\zeta-\tilde\zeta|}{|\xi-\zeta|+\delta}\right)^5 & \text{if }\xi_K\in J.\end{cases} \tag{45}$$

Indeed, the upper part of (45) follows immediately from (33) and (40).

Since by (10)

$$\frac{|\tilde{\xi}-\xi|}{|\xi-\zeta|+\delta} \preceq \left(\frac{|\tilde{\zeta}-\zeta|}{|\xi-\zeta|+\delta}\right)^{1/2},$$

according to (33) and (41), we have

$$\begin{aligned} |f_j(\xi)-\kappa_j(\xi)-v_j(\xi)| &\preceq |\xi-\tilde{\xi}|^{12}\int_H \frac{dm(z)}{|z-\zeta|^2(|\xi-\zeta|+|z-\zeta|)^{12}} \\ &\preceq \left(\frac{|\tilde{\xi}-\xi|}{|\xi-\zeta|+\delta}\right)^{12}\log\frac{|\xi-\zeta|+\delta}{\delta} \\ &\preceq \left(\frac{|\tilde{\zeta}-\zeta|}{|\xi-\zeta|+\delta}\right)^{6}\frac{|\xi-\zeta|+\delta}{\delta} \preceq \left(\frac{|\tilde{\zeta}-\zeta|}{|\xi-\zeta|+\delta}\right)^{5}, \end{aligned}$$

which proves the lower part of (45).

Moreover, according to (33) and (42), for $\xi \in K$ we obtain

$$|\operatorname{grad} v_j(\xi)| \preceq \int_H \frac{dm(z)}{|z-\zeta|^2|\xi-z|} \preceq \frac{1}{\delta^3}. \tag{46}$$

Next, we summarize the properties of the harmonic polynomials Q_j. We relate Q_1 to a characteristic function $\chi(\xi)$ defined as follows. Let for $\xi \in K_u^*$,

$$\chi(\xi) = \chi_{\delta,\zeta,u}(\xi) := \begin{cases} 1 & \text{if } \xi \in K_u^* \cap \overline{G_1}, \\ 0 & \text{if } \xi \in K_u^* \cap G_2. \end{cases}$$

Note that by (11) for any $\xi \in K_u^*$ with $\xi_K \in J$,

$$\frac{\delta}{|\xi-\zeta|+\delta} \preceq \frac{u}{\mu(L(\xi_K,\zeta))+u}. \tag{47}$$

Since for any arbitrary small but fixed $\varepsilon > 0$ one can find $m = m_\varepsilon \asymp 1/u$, $m < 1/u$ such that

$$\delta \preceq |\zeta - \tilde{\zeta}^{\pm}| < \varepsilon\delta,$$

according to Lemma 1, (14), (15), and (43)–(47) the polynomial $Q_1 = Q_{1,m,\zeta}$ satisfies for $\xi \in K_u^*$

$$0 \le Q_1(\xi) \le 1, \tag{48}$$

$$\begin{aligned} |\chi(\xi)-Q_1(\xi)| &\preceq \left(\frac{\delta}{|\xi-\zeta|+\delta}\right)^4 \\ &\preceq \begin{cases} \left(\dfrac{u}{\mu(L(\xi_K,\zeta))+u}\right)^4 & \text{if } \xi_K \in J, \\ \dfrac{1}{m^4} & \text{otherwise.} \end{cases} \end{aligned} \tag{49}$$

Moreover,

$$|\operatorname{grad} Q_1(\xi)| \preceq m^6, \qquad \xi \in B. \tag{50}$$

The inequality (49) implies that

$$\left| \int Q_1 d\mu - \mu(G_1) \right| \preceq u \asymp \frac{1}{m}. \tag{51}$$

Next, the same reasoning, using Lemma 2 instead of Lemma 1, shows that there exists $1/u \preceq m < 1/u$ such that the polynomial $Q_2 = Q_{2,m,\zeta}$ satisfies the inequalities

$$|\operatorname{grad} Q_2(\xi)| \preceq m^6, \qquad \xi \in B, \tag{52}$$

$$Q_2(\xi) \asymp \left(\frac{\delta}{|\xi - \zeta| + \delta} \right)^4 \preceq \begin{cases} \left(\dfrac{u}{\mu(L(\xi_K, \zeta)) + u} \right)^4 & \text{if } \xi \in K_u^*,\ \xi_K \in J, \\ \dfrac{1}{m^4} & \text{otherwise in } K_u^*, \end{cases} \tag{53}$$

