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Some Properties of the Class of Univalent Functions with Negative Coefficients

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ABSTRACT

The main object of this paper is to study some properties of certain subclass of analytic functions with negative coefficients defined by a linear operator in the open unit disc. These properties include the coefficient estimates, closure properties, distortion theorems and integral operators.

Keywords: Analytic Function; Unit Disc; Coefficient Inequality; Closure Properties; Distortion Bound

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in the open unit disc

$$U = \{z \in \mathbb{C} : |z| < 1\},$$

and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots.$$

Let $\mathcal{A}(n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, (n \in \mathbb{N} := \{1,2,3,\cdots\}), \quad (1)$$

which are analytic in the open unit disc. In particular,

$$\mathcal{A}(1) := \mathcal{A}.$$

For two functions $f(z)$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k, (n \in \mathbb{N}),$$

the Hadamard product (or convolution) $(f * g)(z)$ is defined, as usual, by

$$(f * g)(z) := z + \sum_{k=n+1}^{\infty} a_k b_k z^k := (g * f)(z).$$

Let the function $\varphi(a,b;z)$ be given by:

$$\varphi(a,b;z) = z + \sum_{k=n+1}^{\infty} \frac{(a)_{k-1}}{(b)_{k-1}} z^k, (b \neq 0,-1,-2,-3,\cdots),$$

where $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by:

$$(x)_k = \begin{cases} 1 & \text{for } k = 0, x \in \mathbb{C} - \{0\}, \\ x(x+1)\cdots(x+k-1) & \text{for } k \in \mathbb{N} = 1,2,3,\cdots. \end{cases}$$

Carlson and Shaffer [1] introduced a convolution operator on $\mathcal{A}$ involving an incomplete beta function as:

$$L(a,b) f(z) := \varphi(a,b;z) * f(z). \quad (2)$$

Our work here motivated by Catas [2], who introduced an operator on $\mathcal{A}$ as follows:

$$D_l^{m,\lambda} f(z) = z + \sum_{k=n+1}^{\infty} \left(\frac{1 + \lambda(k-1) + l}{1+l} \right)^m a_k z^k,$$

where

$$z \in \mathbb{U}, \lambda \geq 0, m \in \mathbb{Z}, l \geq 0.$$

Now, using the Hadamard product (or convolution), the authors (cf. [3,4]) introduced the following linear operator:

Definition 1.1 *Let*

$$\phi_l^{m,\lambda}(a,b;z) = \sum_{k=n+1}^{\infty} \left(\frac{1 + \lambda(k-1) + l}{1+l} \right)^m \frac{(a)_{k-1}}{(b)_{k-1}} z^k,$$

where

$$(z \in U, b \neq 0,-1,-2,-3,\cdots), \lambda \geq 0, m \in \mathbb{Z}, l \geq 0,$$

and $(x)_k$ is the Pochhammer symbol. *We defines a linear operator* $D_l^{m,\lambda}(a,b) : \mathcal{A}(n) \to \mathcal{A}(n)$ *by the following Hadamard product*:

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$$D_l^{m,\lambda}(a,b)f(z):=\phi_l^{m,\lambda}(a,b;z)*f(z)$$
$$=z+\sum_{k=n+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m\frac{(a)_{k-1}}{(b)_{k-1}}a_kz^k, \quad (3)$$

where

$$(z\in U, b\neq 0,-1,-2,-3,\cdots),\lambda\geq 0, m\in\mathbb{Z}, l\geq 0,$$

and $(x)_k$ the Pochhammer symbol .

