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FIXED POINT THEOREMS FOR GENERALIZED SET-CONTRACTION MAPPINGS AND THEIR APPLICATIONS

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Abstract. In this paper, we introduce a generalized set-valued contraction on topological spaces with respect to a measure of noncompactness. Then, in the setting of metric spaces we give two fixed point theorems for the KKM type set valued mappings which either are generalized μ-set contraction or condensing. Also, some applications in fixed point, coincidence point and maximal element theory are given.

1. Introduction and preliminaries

In 1930 Kuratowski [16] introduced a measure of noncompactness α of bounded sets in a metric space, in order to generalize the Cantor intersection theorem. Let (M, d) be a metric space and $B \subseteq M$ be a bounded set, then

$$\alpha(B) = \inf\{\delta > 0 : B \subseteq \cup_{i=1}^{n} A_i,\ \operatorname{diam}(A_i) < \delta\}.$$

This along with the associated notion of an α-set contraction, entailed useful results in the fixed point theory, and in the theory of differential and integral equations; see [1, 4, 11, 12, 15, 19, 20].

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Definition 1.1. Let X be a nonempty subset of M and $f : X \to X$, then

- f is said to be a α-set contraction if f is bounded and there is a $k \in [0, 1)$ such that $\alpha(fB) \leq k\alpha(B)$ for all bounded subsets B of X.

- f is said to be condensing if f is bounded and if $\alpha(fB) < \alpha(B)$ for all bounded subsets B of X for which $\alpha(B) > 0$.

Darbo [7] showed that if X is a closed, bounded and convex subset of a Banach space and $f : X \to X$ is a continuous α-k-set contraction, then f has a fixed point. Later, Sadovskii [20] introduced the notion of condensing map and by transfinite induction showed that if f is a continuous condensing map, then f has a fixed point. Notice that every α-k-set contraction is a condensing map, but the converse is not true; see [19]. Many authors extended the above results; see [13, 19] and references therein.
Now we recall some definitions and facts which will be used in this paper. Let X be a nonempty set, we denote by 2^X the family of all subsets of X and by $\langle X \rangle$ the family of all nonempty finite subsets of X. Let X and Y be topological spaces, a set-valued map $F : X \multimap Y$ is said to be :

(i) compact if $\overline{F(X)} \subseteq Y$ is compact;
(ii) closed if its graph, $Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ is a closed set in product space $X \times Y$,
(iii) upper semicontinuous, if for each closed set $B \subseteq Y$, $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is closed in X.

It is well-known that if Y is compact Hausdorff and $F(x)$ is closed for each $x \in X$, then F is upper semicontinuous if and only if F is closed.
Let (M, d) be a metric space and let A is a bounded subset of M. Let

$$co(A) = \bigcap\{B \subseteq M : B \text{ is a closed ball in } M \text{ such that } A \subset B\}.$$

If A is a subset of M, then A is called subadmissible, if for each $D \in \langle A \rangle, co(D) \subseteq A$.
An abstract convex space $(E, \mathcal{C})$ consist of a nonempty set E and a family $\mathcal{C}$ of subsets of E such that E and $\emptyset$ belong to $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection. This kind of convexity was widely studied; see [3] and references therein. For any $A \subseteq X$, a natural definition of the $\mathcal{C}$-hull is $co_{\mathcal{C}}(A) = \bigcap B \in \mathcal{C} : A \subseteq B$. We say that A is $\mathcal{C}$-convex (or in brief, convex) if A equals to its $\mathcal{C}$- convex hull. Let X be a nonempty subset of M and Y be a topological space. A set-valued map

