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3 authors, including:

M. Fakhar
University of Isfahan

Jafar Zafarani
Sheikhbahaee University and Uinversity of I…

FIXED POINT THEOREMS FOR THE CLASS S-KKM MAPPINGS IN ABSTRACT CONVEX SPACES

A. AMINI, M. FAKHAR AND J. ZAFARANI

Department of Mathematics, University of Isfahan,
Isfahan 81745-163 Iran

Abstract

We define KKM mappings and S-KKM mappings similar to the case of convex spaces for abstract convex spaces. Some approximate fixed point theorems will be established for the multifunction with S-KKM property on the Φ spaces. We also obtain a new version of Sadovskii's fixed point theorem in topological spaces.

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Keywords : KKM property, abstract convex spaces, Φ-spaces, approximate fixed point, metric spaces, condensing mappings

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1. Introduction

In 1929 Kanster-Kuratowski-Mazurkiewicz [18] established the celebrated KKM theorem, there are many extensions and many applications of this theorem. The most important result for KKM mapping is the famous Fan-KKM theorem [9], which has been used as a very versatile tool in modern nonlinear analysis and from which many far-reaching extensions have been made. Chang and Yen [6] made a systematic study of class KKM mappings. Motivated by their work, Chang et al. [7] introduced the family of multifunctions with S-KKM property. As shown in [7], KKM mappings are contained in S-KKM mappings and generally this inclusion is proper.

In this paper we shall introduce the class S-KKM mappings for abstract convex spaces [cf. 4, 13, 19 ,22], a class of convexity which contains all different concepts of convexity. We obtain also some fixed point theorems for the multifunctions with S-KKM property on the Φ-spaces in the sense of Ben-El-Mechaiekh et al. [3, 4], Horvath [11], and Kim and Park [16]. Furthermore, we establish a new version of Sadovskii's fixed point theorem similar to that of Khamsi [15] in abstract convex spaces. For the remainder of this section we introduce the notations used in this paper and recall some basic facts. Let X be nonempty sets, we shall denote by 2^X the family of all subsets of X, by $\langle X\rangle$ the family of all nonempty finite subsets of

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X. Suppose that Y is a nonempty set and $F : X \multimap Y$ is a multifunction with nonempty values, fibers $F^{-}(y)$ for $y \in Y$ defined by $F^{-}(y) = \{x \in X : y \in F(x)\}$. For topological spaces X and Y, a multifunction $F : X \multimap Y$ is said to be:
(i) compact if $clF(X)$ is compact in Y.
(ii) closed if its graph , $Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ is a closed set.
(iii) F is called upper semi-continuous (u.s.c.), if for each closed set $B \subset Y$, $F^{-}(B) = \{x \in X : F(x) \cap B \neq \varnothing\}$ is closed in X.
A nonempty topological space is acyclic if all of its reduce homology groups over rational vanishes.

2. Abstract convex spaces and fixed point theorems

In this section we define KKM mappings and S-KKM mappings similar to the case of convex spaces for abstract convex spaces. Some approximate fixed point theorems will be established for the multifunctions with S-KKM property on Φ-spaces.

Definition 2.1. An abstract convex space $(X, \mathcal{C})$ consist a nonempty set X and a family $\mathcal{C}$ of subsets of X such that X and $\emptyset$ belong to $\mathcal{C}$ and $\mathcal{C}$ closed under arbitrary intersection. This kind of convexity was widely studied; see [4, 13, 19, 22]. For any $A \subseteq X$, a natural definition of the $\mathcal{C}$-hull is $co_{\mathcal{C}}(A) = \bigcap\{B \in \mathcal{C} : A \subseteq B\}$. We say that A is $\mathcal{C}$-convex (or in brief, convex) if A equals to its $\mathcal{C}$- convex hull.

The following are main examples of abstract convex spaces.