and therefore

$$\int Q_2 d\mu \preceq u \asymp \frac{1}{m}. \tag{54}$$

6. Proof of Theorem 2

We use elements of the method discussed in [3, Sec. 4.4]. As in the previous sections we introduce auxiliary points $\zeta_5 \in L(\zeta_1, \zeta_3)$ and $\zeta_6 \in L(\zeta_2, \zeta_4)$ satisfying (5). Let $\zeta \in L(\zeta_3, \zeta_4)$ be an arbitrary point. We define $\delta = \delta_\zeta \asymp |\zeta - \tilde{\zeta}_u^{\pm}|$ (cf. (9)) such that the harmonic polynomials $Q_j = Q_{j,m,\zeta}$, $j = 1, 2$ from Section 5 satisfy (47)–(54). The Green formula (see [3, p. 7]) implies that for $z \in K_t^*$ and fixed $t > 0$,

$$Q_j(z) = \frac{1}{2\pi} \int_{B_{2t}} \left(\frac{\partial Q_j(\xi)}{\partial \mathbf{n}_\xi} \log \frac{1}{|\xi - z|} - Q_j(\xi) \frac{\partial}{\partial \mathbf{n}_\xi} \log \frac{1}{|\xi - z|} \right) |d\xi|, \tag{55}$$

where $\partial / \partial \mathbf{n}_\xi$ denotes the operator of differentiation with respect to the outward normal to the level line B_{2t} at the point $\xi \in B_{2t}$.

Next, integrating (55) and using the Fubini Theorem we obtain

$$\int Q_j (d\nu - d\mu) = \frac{1}{2\pi} \int_{B_{2t}} \left(U^{\nu - \mu}(\xi) \frac{\partial Q_j(\xi)}{\partial \mathbf{n}_\xi} - Q_j(\xi) \frac{\partial}{\partial \mathbf{n}_\xi} U^{\nu - \mu}(\xi) \right) |d\xi|. \tag{56}$$

We separately estimate integrals of two terms in the right-hand side of (56) as follows.

According to (48), (50), (53), (52), and the Bernstein-Walsh Lemma (cf. [27, p. 77]), we have

$$|Q_j(z)| + |\operatorname{grad} Q_j(z)| \preceq m^6 (1 + 2t)^m, \qquad z \in B_{2t}. \tag{57}$$

Without loss of generality we can assume that

$$a := a(t) = \sup_{\xi \in B_t} U^{\mu-\nu}(\xi) \le \sup_{\xi \in B_t} U^{\nu-\mu}(\xi).$$

The estimate (57) and the mean value property for harmonic functions imply that

$$\begin{aligned}(58)\qquad & \left| \int_{B_{2t}} U^{\nu-\mu}(\xi) \frac{\partial Q_j(\xi)}{\partial \mathbf{n}_\xi} \, |d\xi| \right| \\ &= \left| \int_{B_{2t}} \left(a + U^{\nu-\mu}(\xi)\right) \frac{\partial Q_j(\xi)}{\partial \mathbf{n}_\xi} \, |d\xi| \right| \\ &\preceq m^6 (1+2t)^m \int_{B_{2t}} \left(a + U^{\nu-\mu}(\xi)\right) \, |d\xi| \\ &\preceq m^6 (1+2t)^m \int_{|w|=1+2t} \left(a + U^{\nu-\mu}(\Psi(w))\right) \, |dw| \\ &= 2\pi m^6 (1+2t)^{m+1} a.\end{aligned}$$

Furthermore, since by virtue of (57)

$$\begin{aligned}& \left| \int_{B_{2t}} Q_j(\xi) \frac{\partial}{\partial \mathbf{n}_\xi} U^{\nu-\mu}(\xi) \, |d\xi| \right| \\ &\quad \preceq m^6 (1+2t)^m \int_{B_{2t}} \left| \operatorname{grad} U^{\nu-\mu}(\xi) \right| \, |d\xi| \\ &\quad = m^6 (1+2t)^m \int_{|w|=1+2t} \left| \operatorname{grad}(a + U^{\nu-\mu}(\Psi(w))) \right| \, |dw|,\end{aligned}$$

combined with the Schwarz formula (see [3, p. 4]) and the mean value property for harmonic functions to give