$G : X \multimap Y$ is called a KKM mapping [2, 13], if for each $A \in \langle X \rangle$, $co(A) \cap X \subset G(A)$. More generally let $G : X \multimap Y$, $F : X \multimap Y$ be two set-valued maps. Assume that for each $A \in \langle X \rangle$,

$$F(co(A) \cap X) \subseteq G(A),$$

then G is called a *generalized KKM mapping with respect to F*. If the set-valued map $F : X \multimap Y$ satisfies the requirement that for any generalized KKM mapping $G : X \multimap Y$ with respect to F the family $\{\overline{G(x)} : x \in X\}$ has the finite intersection property, then F is said to have the KKM property. We define

$$KKM(X, Y) := \{F : X \multimap Y : F \text{ has the KKM property}\}.$$

When X is a convex subset of topological vector space E and Y is a topological space, the class $KKM(X, Y)$ was introduced and studied by Chang and Yen [5].
Chen [6] obtained a fixed point theorem for a KKM type, α-k-set contraction set-valued map $F : X \multimap X$, where X is a nonempty bounded nearly subadmissible subset of a complete metric space. Later Dhompongsa et al. [9] obtained a fixed point theorem for condensing KKM type maps by the same technique which Chen used. In [6], the author assumed that $\alpha(co(A)) = \alpha(A)$ for each subset A of X but one may easily come up with a compact set A in l_∞ such that $co(A)$ is not compact [14]. Then $\alpha(A) = 0$ but $\alpha(co(A)) > 0$. Furthermore, the following example shows the proof of Theorem 1 in [6] is not accurate.

Examples 1.2. Let $(\mathbb{R}, d)$ be the real line provided with the discreet metric space and $X = [-1, 1]$. Define $F : [-1, 1] \multimap [-1, 1]$ by $F(x) = \{0\}$ for each $x \in [-1, 1]$. Clearly $F \in KKM(X, X)$ is a α-$\frac{1}{2}$-contraction. Let $y = 1$, $X_0 = X$ and $X_{n+1} = co(F(X_n) \cup \{1\}) \cap X$. Then $X_n = [-1, 1]$ for each n and $\alpha(X_n) = 1 \not\to 0$ which contradicts the claimed in the proof of [7, Theorem 1].

In this paper we not only give correct versions of the main results of Chen [6, Theorem 1] and Dhompongsa et al [9], but also generalize their results for the new set-valued contractions.

2. Main results

In this section we introduce a generalized set contraction in topological spaces with respect to a measure of noncompactness. Then, we present fixed point theory for the KKM type set-valued map F which is either set contraction or condensing. In both cases the set-valued maps are not necessarily compact.

Let E be a topological space. A measure of noncompactness is simply any functional $\mu : 2^E \to [0, \infty]$ such that:

(i) $\mu(\overline{A}) = \mu(A)$ for all $A \in 2^E$;
(ii) $\mu(A) = 0$ if and only if A is precompact;
(iii) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$.

A sequence $\{A_n\}_{n=1}^{\infty}$ of nonempty closed subsets of E is called μ- descending if $A_{n+1} \subseteq A_n$ for each n and $\lim_{n\to\infty} \mu(A_n) = 0$. We say that μ has the Kuratowski property, if the intersection $A = \bigcap_{n\in\mathbb{N}} A_n$ is nonempty and compact for any mu-descending sequence $\{A_n\}_{n=1}^{\infty}$. Notice that, if E is a complete metric space, then the Hausdorff measure of non-compactness and the Kuratowski measure of noncompactness have the Kuratowski property [1].

Examples 2.1. Let E be a Banach space. Let $\mathcal{B}(E)$ and $\mathcal{W}(E)$ denote the families of the bounded subsets of E and of the weakly compact subsets of E. The *weak non-compactness measure* of $B \in B(E)$ [8] is defined by

$$\beta(B) = \inf\{\epsilon > 0 : \exists\ A \in \mathcal{WC}(E), B \subseteq A + \epsilon B_1(0)\}.$$

The weak non-compactness measure is a measure of noncompactness with the Kuratowski property [8, 17].

For some other examples of measure of non-compactness with the Kuratowski property one can refer to [1].