Examples 2.2.(a) Let (M, d) be a bounded metric space, $N(x, r) = \{y \in M : d(x, y) < r\}$, $B(x, r) = \{y \in M : d(x, y) \leq r\}$ and A be a subset of M. Then:
(i) $co(A) = \bigcap\{B \subseteq M : B$ is a closed ball in M such that$A \subset B\}$
(ii) $\mathcal{A}(M) = \{A \subseteq M : A = co(A)\}$ i.e. $A \in \mathcal{A}(M)$ if and only if A is an intersection of closed balls containing A. In this case, we say that A is admissible subset of M.
(iii) A is called subadmissible, if for each $D \in \langle A \rangle$, $co(D) \subseteq A$. Obviously , if A is an admissible subset of M, then A must be subadmissible. One can easily show that $\mathcal{A}(M) = \mathcal{C}$ is an abstract convexity structure for M. Hence, $(M, \mathcal{A}(M))$ is an abstract convex space.
(b) There are some more examples of abstract convex spaces namely K-convex structure [19], hyperconvex spaces [14], C-spaces [11, 12], L-spaces [4], G-convex spaces [20] and mc-spaces [19]. For further information about these structures and spaces, one can refer to [19] and references therein.

Remark 2.3. *The distinction between abstract convex spaces with other kinds of convexity which are mainly mentioned in the part (b) of the above example is that we do not consider the existence of a continuous function from a simplexe to $co_{\mathcal{C}}(A)$ for each $A \in \langle M \rangle$.*

Motivated by the work of Chang and Yen [6], we define in a similar way the class of multifunctions with KKM property. Let $(X, \mathcal{C})$ be an abstract convex space and Y a topological space. If $F : X \multimap Y$ and $G : X \multimap Y$ are two multifunctions such that for any $A \in \langle X \rangle$ $F(co_{\mathcal{C}}(A)) \subseteq \bigcup_{x \in A} G(x)$, then G is said to be a $\mathcal{C}$-KKM mapping with respect to F. A multifunction $F : X \multimap Y$ is said to have the KKM property with respect to $\mathcal{C}$ if for any $\mathcal{C}$-KKM map $G : X \multimap Y$ with respect to F, the family $\{clG(x) : x \in X\}$ has the finite intersection property. We let $\text{KKM}_{\mathcal{C}}(X, Y) := \{F : X \multimap Y : F$ has the KKM property with respect to $\mathcal{C}\}$. Similar to the work of Chang et al. [7], we introduce the family of multifunctions with S-KKM property as follows. Let Z be nonempty set, $(X, \mathcal{C})$ an abstract convex space, and Y a topological space. If $S : Z \multimap X$, $F : X \multimap Y$ and $G : X \multimap Y$ are three mappings satisfying :

$$F(co_{\mathcal{C}}(S(A))) \subseteq \bigcup_{x \in A} G(x)$$

for each $A \in \langle Z \rangle$, then G is called a $\mathcal{C}$-S-KKM mapping with respect to F. If the multifunction $F : X \multimap Y$ satisfies the requirement that for any $\mathcal{C}$-S-KKM mapping G with respect to F, the family $\{clG(x) : x \in X\}$ has the finite intersection property, then F is said to have the S-KKM property with respect to $\mathcal{C}$. We define

$$S\text{-KKM}_{\mathcal{C}}(Z, X, Y) := \{F : X \multimap Y : F \text{ has the } S\text{-KKM property}\}.$$

One can show that, when S is the identity mapping I, then $S\text{-KKM}_{\mathcal{C}}(X, X, Y) = \text{KKM}_{\mathcal{C}}(X, Y)$. Moreover, $\text{KKM}_{\mathcal{C}}(X, Y)$ is contained in $S\text{-KKM}_{\mathcal{C}}(Z, X, Y)$ for any $S : Z \multimap X$ and generally this inclusion is proper; see [7].

In order to establish the main result of this paper for the mappings with the S-KKM property, we define the Φ-maps and the Φ-spaces.

Definition 2.4. (a). Let $(X, \mathcal{C})$ be an abstract convex space and Y a topological space. A map $T : Y \multimap X$ is called a Φ-map if there exists a map $G : Y \multimap X$ such that

(i) for each $y \in Y$, $A \in \langle (G(y) \rangle$ implies $co_{\mathcal{C}}(A) \subseteq T(y)$; and
(ii) $Y = \bigcup\{IntG^{-}(x) : x \in X\}$.