Special cases of this operator include:

- $D_0^{m,0}(a,b)f(z)=D_l^{0,\lambda}(a,b)f(z)=L(a,b)f(z)$, see [1].
- the Catas drivative operator [2]: $D_l^{m,\lambda}(1,1)f(z)$.
- the Ruscheweyh derivative operator [5] in the cases:
$$D_0^{0,0}(\beta+1,1)f(z)=D^{\beta}f(z);\beta\geq -1.$$
- the Salagean derivative operator [6]: $D_0^{m,1}(1,1)f(z)$.
- the generalized Salagean derivative operator introduced by Al-Oboudi [7]: $D_0^{m,\lambda}(1,1)f(z)$.
- Note that:
$$D_0^{0,\lambda}(a,b)f(z)=L(a,b)f(z),$$
$$D_l^{1,\lambda}(a,b)f(z)=\left(\frac{1-\lambda+l}{1+l}\right)L(a,b)f(z)$$
$$+\frac{\lambda z}{1+l}\left(L(a,b)f(z)\right)=D_\lambda\left(L(a,b)f(z)\right),\lambda\geq 0.$$

Let $\mathcal{T}(n)$ denote the class of functions $f(z)$ of the form

$$f(z)=z=\sum_{k=n+1}^{\infty}a_kz^k, a_k\geq 0, n\in\mathbb{N}, \quad (4)$$

which are analytic in the open unit disc.

Following the earlier investigations by [8] and [9], we define (n,η)-neighborhood of a function $f(z)\in\mathcal{T}(n)$ by

$$N_{n,\eta}(f)$$
$$=\left\{g(z)=z-\sum_{k=n+1}^{\infty}b_kz^k\in\mathcal{T}(n):\sum_{k=n+1}^{\infty}k\left|a_k-b_k\right|\leq\eta\right\}$$

or,

$$N_{n,\eta}(h):=\left\{g(z)=z-\sum_{k=n+1}^{\infty}b_kz^k\in\mathcal{T}(n):\sum_{k=n+1}^{\infty}k\left|b_k\right|\leq\eta\right\},$$

where $h(z)=z.$

Let $\mathcal{S}_n^*(\alpha)$ denote the subclass of $\mathcal{T}(n)$ consisting of functions which satisfy

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha, z\in U, 0\leq\alpha<1.$$

A function $f(z)$ in $\mathcal{S}_n^*(\alpha)$ is said to be starlike of order α in U .

A function $f(z)\in\mathcal{T}(n)$ is said to be convex of order α it it satisfies

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha, z\in U, 0\leq\alpha<1.$$

We denote by $\mathcal{C}_n(\alpha)$ the subclass of $\mathcal{T}(n)$ consisting of all such functions [10].

The unification of the classes $\mathcal{S}_n^*(\alpha)$ and $\mathcal{C}_n(\alpha)$ is provided by the class $\mathcal{T}_n(\alpha,\gamma)$ of functions $f(z)\in\mathcal{T}(n)$ which also satisfy the following inequality

$$\operatorname{Re}\left(\frac{zf'(z)+\gamma z^2f''(z)}{\gamma zf'(z)+(1-\gamma)f(z)}\right)>\alpha,$$
$$z\in U, 0\leq\alpha<1, 0\leq\gamma\leq 1.$$

The class $\mathcal{T}_n(\alpha,\gamma)$ was investigated by Altintas [11].

Now, by using $D_l^{m,\lambda}(a,b)$ we will define a new class of starlike functions.

Definition 1.2 *Let*

$$0\leq\alpha<1, 0\leq\gamma\leq 1,$$
$$b\neq 0,-1,-2,-3,\cdots, m\in\mathbb{Z}, l\geq 0, \lambda\geq 0.$$

A function f *belonging to* $\mathcal{T}(n)$ *is said to be in the class* $\mathcal{T}_{\lambda,l}^m(n,\alpha,\gamma,a,b)$ *if and only if*

$$\operatorname{Re}\left\{\frac{(1-\gamma)z\left(D_l^{m,\lambda}(a,b)f(z)\right)'+\gamma z\left(D_l^{m+1,\lambda}(a,b)f(z)\right)'}{(1-\gamma)z\left(D_l^{m,\lambda}(a,b)f(z)\right)+\gamma z\left(D_l^{m+1,\lambda}(a,b)f(z)\right)}\right\}$$
$$>\alpha, z\in U. \quad (6)$$

