Aequationes Math. 62 (2001) 310–320
0001-9054/01/030310-11 $ 1.50+0.20/0

Aequationes Mathematicae

On Ky Fan-type inequalities

Horst Alzer, Stephan Ruscheweyh* and Luis Salinas*

Dedicated to Professor Ludwig Reich on his 60th birthday

Summary. Let A_n, G_n, H_n (respectively, A'_n, G'_n, H'_n) denote the unweighted arithmetic, geometric, harmonic means of $x_1, \dots, x_n$ (respectively, $1 - x_1, \dots, 1 - x_n$), where $x_j \in (0, 1/2]$ $(j = 1, \dots, n)$. In 1984, Wang and Wang established

$$\left(\frac{G_n}{G'_n}\right)^n \le \left(\frac{A_n}{A'_n}\right)^{n-1} \frac{H_n}{H'_n},$$

which refines the well-known Ky Fan inequality $G_n/G'_n \le A_n/A'_n$. The validity of the converse inequality

$$\left(\frac{H_n}{H'_n}\right)^{n-1} \frac{A_n}{A'_n} \le \left(\frac{G_n}{G'_n}\right)^n \tag{0.1}$$

was conjectured in 1988. In this paper we give a proof for (0.1).

Mathematics Subject Classification (2000). 26D15.

Keywords. Inequalities of Fan and Wang-Wang, mean values.

1. Introduction

In 1961, E. F. Beckenbach and R. Bellman [8, p. 5] presented in their well-known monograph "Inequalities" a remarkable counterpart of the classical arithmetic mean-geometric mean inequality, which is due to Ky Fan:

$$\frac{G_n}{G'_n} \le \frac{A_n}{A'_n}, \tag{1.1}$$

where

$$A_n = \frac{1}{n}\sum_{j=1}^{n} x_j, \quad G_n = \prod_{j=1}^{n} x_j^{1/n}, \quad A'_n = \frac{1}{n}\sum_{j=1}^{n}(1 - x_j), \quad G'_n = \prod_{j=1}^{n}(1 - x_j)^{1/n},$$

* Partially supported by FONDECYT grants 1980015 and 7980001.

and $x_j \in (0, 1/2]$ $(j = 1, \ldots, n)$. The sign of equality is valid in (1.1) if and only if $x_1 = \cdots = x_n$.

In numerous publications new proofs, interesting extensions, refinements, and various related results of Ky Fan's inequality are given. We refer to the survey article [4] and the recently published papers [5, 6, 7, 9, 10, 11, 12, 13, 14, 16, 18].

In 1984, W.-L. Wang and P.-F. Wang [17] proved a striking companion of (1.1) involving geometric and harmonic means:

$$\frac{H_n}{H_n'} \leq \frac{G_n}{G_n'}, \tag{1.2}$$

where

$$H_n = \left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{x_j}\right)^{-1}, \quad H_n' = \left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{1-x_j}\right)^{-1},$$

and $x_j \in (0, 1/2]$ $(j = 1, \ldots, n)$. Equality holds in (1.2) if and only if $x_1 = \cdots = x_n$. We remark that (1.1) and (1.2) were originally established by forward and backward induction – a method introduced by Cauchy in 1821 to give one of the first proofs for the arithmetic mean – geometric mean inequality.

Wang and Wang [17] also showed that the three classical mean values can be connected in one inequality. The following elegant sharpening of (1.1) is valid for all $x_j \in (0, 1/2]$ $(j = 1, \ldots, n)$:

$$\left(\frac{G_n}{G_n'}\right)^n \leq \left(\frac{A_n}{A_n'}\right)^{n-1}\frac{H_n}{H_n'}. \tag{1.3}$$

If $n = 1, 2$, then the sign of equality holds in (1.3), and, if $n \geq 3$, then equality is valid if and only if $x_1 = \cdots = x_n$. A short and simple proof of (1.3) is given in [1].

We point out that (1.3) is closely related to the inequalities

$$(H_n)^{n-1}A_n \leq (G_n)^n \leq (A_n)^{n-1}H_n, \tag{1.4}$$

which hold for all real numbers $x_j > 0$ $(j = 1, \ldots, n)$. Double-inequality (1.4) was proved by W. Sierpiński [15] in 1909.

In view of (1.4) it is natural to ask whether the following converse of (1.3) is true:

$$\left(\frac{H_n}{H_n'}\right)^{n-1}\frac{A_n}{A_n'} \leq \left(\frac{G_n}{G_n'}\right)^n. \tag{1.5}$$

The validity of (1.5) was conjectured already in 1988 in [1], but neither a proof nor a disproof can be found in the literature. It is the main purpose of this paper to establish inequality (1.5). In the next section we provide some technical lemmas, which we need to prove our main result. A proof for (1.5) is given in the third section. In the final part, we present an open problem concerning additive analogues of (1.3) and (1.5), and we establish a recently published conjecture of Mercer on Ky Fan's inequality.