(b) An abstract convex space $(X, \mathcal{C})$ is called a Φ-space if X is a uniform space and for each entourage V there is a Φ-map $T : X \multimap X$ such that $Gr(T) \subseteq V$.

The concepts of Φ-maps and Φ-spaces are originated from Ben-El-Mechaiekh et al. [3], Horvath [11] and motivated by the works of Fan and Browder [5]. This notions also have been studied by Ben El- Mechaiekh et al. [4], and more recently by Park [20] and Kim and Park [16]. Let $(X, \mathcal{C})$ be a Φ-space and $F : X \multimap X$. We say that F has the approximate fixed point property if for any $U \in \mathcal{U}$ where $\mathcal{U}$ is a basis of the uniform structure of X, there exists $x \in X$ such that $U[x] \bigcap F(x) \neq \varnothing$.

Theorem 2.5. *Let $(X, \mathcal{C})$ be a Φ-space and $S : X \to X$ a surjective function. Suppose that $F \in S$-$KKM_{\mathcal{C}}(X, X, X)$ is compact, then F has the approximate fixed point property.*

Proof. Let $\mathcal{U}$ be a basis of the uniform structure of X. Suppose that $U \in \mathcal{U}$, then there is a Φ-map $T : X \multimap X$ such that $Gr(T) \subseteq U$. Since T is a Φ-map, then there exists a map $G : X \multimap X$ such that $X = \bigcup_{x \in X} IntG^{-}(x)$. If $K = clF(X)$, then K is compact and so there is a finite subset A of X such that

$$K \subseteq \bigcup_{x \in A} IntG^{-}(S(x)). \qquad (1)$$

Now, define $R(x) = K \setminus IntG^{-}(S(x))$ for any $x \in X$. From (1) we deduce that R is not a generalized $\mathcal{C}$-S-KKM mapping with respect to F. Hence, there exists $B = \{x_1, ..., x_m\} \subseteq X$ such that $F(co_{\mathcal{C}}(S(B)) \nsubseteq \bigcup_{i=1}^{m} R(x_i)$. Thus, there are a point $x \in co_{\mathcal{C}}(S(B))$ and a point $y \in F(x)$ such that $y \notin \bigcup_{i=1}^{m} R(x_i)$. Consequently, $y \in \bigcap_{i=1}^{m} IntG^{-}(S(x_i))$ and so $S(x_i) \in G(y)$ for $i = 1, ..., m$. Therefore, $co_{\mathcal{C}}(S(B)) \subseteq T(y)$ and since $x \in co_{\mathcal{C}}(S(B))$, then $x \in T(y)$, i.e., $(y, x) \in Gr(T)$. But $Gr(T) \subseteq U$, then $y \in U[x] \bigcap F(x)$. □

By the above theorem we obtain the following fixed point theorem.

Corollary 2.6. *Suppose that all of the assumptions of the above theorem hold and F is closed, then F has a fixed point.*

Remark 2.7. **(a)** *As G-convex spaces are abstract convex spaces, and by Lemma 2.5 of [8] any better admissible mapping which is upper semicontinuous, compact and closed valued has the KKM property, hence the above corollary refines the main results of [16, Theorem 4.2] and [20, Theorem 3.3] in our context.*
(b) *Horvath [12] found that hyperconvex spaces are a particular type of C-spaces, hence they are G-convex spaces. In [8, Lemma 2.7] it has been shown that those multifunctions defined on G-convex spaces which are closed, compact and acyclic valued have the KKM property. Hence, the above corollary improves Theorems 2.1 and 2.2 of Wu et al. [23].*
(c) *By the Lemma 2.6 of [8] and its remark when $(X, \mathcal{U}, \Gamma)$ is a uniform L-space, the class of u.s.c., closed values, compact approachable mappings is contained in $KKM(X, X)$. Hence, the above theorem and its corollary improve Theorem 4.1 of [4].*

By a similar proof as that it was given by Chang et al. [7 , Proposition 2.3(ii)], we can obtain the following lemma.