Remark 1.3 *The class* $\mathcal{T}_{\lambda,l}^m(n,\alpha,\gamma,a,b)$ *is a generalization of the following subclasses*:

i) $\mathcal{T}_{1,0}^0(1,\alpha,0,1,1)\equiv\mathcal{T}^*(\alpha)\equiv\mathcal{S}_1^*(\alpha)$ and $\mathcal{T}_{1,0}^1(1,\alpha,0,1,1)\equiv\mathcal{C}(\alpha)\equiv\mathcal{C}_1(\alpha)$ defined and studied by [12];

ii) $\mathcal{T}_{1,0}^0(1,\alpha,0,1,1)$ and $\mathcal{T}_{1,0}^1(1,\alpha,0,1,1)$ studied by [13] and [14];

iii) $\mathcal{T}_{1,0}^m(1,\alpha,0,1,1)\equiv\mathcal{T}(m,\alpha)$ studied by [15];

iv) $\mathcal{T}_{1,0}^0(n,\alpha,\gamma,1,1)$ studied by [16].

Now, we shall use the same method by [17] to establish certain coefficient estimates relating to the new introduced class.

2. Coefficient Estimates

Theorem 2.1 *Let the function* f *be defined by* (1). *Then* f *belongs to the class* $\mathcal{T}_{\lambda,l}^m(n,\alpha,\gamma,a,b)$ *if and only if*

$$\sum_{k=n+1}^{\infty}c_k(m,\lambda,l,a,b)(k-\alpha)\left[1+l+\gamma\lambda(k-1)\right]a_k$$
$$\leq(1+l)(1-\alpha), \quad (7)$$

where

$$c_k(m,\lambda,l,a,b) = \left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m \frac{(a)_{k-1}}{(b)_{k-1}}. \quad (8)$$

The result is sharp and the extremal functions are

$$f_k(z) = z - \frac{(1+l)(1-\alpha)}{c_k(m,\lambda,l,a,b)(k-\alpha)\left[1+l+\gamma\lambda(k-1)\right]} z^k, \quad (9)$$

$k \geq n+1.$

Proof: Assume that the inequality (7) holds and let $|z|=1$. Then we have

$$\left|\frac{(1-\gamma)z\left(D_l^{m,\lambda}(a,b)f(z)\right)' + \gamma z\left(D_l^{m+1,\lambda}(a,b)f(z)\right)'}{(1-\gamma)z\left(D_l^{m,\lambda}(a,b)f(z)\right) + \gamma z\left(D_l^{m+1,\lambda}(a,b)f(z)\right)} - 1\right|$$

$$= \left|\frac{\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]\frac{(a)_{k-1}}{(b)_{k-1}}(k-1)a_k z^{k-1}}{1-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]\frac{(a)_{k-1}}{(b)_{k-1}}a_k z^{k-1}}\right|$$

$$\leq 1 + \frac{\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]\frac{(a)_{k-1}}{(b)_{k-1}}ka_k - 1}{1-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]\frac{(a)_{k-1}}{(b)_{k-1}}a_k} \leq 1-\alpha.$$

Consequently, by the maximum modulus theorem one obtains

$$f(z) \in \mathcal{T}_{\lambda,l}^m(n,\alpha,\gamma,a,b).$$

Conversely,suppose that

$$f(z) \in \mathcal{T}_{\lambda,l}^m(n,\alpha,\gamma,a,b).$$

Then from (6) we find that

$$\operatorname{Re}\left\{\frac{z-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]\frac{(a)_{k-1}}{(b)_{k-1}}ka_k z^k}{z-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]\frac{(a)_{k-1}}{(b)_{k-1}}a_k z^k}\right\} > \alpha.$$