Definition 2.2. Let E be a topological space, μ a measure of noncompactness on E and $F : E \multimap E$ a set-valued map. Then,
F is said to be

- μ-k set contraction, if $0 < k < 1$ and $\mu(F(A)) \leq k\mu(A)$ for all $A \in 2^E$;

- generalized μ-set contraction if, for each $\epsilon > 0$, there exists $\delta > 0$ such that for $A \subseteq E$ with $\epsilon \leq \mu(A) < \epsilon + \delta$, there exists $n \in \mathbb{N}$ such that $\mu(F^n(A)) < \epsilon$.

Now, we obtain the relationships between the above notions. In the following we assume μ is a measure of noncompactness on E and $F : E \multimap E$ is a set-valued map.

Proposition 2.3. *If F is a μ-k set contraction on E, then F is a generalized μ-set contraction.*

Proof. Assume that $\epsilon > 0$ is arbitrary and $0 < \delta \leq \frac{(1-k)\epsilon}{k}$. If $A \subseteq E$ and $\epsilon \leq \mu(A) < \epsilon + \delta$, then

$$\mu(F(A)) \leq k\mu(A) < k(\epsilon + \delta) \leq \epsilon.$$

□

The following provides an example of a generalized μ-set contraction which is not a μ-k set contraction.

Examples 2.4. Suppose that $M = \{1\} \bigcup \{2n, 3n : n \in \mathbb{N}\}$ and d is discrete metric. Assume that $F : M \multimap M$ is defined as follows:

$$F(x) = \begin{cases} \{1\} & \text{if } x = 1, \\ \{3(2n+1)\} & \text{if } x = 2n, \\ \{1\} & \text{if } x = 3n \text{ and } n \text{ is odd.} \end{cases}$$

If α is the Kuratowski measure of noncompactness, then $\alpha(F^2(M)) = 0$. Therefore, F is a generalized α-set contraction. But if $A = \{2n : n \in \mathbb{N}\}$, then $\alpha(A) = \alpha(F(A)) = 1$ and so F is not μ-k-set contraction.

The following lemmas are essential in proving our main result.

Lemma 2.5. *Let E be a topological space, and μ a measure of noncompactness on E. Suppose that F is a generalized μ-set contraction on E. Then for every subset A of E which $F(A) \subseteq A$ and $\mu(A) < \infty$, we have*

$$\lim \mu(F^n(A)) = 0.$$

Proof. Since $F(A) \subseteq A$, so $F^{n+1}(A) \subseteq F^n(A)$ for each $n \in \mathbb{N}$. Thus $\mu(F^{n+1}(A)) \leq \mu(F^n(A))$, which implies that $\{\mu(F^n(A))\}_n$ is a decreasing sequence of non-negative real numbers and therefore tends to a limit $r \geq 0$. Now we show that $r = 0$. On the contrary, assume that $r > 0$. Since F is a generalized μ-set contraction, then by definition, there exists $\delta > 0$ such that for $B \subseteq X$ with $r \leq \mu(B) < r + \delta$, there exists $n \in \mathbb{N}$ such that $\mu(F^n(B)) < r$. But there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$, $r \leq \mu(F^j(A) < r + \delta$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\mu(F^{n_0+j_0}(A)) < r,$$

which is a contradiction and so $r = 0$. □

Remark 2.6. It is easy to see that if $\lim \mu(F^n(A)) = 0$ for any subset A of E, then F is a generalized μ-set contraction on E.

Lemma 2.7. *Let μ be a measure of noncompactness on E with the Kuratowski property. Let X be a nonempty subset of E with $\mu(X) < \infty$. Suppose that $F : X \multimap X$ is a generalized μ-set contraction with nonempty compact values and $\overline{F(X)} \subseteq X$. Then there exists a precompact subset K of X with $\overline{K} \subseteq X$ such that $F(K) \subseteq K$.*