Lemma 2.8. *Let $(X, \mathcal{C})$ be an abstract convex space, Z a nonempty set and Y, W are two topological spaces. Suppose that $S : Z \multimap X$, $T \in S\text{-}KKM_{\mathcal{C}}(Z, X, Y)$ and $f : Y \to W$ is continuous, then $fT \in S\text{-}KKM_{\mathcal{C}}(Z, X, W)$.*

As a consequence of Corollary 2.6 and Lemma 2.8, we obtain a Schauder type fixed point theorem for abstract convex spaces.

Corollary 2.9. *Let $(X, \mathcal{C})$ be a Hausdorff Φ-space. Suppose that the identity mapping $I : X \to X$ belongs to $KKM_{\mathcal{C}}(X, X)$, then any continuous mapping $f : X \to X$ such that $clf(X)$ is compact, has a fixed point.*

Examples 2.10. **(a)** By Fan's theorem [9], the identity mapping on a convex subset X of a topological vector space is an elements of $KKM(X, X)$. Horvath in [11] has established that in C-spaces, LC-spaces and LC-metric spaces M, $I \in KKM(M, M)$. Khamsi [14] has shown that in hyperconvex spaces identity map has the KKM property. Park [20] has shown that when $(X, D; \Gamma)$ is G-convex space, then $I \in$ KKM(X, X). Similar result has been obtained By Ben-El-Mechaiekh et al. in [4] for L-spaces.

(b) Let E be a metric topological vector space in which all balls of the metric are convex. As the identity mapping belongs to $KKM(X,X)$ for each convex subset X of E. Hence the identity mapping also belongs to $KKM(X,X)$ with respect to metric of E.

Now, we obtain the following theorem for existence of approximate fixed point for a wide class of uniform topological spaces. This result improves Corollary 4.3 of Ben-El-Mechaiekh et al. [4].

Theorem 2.11. *Let $(X,\mathcal{C})$ be an abstract convex space supply with uniform space with basis $\mathcal{U}$. Assume that for each $U\in\mathcal{U}$, there exists $V\in\mathcal{U}$, $V\subseteq U$ such that for each $x\in X$ and each $A\in\langle V[x]\rangle$, $co_{\mathcal{C}}(A)\subseteq U[x]$. Suppose that $S:X\to X$ is a surjective function and $F\in S\text{-}KKM_{\mathcal{C}}(X,X,X)$ such that $clF(X)$ is a totally bounded, then F has an approximate fixed point.*

Proof. For each $U\in\mathcal{U}$, it is enough to set $T(x)=U[x]$ and $G(x)=V[x]$ for each $x\in X$. Hence, $(X,\mathcal{C},\mathcal{U})$ becomes a Φ-space. Now by a similar proof as that of Theorem 2.5 one can obtain the result. □

Remark 2.12. **(a)** *If X is a convex subset of a topological vector space and V is a symmetric convex open neighborhood of 0. Then for V, we can define a Φ-mapping T as $T(x)=G(x)=\{y\in X: x-y\in V\}$. Hence, $Gr(T)\subseteq V$, therefore Theorem 2.11 implies that any $F\in KKM(X,X)$ such that $clF(X)$ is totally bounded has an approximate fixed point with respect to V. Hence, we reobtain Theorem 2 of [21].*

(b) *Following Ben-El-Mechaiekh et al. [4], Park [20] and Yuan [24], we call a uniform abstract convex space $(X,\mathcal{C})$ is a locally abstract space if X is a Hausdorff uniform space with the basis $\mathcal{U}$ of symmetric entourages such that for each $U\in\mathcal{U}$, and each $x\in X$, the set $U[x]=\{y\in X:(x,y)\in U\}$ is $\mathcal{C}$-convex set. In this case Theorem 2.11 implies that, if $F\in KKM_{\mathcal{C}}(X,X)$ such that $clF(X)$ is a totally bounded, then F has the approximate fixed point property.*
As a consequence of the above theorem we deduce Theorem 2.1 of Amini et al. [1].