Choose values of z on the real axis such that

$$\frac{(1-\gamma)z\left(D_l^{m,\lambda}(a,b)f(z)\right)' + \gamma z\left(D_l^{m+1,\lambda}(a,b)f(z)\right)'}{(1-\gamma)z\left(D_l^{m,\lambda}(a,b)f(z)\right) + \gamma z\left(D_l^{m+1,\lambda}(a,b)f(z)\right)}$$

is real. Letting $z \to 1^-$ through real values, we obtain

$$\operatorname{Re}\left\{\frac{1-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]\frac{(a)_{k-1}}{(b)_{k-1}}ka_k}{1-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]\frac{(a)_{k-1}}{(b)_{k-1}}a_k}\right\} \geq \alpha,$$

or, equivalently

$$1-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]\frac{(a)_{k-1}}{(b)_{k-1}}ka_k$$

$$\geq \alpha\left\{1-\sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m\left[\frac{1+l+\gamma\lambda(k-1)}{1+l}\right]\frac{(a)_{k-1}}{(b)_{k-1}}a_k\right\},$$

which gives (7).

Remark 2.2 *In the special case* $a = b = 1$, *Theorem* 2.1 *yields a result given earlier by* [17].

Remark 2.3 *In the special case* $\lambda = 1,\ l = 0,$ $a = b = 1$, *Theorem* 2.2 *yields a result given earlier by* [6].

Theorem 2.4 *Let the function* f *defined by* (3) *be in the class* $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$. *Then*

$$\sum_{k=n+1}^{\infty} a_k \le \frac{(1+l)(1-\alpha)}{c_{n+1}(m,\lambda,l,a,b)(1+l+\gamma\lambda n)(n+1-\alpha)}, \quad (10)$$

and

$$\sum_{k=n+1}^{\infty} k a_k \le \frac{(1+l)(1-\alpha)(n+1)}{c_{n+1}(m,\lambda,l,a,b)(1+l+\gamma\lambda n)(n+1-\alpha)}. \quad (11)$$

The equality in (10) and (11) is attained for the function f given by (9).

Proof: By using Theorem 2.2, we find from (6) that

$$\begin{aligned}&(1+l+\gamma\lambda n)(n+1-\alpha)c_{n+1}(m,\lambda,l,a,b)\sum_{k=n+1}^{\infty} a_k \\ &\le \sum_{k=n+1}^{\infty} c_k(m,\lambda,l,a,b)(k-\alpha)\left[1+l+\gamma\lambda(k-1)\right]a_k \\ &\le (1+l)(1-\alpha),\end{aligned}$$

which immediately yields the first assertion (10) of Theorem 2.3.

On the other hand, taking into account the inequality (6), we also have

$$\begin{aligned}&(1+l+\gamma\lambda n)c_{n+1}(m,\lambda,l,a,b)\sum_{k=n+1}^{\infty}(k-\alpha)a_k \\ &\le (1+l)(1-\alpha),\end{aligned}$$

that is

$$\begin{aligned}&(1+l+\gamma\lambda n)c_{n+1}(m,\lambda,l,a,b)\sum_{k=n+1}^{\infty} k a_k \\ &\le (1+l)(1-\alpha)+\alpha(1+l+\gamma\lambda n)c_{n+1}(m,\lambda,l,a,b)\sum_{k=n+1}^{\infty} a_k,\end{aligned}$$

which, in view of the coefficient inequality (10), can be put in the form

$$\begin{aligned}&(1+l+\gamma\lambda n)c_{n+1}(m,\lambda,l,a,b)\sum_{k=n+1}^{\infty} k a_k \\ &\le (1+l)(1-\alpha)+\alpha(1+l+\gamma\lambda n)c_{n+1}(m,\lambda,l,a,b) \\ &\frac{(1+l)(1-\alpha)}{c_{n+1}(m,\lambda,l,a,b)(1+l+\gamma\lambda n)(n+1-\alpha)},\end{aligned}$$

and this completes the proof of (11).