$$\begin{aligned}& \int_{|w|=1+2t} \left| \operatorname{grad}(a + U^{\nu-\mu}(\Psi(w))) \right| \, |dw| \\ &\quad \preceq \int_{|w|=1+2t} \int_{|\tau|=1+t} \left(a + U^{\nu-\mu}(\Psi(\tau))\right) \frac{|d\tau||dw|}{|\tau - w|^2} \\ &\quad = \int_{|\tau|=1+t} \left(a + U^{\nu-\mu}(\Psi(\tau))\right) \left(\int_{|w|=1+2t} \frac{|dw|}{|\tau - w|^2} \right) |d\tau| \\ &\quad \preceq \int_{|\tau|=1+t} \left(a + U^{\nu-\mu}(\Psi(\tau))\right) \, |d\tau| = 2\pi(1+t)a \asymp a,\end{aligned}$$

we have

$$(59)\qquad \left| \int_{B_{2t}} Q_j(\xi) \frac{\partial}{\partial \mathbf{n}_\xi} U^{\nu-\mu}(\xi) \, |d\xi| \right| \preceq m^6 (1+2t)^m a.$$

Hence, (56), (58), and (59) yield

$$\left|\int Q_j(d\nu - d\mu)\right| \preceq m^6(1+2t)^m a. \tag{60}$$

We derive two corollaries from (60). Since ζ_3 and ζ_4 are arbitrary points on L, the inequality (60) is also true for the polynomial $Q_2 = Q_{2,m,\zeta}$ with $\zeta \in L(\zeta_5, \zeta_6)$. Therefore, if $J \subset L(\zeta_5, \zeta_6)$ satisfies $\mu(J) \le 1/m$, then

$$\nu(\overline{S(J)}) \preceq \frac{1}{m} + m^6(1+2t)^m a =: C. \tag{61}$$

Indeed, by (53), (54) and (60) for Q_2 with any fixed $\zeta \in J$ we obtain

$$\nu(\overline{S(J)}) \preceq \int Q_2\, d\nu = \int Q_2(d\nu - d\mu) + \int Q_2\, d\mu \preceq C.$$

The second corollary concerns the following case. Let $\zeta \in L(\zeta_3, \zeta_4)$ and let m be sufficiently large. We divide $L(\zeta, \zeta_6)$ by points $\zeta =: \eta_1, \eta_2, \ldots, \eta_k := \zeta_6$ moving along $L(\zeta, \zeta_6)$ from ζ to ζ_6 such that

$$\mu(L(\eta_1, \eta_2)) = \ldots = \mu(L(\eta_{k-2}, \eta_{k-1})) = \frac{1}{m}, \qquad \mu(L(\eta_{k-1}, \eta_k)) \le \frac{1}{m}.$$

By virtue of (61),

$$\nu(\overline{S(L(\eta_j, \eta_{j+1}))}) \preceq C, \qquad j = 1, \ldots, k. \tag{62}$$

Note that the unbounded Jordan curve $\overline{S(\{\zeta\})}$ divides $\mathbb{C}$ into two Jordan domains $V_1 = V_1(\zeta)$ and $V_2 = V_2(\zeta)$ (numbered such that $\zeta_j \in V_j$, $j = 1, 2$). According to (48), (49), and (62) for $Q_1 = Q_{1,m,\zeta}$ we have

$$\begin{aligned}\int Q_1\, d\nu &\le \int_{\overline{V_1}} Q_1\, d\nu + \sum_{j=1}^{k-1} \int_{\overline{S(L(\eta_j,\eta_{j+1}))}} Q_1\, d\nu + \frac{c_1}{m^4} \\ &\le \nu(\overline{V_1}) + c_2 C \sum_{j=1}^{k-1} \frac{1}{j^4} + \frac{c_1}{m^4} \\ &\le \nu(\overline{V_1}) + c_3 C.\end{aligned}$$

Since, in addition, by (51) and (60),

$$\int Q_1\, d\nu \ge \int Q_1\, d\mu - c_4 C \ge \mu(V_1) - c_5 C,$$

we obtain

$$\nu(\overline{V_1}) - \mu(V_1) \ge -c_6 C. \tag{63}$$

If we replace ζ_6 by ζ_5 and V_1 by V_2 in the above discussion, we obtain

$$\nu(\overline{V_2}) - \mu(V_2) \ge -c_7 C. \tag{64}$$

Since

$$\nu(\overline{V_1}) + \nu(\overline{V_2}) - 1 = \nu(\overline{S(\{\zeta\})}) \qquad \text{and} \qquad \mu(V_1) + \mu(V_2) = 1,$$

the inequalities (61), (63), and (64) yield

$$|\nu(\overline{V_1}) - \mu(V_1)| \preceq C. \tag{65}$$

Furthermore, since for an arbitrary arc $J = L(\zeta', \zeta'') \subset L(\zeta_3, \zeta_4)$ such that $\zeta' \in L(\zeta_3, \zeta'')$ we have

$$\begin{aligned} \nu(\overline{S(J)}) &= \nu(\overline{V_1(\zeta'')}) - \nu(V_1(\zeta')), \\ \mu(J) &= \mu(\overline{V_1(\zeta'')}) - \mu(V_1(\zeta')), \end{aligned}$$

the estimates (61) and (65) imply (2).