Proof. Let $x_0 \in X$, $X_0 = X$ and $X_{n+1} = F(X_n) \cup \{x_0\}$ for all $n \in \mathbb{N} \cup \{0\}$. Clearly $X_{n+1} \subseteq X_n$. If $K = \bigcap_{n=0}^{\infty} X_n$, then since $x_0 \in K$, K is nonempty and $F(K) \subseteq K$. Furthermore, we have

$$\mu(X_1) = \mu(F(X_0))$$
$$\mu(X_2) = \mu(F(X_1)) = \mu(F^2(X_0) \cup F(x_0))$$
$$= \max\{\mu(F^2(X_0)), \mu(F(x_0))\} = \mu(F^2(X_0))$$
$$\vdots$$
$$\mu(X_n) = \mu(F^n(X)).$$

Therefore, by Lemma **??**, $\mu(\overline{X_n}) = \mu(F^n(X_0)) \to 0$. Since μ has the Kuratowski property, then $\bigcap_{n=0}^{\infty} \overline{X_n}$ is compact. Moreover, since $\overline{F(X)} \subseteq X$, then $\bigcap_{n=0}^{\infty} \overline{X_n} \subseteq X$. Hence, K is a precompact subset of X and $\overline{K} \subseteq X$. □

A slight modification of the proof of the Lemma 2.7 yields the following.

Lemma 2.8. *Let $(E, \mathcal{C})$ be an abstract convex space and μ be a measure of noncompactness on E with the Kuratowski property. Assume that $\mu(co_{\mathcal{C}}(A)) = \mu(A)$ for each $A \subseteq E$. Let X be nonempty $\mathcal{C}$-convex subset of E and $F : X \multimap X$ be a generalized μ-set contraction, with nonempty compact values. Suppose that $F(co_{\mathcal{C}}(A)) \subseteq co_{\mathcal{C}}(F(A))$ for any subset A of X and $\overline{F(X)} \subseteq X$. Then, there exists a precompact and $\mathcal{C}$-convex subset K of X such that $F(K) \subseteq K$.*

The following result generalizes Theorem 1 in [6].

Theorem 2.9. *Let (M, d) be a metric space, μ a measure of noncompactness on M with the Kuratowski property and X be a nonempty subset of M with $\mu(X) < \infty$. Suppose that $F \in KKM(X, X)$ is a closed generalized μ-set contraction, with nonempty compact values and $\overline{F(X)} \subseteq X$. Then F has a fixed point.*

Proof. According to Lemma 2.7, there exists a nonempty precompact subset K of X such that $F(K) \subseteq K$. Let $Y = \overline{F(K)}$. Then, for each $\epsilon > 0$, there exists a finite subset A of K such that $Y \subseteq \bigcup_{x \in A} N(x, \epsilon)$. Now define a mapping $G : K \multimap Y$ by $G(x) = Y \setminus N(x, \epsilon)$, for all $x \in K$, then $G(x)$ is closed for each $x \in K$ and $\bigcap_{x \in A} G(x) = \emptyset$. Since $Y = \overline{F(K)} \subseteq \overline{K} \subseteq X$ and $F \in KKM(X, X)$ then $F_| K \in KKM(K, Y)$. Thus G is not a generalized KKM mapping with respect to $F_| K$. Therefore, there exists a finite subset $B = \{x_0, x_1, ..., x_m\}$ of K such that $F(co(B) \cap K) \nsubseteq \bigcup_{i=0}^{m} G(x_i)$. So there exists $x_\epsilon \in F(co(B) \cap K)$ such that $x_\epsilon \notin \bigcup_{i=0}^{m} G(x_i)$. From the definition G it follows that $x_\epsilon \in N(x_i, \epsilon)$ for all $i \in \{0, 1, ..., m\}$. Hence, $x_i \in N(x_\epsilon, \epsilon)$ for all $i \in \{0, 1, ..., m\}$. If $x_\epsilon \in F(x'_\epsilon)$ for some $x'_\epsilon \in co(B) \cap K$, then

$x'_\epsilon \in B(x_\epsilon, \epsilon) \cap K$, which implies that $x_\epsilon \in F(x'_\epsilon) \cap B(x'_\epsilon, \epsilon')$. Now since Y is a compact subset of X, we may assume that x_ϵ converges to some $x \in Y$ as $\epsilon \to 0$. Consequently, x'_ϵ also converges to x as $\epsilon \to 0$. Since F is closed, then $x \in F(x)$. □

Now we give an application to fixed point theory of contractive mappings. Recall that a mapping $f : X \to X$, where X is subset of a Banach space $(E, \|\|)$, is called contractive if, $\|f(x) - f(y)\| < \|x - y\|$ for each $x \neq y \in X$.