3. Closure Theorem

Theorem 3.1 *Let the function* $f_j(z),(j=1,2,\cdots,m)$ *be defined by*

$$f_j(z) = z - \sum_{k=n+1}^{\infty} a_{kj} z^k,$$

for $z \in U$, be in the class $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$ then the function $h(z)$ defined by

$$h(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, (a_{kj} > 0),$$

also belongs to the class $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$, where

$$b_k = \frac{1}{m}\sum\nolimits_{j=n}^{m} a_{kj}.$$

Proof: Since $f_j(z) \in \mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$, it follows from Theorem 2.1, that

$$\begin{aligned}&\sum_{k=n+1}^{\infty} c_k(m,\lambda,l,a,b)(k-\alpha)\left[1+l+\gamma\lambda(k-1)\right]a_{kj} \\ &< (1+l)(1-\alpha),(j=1,2,\cdots,m).\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{k=n+1}^{\infty} c_k(m,\lambda,l,a,b)(k-\alpha)\left[1+l+\gamma\lambda(k-1)\right]b_k \\ &= \sum_{k=n+1}^{\infty} c_k(m,\lambda,l,a,b)(k-\alpha)\left[1+l+\gamma\lambda(k-1)\right]\left(\frac{1}{m}\sum_{j=n}^{m} a_{kj}\right) \\ &= \frac{1}{m}\sum_{j=n}^{m}\left(\sum_{k=n+1}^{\infty} c_k(m,\lambda,l,a,b)(k-\alpha)\left[1+l+\gamma\lambda(k-1)\right]a_{kj}\right) \\ &< (1+l)(1-\alpha).\end{aligned}$$

Hence by Theorem 2.1, $h(z) \in \mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$ also.

Morever, we shall use the same method by [17] to prove the distrotion Theorems.

4. Distortion Theorems

Theorem 4.1 *Let the function* f *defined by* (1) *be in the class* $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$. *Then we have*

$$\begin{aligned}&\left|D_l^{i,\lambda}(a,b)f(z)\right| \\ &\ge |z| - \frac{(1+l)(1-\alpha)}{c_k(m-i,\lambda,l,a,b)(n+1-\alpha)(1+l+\gamma\lambda n)}|z|^{n+1},\end{aligned} \quad (12)$$

and

$$\begin{aligned}&\left|D_l^{i,\lambda}(a,b)f(z)\right| \\ &\le |z| + \frac{(1+l)(1-\alpha)}{c_k(m-i,\lambda,l,a,b)(n+1-\alpha)(1+l+\gamma\lambda n)}|z|^{n+1},\end{aligned} \quad (13)$$

for $z \in U$, where $0 \le i \le m$ and $c_k(m-i,\lambda,l,a,b)$ is given by (8).

The equalities in (12) and (13) are attained for the function f given by

$$\begin{aligned}&f_{n+1}(z) \\ &= z - \frac{(1-\alpha)(1+l)^{m+1}}{(1+\lambda n+l)^m(n+1-\alpha)(1+l+\gamma\lambda n)} z^{n+1}.\end{aligned} \quad (14)$$

Proof: Note that $f \in \mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$ if and only if $D_l^{i,\lambda}(a,b)f(z) \in \mathcal{T}_{\lambda,l}^{m-i}(n,\alpha,\gamma,a,b)$, where

$$D_l^{i,\lambda}(a,b)f(z) = z - \sum_{k=n+1}^{\infty} c_k(i,\lambda,l,a,b)a_k z^k.$$

By Theorem 2.2, we know that

$$\begin{aligned}&c_k(m-i,\lambda,l,a,b)(n+1-\alpha)\\&\cdot(1+l+\gamma\lambda n)\sum_{k=n+1}^{\infty}c_k(i,\lambda,l,a,b)a_k\\&\le \sum_{k=n+1}^{\infty}c_k(m,\lambda,l,a,b)(k-\alpha)[1+l+\gamma\lambda(k-1)]a_k\\&\le(1+l)(1-\alpha),\end{aligned}$$

that is

$$\begin{aligned}&\sum_{k=n+1}^{\infty}c_k(i,\lambda,l,a,b)a_k\\&\le\frac{(1+l)(1-\alpha)}{c_k(m-i,\lambda,l,a,b)(n+1-\alpha)(1+l+\gamma\lambda n)}.\end{aligned}$$

The assertions of (12) and (13) of Theorem 4.1 follow immediately. Finally, we note that the equalities (12) and (13) are attained for the function f defined by

$$\begin{aligned}&D_l^{i,\lambda}(a,b)f(z)\\&=z-\frac{(1+l)(1-\alpha)}{c_k(m-i,\lambda,l,a,b)(n+1-\alpha)(1+l+\gamma\lambda n)}z^{n+1}.\end{aligned}$$

This completes the proof of Theorem 4.1.