Corollary 2.13. *Let (M,d) be a metric space X, a nonempty subadmissible subset of M and $S:X\to X$ a surjective function. Suppose that $F\in S\text{-}KKM_{\mathcal{C}}(X,X,X)$ such that $clF(X)$ is totally bounded, then F has the approximate fixed point property.*

Proof. For each $\lambda\in\mathbb{R}^+$, set $U_\lambda=\{(x,y)\in X\times X: d(x,y)\leq\lambda\}$, $V_\lambda=\{(x,y)\in X\times X: d(x,y)<\lambda\}$. Let $\mathcal{U}=\{U_\lambda,V_\lambda:\lambda\in\mathbb{R}^+\}$, then all of the conditions of Theorem 2.11 are fulfilled and the proof is complete. □

Remark 2.14. *Similarly to Corollary 2.6, in Theorem 2.11 and Corollary 2.13, when F is closed and compact, we can obtain a fixed point for the multifunction F.*

Recently Khamsi [15] obtained an abstract formulation of Sadovski's fixed point theorem for continuous functions, using convexity structure. Here we will obtain an analogous result for multifunctions which are u.s.c. and have the KKM property. Let X be a topological space and $\mathcal{C}$ a family of closed subsets of X such that $\emptyset$ and X belong to $\mathcal{C}$. We will say that
(1) $\mathcal{C}$ has the intersection property(IP) if and only if $\cap A_i \in \mathcal{C}$ provided $A_i \in \mathcal{C}$.
(2) $\mathcal{C}$ has the chain intersection property(CIP) if and only if $\cap A_i \in \mathcal{C}$ provided (A_i) is a decreasing chain of elements of $\mathcal{C}$.
Suppose that $\mathcal{C}$ has (IP) and $A \subseteq X$, by $\mathcal{C}(A)$ we mean $\{B \in \mathcal{C} : A \subseteq B\}$ and $\mathcal{C}$-hull of A as in Definition 2.1 will be denoted by $co_{\mathcal{C}}(A)$. If $\mathcal{C}$ has (CIP), then the subfamily $\mathcal{C}(A)$ satisfies the assumptions of Zorn's lemma. Therefore, $\mathcal{C}(A)$ has minimal element. We will still use the notation $co_{\mathcal{C}}(A)$ to designate such minimal element.

Examples 2.15. **(a)** Let (M, d) be a bounded hyperconvex metric space. Set

$$\mathcal{H} = \{H \subset M : H \neq \emptyset \text{ and is hyperconvex}\}.$$

By a result of Baillon [2], $\mathcal{H}$ satisfies CIP (but fails to satisfy IP, i.e. the intersectoin of two hyperconvex is not necessarily hyperconvex). Khamsi [15] has proved that $\alpha : 2^M \to [0, \infty)$, the Kuratowski measure of noncompactness defined by

$$\alpha(A) = \inf\{\varepsilon > 0; A \subset \bigcup_{i=1}^{n} A_i \ , A_i \subset M,\ diam(A_i) \leq \varepsilon\}$$

is a measure of noncompactness for $\mathcal{H}$.
(b) It is trivial that the family $\mathcal{A}(M)$ of admissible sets satisfies (IP).

Henceforth let $\mathcal{C}$ stand for a family of closed subsets of M with the (IP) or (CIP) such that $\emptyset$ and M belong to $\mathcal{C}$.

Definition 2.16. We will say that $\mathcal{C}$ satisfies the property (K)(for Kakutani) if and only if for each $C \in \mathcal{C}$ nonempty compact and any $F : C \multimap C$ which is u.s.c., nonempty closed values with KKM property with respect to $\mathcal{C}$ has a fixed point. In Corollary 2.13 and its Remark 2.14, we have shown that if M is metric, then the family $\mathcal{A}(M)$ of admissible subsets of M satisfies (K).