Corollary 2.10. *Let X be a weakly closed bounded subset of a Banach space E and β be the weak measure of noncompactness on E. Suppose that $f : X \to X$ is a cotractive generalized β-set contraction and weakly continuous. Then f has a fixed point.*

Proof. The existence of a nonempty relatively weakly compact K which is f-invariant is insured by Lemma 2.7. Since f is a weakly continuous mapping the $f(\overline{K}^w) \subseteq \overline{f(K)}^w$. Therefore, $f : \overline{K}^w \to X$ is a weakly compact and weakly sequentially closed map. Hence, the Corollary 2.3 of [18] completes the proof. □

Now by a standard technique, we will obtain the above result for μ-condensing mappings.

Lemma 2.11. *Let E be a topological space, μ be a measure of noncompactness on E and X be a nonempty closed subset of E with $\mu(X) < \infty$. Let $F : X \multimap X$ be a μ-condensing set-valued map. Then there exists a compact subset K of X such that $F(K) \subseteq K$.*

Proof. Fix $x \in X$ and let Σ denotes the family of all closed subset D of X for which $x \in D$ and $F(D) \subseteq D$. Σ is nonempty since $X \in \Sigma$. Now set

$$B = \bigcap_{D \in \Sigma} D,$$

and let

$$K = \overline{F(B) \bigcup \{x\}}.$$

Since $x \in B$ and $F(B) \subseteq B$, then $K \subseteq B$ and so $F(K) \subseteq F(B) \subseteq K$. Therefore, $B = K$. We now have $F(B) = F(K) \subseteq K$ and

$$\mu(K) = \mu(\overline{F(B) \bigcup \{x\}}) = \mu(F(B) \bigcup \{x\}) = \max\{\mu(F(B)), \mu\{x\}\}$$
$$= \max\{\mu(F(B)), 0\} = \mu(F(B)) = \mu(F(K)).$$

Since F is μ-condensing then $\mu(K) = 0$. Therefore K is compact. □

The proof of the following theorem, which generalize the main result of [9], is similar to the proof of Theorem **??**.

Theorem 2.12. *Let (M, d) be a metric space, μ be a measure of non-compactness on M and X be a nonempty closed subset of M with $\mu(X) < \infty$. If $F \in KKM(X, X)$ is a closed μ-condensing set-valued map, then F has a fixed point.*

3. SOME APPLICATIONS

In this section by applying Theorem 2.9, we obtain a coincidence theorem, a Fan-Browder type fixed point theorem and an intersection theorem for generalized set contraction mappings. In the first step we obtain a coincidence theorem.

Theorem 3.1. *Let μ be a measure of non-compactness with the Kuratowski property on the metric space M and X be a nonempty subadmissible subset of M. Suppose that $F, G : X \multimap X$ are two set-valued mappings satisfying the following conditions:*

1. *$F \in KKM(X, X)$;*
2. *G has nonempty subadmissible values and for every compact subset C of X and any $y \in X$, $G^{-}(y) \cap C$ is open in C;*
3. *F is a generalized μ-set contraction map with compact values and $\overline{F(X)} \subseteq X$.*

Then there exists $x_0, y_0 \in X$ such that $y_0 \in F(x_0)$ and $x_0 \in G(y_0)$.