Remark 4.2 *In the special case* $a=b=1$, *Theorem* 4.1 *yields a result given earlier by* [17].

Corollary 4.3 *Let the function* f *defined by* (1) *be in the class* $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$. *Then we have*

$$\begin{aligned}&|f(z)|\\&\ge|z|-\frac{(1+l)(1-\alpha)}{c_k(m,\lambda,l,a,b)(n+1-\alpha)(1+l+\gamma\lambda n)}|z|^{n+1},\end{aligned}\tag{15}$$

and

$$\begin{aligned}&|f(z)|\\&\le|z|+\frac{(1+l)(1-\alpha)}{c_k(m,\lambda,l,a,b)(n+1-\alpha)(1+l+\gamma\lambda n)}|z|^{n+1},\end{aligned}\tag{16}$$

for $z\in U$. The equalities in (15) and (16) are attained for the function f_{n+1} given in (14).

Corollary 4.4 *Let the function* f *defined by* (1) *be in the class* $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$. *Then we have*

$$\begin{aligned}&|f'(z)|\\&\ge1-\frac{(1+l)(1-\alpha)(n+1)}{c_k(m,\lambda,l,a,b)(n+1-\alpha)(1+l+\gamma\lambda n)}|z|^{n},\end{aligned}\tag{17}$$

and

$$\begin{aligned}&|f'(z)|\\&\le1+\frac{(1+l)(1-\alpha)(n+1)}{c_k(m,\lambda,l,a,b)(n+1-\alpha)(1+l+\gamma\lambda n)}|z|^{n},\end{aligned}\tag{18}$$

for $z\in U$. The equalities in (17) and (18) are attained for the function f_{n+1} given in (14).

Corollary 4.5 *Let the function* f *defined by* (3) *be in the class* $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$. *Then the unit disc is mapped onto a domain that contains the disc*

$$\begin{aligned}&|w|\\&<\frac{c_k(m,\lambda,l,a,b)(n+1-\alpha)(1+l+\gamma\lambda n)-(1+l)(1-\alpha)}{c_k(m,\lambda,l,a,b)(n+1-\alpha)(1+l+\gamma\lambda n)}.\end{aligned}$$

The result is sharp with the extremal function f_{n+1} given in (14).

5. Integral Operators

Theorem 5.1 *Let the function* $f(z)$ *defined by* (1) *be in the class* $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$ *and let* c *be a real number such that* $c>-1$. *Then* $F(z)$, *defined by*

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,\mathrm{d}t,$$

also belongs to the class $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$

Proof: From the representation of $F(z)$, it is obtained that

$$F(z)=z-\sum_{k=n+1}^{\infty}b_k z^k,(n\in\mathbb{N}),$$

where

$$b_k=\left(\frac{c+1}{k+c}\right)a^k.$$

Therefore

$$\begin{aligned}&\sum_{k=n+1}^{\infty}c_k(m,\lambda,l,a,b)(k-\alpha)[1+l+\gamma\lambda(k-1)]b_k\\&\cdot\sum_{k=n+1}^{\infty}c_k(m,\lambda,l,a,b)(k-\alpha)[1+l+\gamma\lambda(k-1)]\left(\frac{c+1}{k+c}\right)a^k\\&\le\sum_{k=n+1}^{\infty}c_k(m,\lambda,l,a,b)(k-\alpha)[1+l+\gamma\lambda(k-1)]a^k\\&\le(1+l)(1-\alpha),\end{aligned}$$

since $f(z)$ belongs to $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$ so by virtue of Theorem 2.1, $F(z)$ is also element of $\mathcal{T}_{\lambda,l}^{m}(n,\alpha,\gamma,a,b)$

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