References

1. J.-E. Andersson, *On the degree of polynomial and rational approximation of holomorphic functions*, Dissertation, Göteborg, 1975.
2. V. V. Andrievskii, V. I. Belyi and V. K. Dzjadyk, *Conformal Invariants in Constructive Theory of Functions of Complex Variable*, Atlanta, Georgia, World Federation Publisher, 1995.
3. V. V. Andrievskii and H.-P. Blatt, *Discrepancy of Signed Measures and Polynomial Approximation*, Springer-Verlag, Berlin/New York, 2002.
4. J. H. Curtiss, Faber polynomials and Faber series, *Amer. Math. Monthly* **78** (1971), 577–596.
5. V. K. Dzjadyk, *Introduction to the Theory of Uniform Approximation of Functions by Polynomials*, (in Russian), Nauka, Moskow, 1977.
6. M. Eiermann and R. S. Varga, Zeros and local extreme points of Faber polynomials associated with hypocicloidal domains, *Electron. Trans. Numer. Anal.* **1** (1993), 49–71.
7. P. Erdős and Turán, On the problem in the theory of uniform distribution I and II, *Indag. Math.* **10** (1948), 370–378, 406–413.
8. L. Frerick and J. Müller, Polynomial approximation and maximal convergence on Faber sets, *Constr. Approx.* **10** (1994), 427–438.
9. D. Gaier, *Vorlesungen über Approximation im Komplexen*, Birkhäuser-Verlag, Basel, 1980.
10. ———, The Faber operator and its boundednees, *J. Approx. Theory* **101** (1999), 265–277.
11. A. W. Goodman, A note on the zeros of Faber polynomials, *Proc. Amer. Math. Soc.* **49** (1975), 407–410.
12. M. X. He and E. B. Saff, The zeros of Faber polynomials for an m-cusped hypocycloid, *J. Approx. Theory* **78** (1994), 410–432.
13. T. Kövari and Ch. Pommerenke, On Faber polynomials and Faber expansions, *Math. Z.* **99** (1967), 193–206.
14. A. B. J. Kuijlaars, The zeros of Faber polynomials generated by an m-star, *Math. Comp.* **65** (1996), 151–156.
15. A. B. J. Kuijlaars and E. B. Saff, Asymptotic distribution of the zeros of Faber polynomials, *Math. Proc. Camb. Phil. Soc.* **118** (1995), 437-447.
16. O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane, 2nd ed.*, Springer-Verlag, New York, 1973.
17. J. Liesen, On the location of the zeros of Faber polynomials, *Analysis (Munich)* **20** (2000), 157–162.
18. E. Miña Díaz, *Assymptotics for Faber polynomials and polynomials orthogonal over regions in the complex plane*, Dissertation, Nashville, Tennessee, 2006.
19. Ch. Pommerenke, Konforme Abbildung und Fekete-Punkte, *Math. Z.* **89** (1965), 422–438.

20. T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995.
21. E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer-Verlag, New York/Berlin, 1997.
22. V. I. Smirnov and N. A. Lebedev, *Functions of a Complex Variable: Constructive Theory*, M.I.T. Press, Cambrige, Mass., 1968.
23. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.
24. P. K. Suetin, *Series of Faber Polynomials*, Gordon and Breach Science Publications, Amsterdam, 1998.
25. J. L. Ullman, Studies in Faber polynomials I, *Trans. Amer. Math. Soc.* **73** (1960), 515–528.
26. ———, The location of the zeros of the derivatives of Faber polynomials, *Proc. Amer. Math. Soc.* **34** (1972), 422–424.
27. J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Plane, 5th ed.*, American Mathematical Society, Providence, RI, 1969.
28. H. Widom, Extremal polynomials associated with a system of curves in the complex plane, *Adv. Math.* **3** (1969), 127–232.

Vladimir V. Andrievskii E-MAIL: andriyev@math.kent.edu
ADDRESS: *Kent State University, Department of Mathematical Sciences, Kent, OH 44242, U.S.A.*

Hans-Peter Blatt E-MAIL: hans.blatt@ku-eichstaett.de
ADDRESS: *Katholische Universität Eichstätt-Ingolstadt, Ostenstrasse 26–28, 85072 Eichstätt, Germany.*