Proof. By Lemma 2.7, there is a compact subset K of X such that $F(K) \subseteq K$. Since $G(x) \neq \emptyset$, then $X = \cup_{x \in X} G^{-}(x)$ and so $K = \bigcup_{x \in K} G^{-}(x) \cap K$. But K is compact thus there exist $x_1, ..., x_n \in X$ such that $\overline{K} = \bigcup_{i=1}^{n} G^{-}(x) \cap \overline{K}$. If $T : X \multimap \overline{K}$ is defined by $T(x) = K \setminus (G^{-}(x) \cap K)$ for each $x \in X$, then $\cap_{i=1}^{n} T(x_i) = \emptyset$. Therefore, T is not a generalized KKM mapping with respect to F. Hence, there exists a finite subset $A = \{a_1, ..., a_m\}$ of X such that $F(co(A) \cap K) \nsubseteq T(A)$. Therefore, there exist $x_0 \in co(A) \cap K$ and $y_0 \in F(x_0)$ such that $y_0 \notin T(A)$. Thus, $y_0 \in G^{-}(a_i) \cap K$ for $i = 1, ..., m$ and so $a_i \in G(y_0)$ for $i = 1, ..., m$. Since $G(y_0)$ is subadmissible, then $co(A) \subseteq G(y_0)$ and so $x_0 \in G(y_0)$. □

Remark 3.2.
- (a) In the above theorem, instead of (3), we can assume the following condition: $[(3)']$ G is a generalized μ-set contraction with compact values and $\overline{G(X)} \subseteq X$.
- (b) If the identity mapping $I \in KKM(X, X)$ and condition $(3)'$ is satisfied, then we have a Fan-Browder fixed point theorem for G.

As an application of Lemma 2.8 we have the following maximal element theorem in Banach spaces.

Theorem 3.3. *Let X be a nonempty closed, convex and bounded subset of a Banach space E. Suppose that α is the Hausdroff or the Kuratowski measure of noncompactness on E. Assume that $T : X \multimap X$ is a set-valued mapping such that*

(i) *T is a generalized α-set contraction map with compact values;*
(ii) *$T(tx + (1-t)y) \subseteq tT(x) + (1-t)T(y)$ for all $x, y \in X$ and $t \in [0,1]$;*
(iii) *for every compact subset C of X and any $y \in X$, $T^{-}(y) \cap C$ is open in C;*
(iv) *$x \notin conv(T(x))$ for any $x \in X$.*

Then there exists $\bar{x} \in X$ such that $T(\bar{x}) = \emptyset$.

Proof. By Lemma [2.8] there exists a nonempty, convex and precompact subset K of X such that $T(K) \subseteq K$. Suppose that $T(x) \neq \emptyset$ for all $x \in X$. Since $I \in KKM(X, X)$, then by the same proof as that of the above theorem there exists $\overline{x} \in X$ such that $\overline{x} \in conv(T(\overline{x}))$. This is a contraction with condition (iv). □

In the following we give another application of Theorem 2.9.

Theorem 3.4. *Let μ be a measure of non-compactness with the Kuratowski property on the metric space M and X be a nonempty subset of M with $\mu(X) < \infty$. Assume that $F \in KKM(X, X)$ is a generalized μ-set contraction map with compact values and $\overline{F(X)} \subseteq X$. Suppose that $G : X \multimap X$ is a set-valued mapping with closed values and G is a generalized KKM mapping with respect to F. Then, there exists a precompact subset K of X such that*

$$\overline{F(K)} \cap (\bigcap_{x \in K} G(x)) \neq \emptyset.$$

Proof. By Lemma 2.7, there is a precompact subset K of X such that $F(K) \subseteq K$. Now, if $T : K \multimap X$ is defined as the following:

$$T(x) = \overline{F(K)} \cap G(x), \ \forall\ x \in K.$$

Then T is a generalized KKM mapping with respect to F and $T(x)$ is compact for any $x \in K$. Therefore,

$$\overline{F(K)} \cap (\bigcap_{x \in K} G(x)) \neq \emptyset.$$

□

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