Motivated by the concept of c-measure of noncompactness of Hahn [10] for topological vector spaces, we define this notion in a similar way for a topological space X with respect to the family $\mathcal{C}$. Let γ be a cone in a vector space with partial ordering $\leq$ and $\mathcal{M}$ a collection of nonempty subsets of a topological space X with the property that for any $A \in \mathcal{M}$, the sets $co_{\mathcal{C}}(A)$, $\overline{A}$, $A \cup \{x\}$, $(x \in X)$, and every subset of A belong to $\mathcal{M}$. Let c be a real number with $c \geq 1$. A function $\Psi : \mathcal{M} \to \gamma$ is called a c-measure of noncompactness with respect to $\mathcal{C}$, provided that the following conditions hold for any $Z \in \mathcal{M}$:

(1) $\Psi(co_{\mathcal{C}}(Z)) \leq c\Psi(Z)$;
(2) if $x \in X$, then $\Psi(Z \cup \{x\}) = \Psi(Z)$;
(3) if $Z_1 \subset Z$, then $\Psi(Z_1) \leq \Psi(Z)$;
(4) $\Psi(\overline{Z}) = \Psi(Z)$.

If $F : X \to \mathcal{M}$, then F is called Ψ-pseudocondensing mapping if, whenever $\Psi(Z) \leq c\Psi(F(Z))$ for $Z \in \mathcal{M}$, then Z is relatively compact.
In particular, if $c = 1$, then F is called Ψ-condensing.

Theorem 2.17. *Let X be a Hausdorff topological space and the family $\mathcal{C}$ has the property (K). Then for any nonempty $C \in \mathcal{C} \cap \mathcal{M}$, any u.s.c. $F : C \multimap C$ which is Ψ-pseudocondensing mapping, nonempty closed values and $F \in KKM_{\mathcal{C}}(C, C)$ has a fixed point.*

Proof. The proof is similar to that given in the proof of Theorem 1 of Khamsi [15]. First let us give the proof of this theorem when $\mathcal{C}$ satisfies IP. Let $x \in C$ and define

$$\mathcal{C}(x, F) := \{D \in \mathcal{C}; x \in D, F(D) \subseteq D\}.$$

Let $C(x) := \bigcap_{D \in \mathcal{C}(x,F)} D$, since $C \in \mathcal{C}(x, F)$, then $C(x)$ does exist. It is easy to see that $C(x)$ is F-invariant (i.e. $F(C(x)) \subset C(x)$) and is not empty since $x \in C(x)$. Let us show that $C(x)$ is compact. Indeed, we have $co_{\mathcal{C}}(F(C(x)) \cup x) \subset C(x)$. Hence we deduce

$$F(co_{\mathcal{C}}(F(C(x)) \cup x) \subset F(C(x)) \subset co_{\mathcal{C}}(F(C(x)) \cup x).$$

By minimality of $C(x)$, we deduce that $C(x) = co_{\mathcal{C}}(F(C(x)) \cup x)$, so

$$\Psi(co_{\mathcal{C}}(F(C(x)) \cup x)) \leq c\Psi(F(C(x)) \cup x)) = c\Psi(F(C(x)))$$

Therefore, we have $\Psi(C(x)) \leq c\Psi(F(C(x)))$. Since F is Ψ-psedocondensing and $C(x)$ is closed, we deduce that $C(x)$ is compact. Since $C(x)$ is closed the restriction of F to $C(x)$ is closed values and $\mathcal{C}$-KKM map. Using the property (K), we conclude that F has a fixed point.
When $\mathcal{C}$ satisfies the property CIP, we consider the family

$$\mathcal{C}(C(x), F) = \{D \in \mathcal{C}; F(C(x)) \cup x \subset D, \text{and } F(D) \subset D\}.$$

This family is not empty since $C(x) \in \mathcal{C}(C(x), F)$. Let $C^*(x)$ be a minimal of $\mathcal{C}(C(x), F)$. This element exists since $\mathcal{C}$ satisfies CIP. By minimality of $C(x)$, we conclude that $C(x) = C^*(x)$. In other words, $co_{\mathcal{C}}(F(C(x)) \cup x) = C(x)$. The end of the proof is similar to the case described above. □

As a corollary, we get the following result which improves the results of Khamsi [15] and Kirk and Shin [17].

Corollary 2.18. *Let H be a bounded hyperconvex metric space and $F : H \to \mathcal{H}$ a closed α-condensing such that $F \in KKM_{\mathcal{H}}(H, H)$. Then F has a fixed point.*

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