2. Lemmas

In this section, we prove three elementary inequalities for sums of four ratios.

Lemma 1. *Let x, a, u, and s be real numbers with $0 < x \le a \le 1/2$, $u \ge 1$, and $s \ge 1$. Then we have*

$$\Gamma(u,s,x,a) := \frac{(u+s-1)x}{u+s\frac{1-x}{1-a}} + \frac{(u+s-1)(1-x)}{u+s\frac{x}{a}} + \frac{1-x}{u+s\frac{a}{x}} + \frac{x}{u+s\frac{1-a}{1-x}} \ge 1, \quad (2.1)$$

with equality if and only if $x = a$ or $u = s = 1$.

Proof. Let $x < a$ and let $u > 1$ or $s > 1$. We have

$$\Gamma(u,s,x,a) - 1 = s(x-a)\Delta(u,s,x,a), \quad (2.2)$$

where

$$\Delta(u,s,x,a) = x\frac{(s-1)(1-a)+(u-1)(1-x)}{[s(1-a)+u(1-x)][s(1-x)+u(1-a)]} - (1-x)\frac{(s-1)a+(u-1)x}{(sa+ux)(sx+ua)}.$$

Since $sx + ua < s(1-x) + u(1-a)$, we obtain

$$\begin{aligned} \Delta(u,s,x,a)[s(1-x)+u(1-a)] &< x\frac{(s-1)(1-a)+(u-1)(1-x)}{s(1-a)+u(1-x)} \\ &\quad - (1-x)\frac{(s-1)a+(u-1)x}{sa+ux} \\ &= \frac{\Phi(u,s,x,a)}{[s(1-a)+u(1-x)](sa+ux)}, \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} \Phi(u,s,x,a) &= -(1-2x)[s(s-1)a(1-a)+u(u-1)x(1-x)+ux(s-1)(1-a)] \\ &\quad - u(s-1)(1-x)(a-x) - s(u-1)(1-2a)x(1-x) \\ &\le 0. \end{aligned} \quad (2.4)$$

From (2.2)–(2.4) we conclude that (2.1) holds with “>” instead of “≥”. □

Lemma 2. *Let x, a, u, and v be real numbers with $0 < x \le a \le 1/2$, $1 \le u \le v$. Then we have*

$$\begin{aligned}\Sigma(u,v,x,a) :=& \frac{vx}{u+(v-u)\frac{1-x}{1-a}+2(1-x)} + \frac{v(1-x)}{u+(v-u)\frac{x}{a}+2x} \qquad (2.5)\\ &+ \frac{1-x}{u+(v-u)\frac{a}{x}+\frac{1}{2x}} + \frac{x}{u+(v-u)\frac{1-a}{1-x}+\frac{1}{2(1-x)}}\\ &\ge 1,\end{aligned}$$

with equality if and only if $x = a = 1/2$ or $u = v = 1$.

Proof. Let $x \in (0, 1/2)$ and $v > 1$ be fixed real numbers, and let $M_0 = \{(a,u) \in \mathbb{R}^2 \mid x \le a \le 1/2,\ 1 \le u \le v\}$. Then we set $S(a,u) := \Sigma(u,v,x,a)$, where $(a,u) \in M_0$. Further, let $\alpha^* = (a^*, u^*)$ be the absolute minimum of S. If α^* is an interior point of M_0, then we have

$$\begin{aligned}0 &= \left(\frac{1}{v-u}\frac{\partial S(a,u)}{\partial a} + \frac{1}{a-x}\frac{\partial S(a,u)}{\partial u}\right)\Big|_{(a,u)=(a^*,u^*)}\\ &= v(x-a^*)\Big[x\Big((1-a^*)[u^*+2(1-x)]+(v-u^*)(1-x)\Big)^{-2}\\ &\qquad + (1-x)\Big(a^*(u^*+2x)+(v-u^*)x\Big)^{-2}\Big]\\ &< 0.\end{aligned}$$

This implies that α^* is a boundary point of M_0. Hence,

$$\alpha^* \in \{(x,u^*), (1/2,u^*), (a^*,1), (a^*,v)\}.$$

Using Lemma 1 we obtain

$$S(x,u^*) = S(a^*,v) = \Gamma\left(v,1,x,\frac{1}{2}\right) > 1$$

and

$$S\left(\frac{1}{2},u^*\right) = \Gamma\left(u^*, v-u^*+1, x, \frac{1}{2}\right) > 1.$$

For $S(a^*,1)$ we have the representation

$$S(a^*,1) = b(a^*-x)[(1-x)D_1 + xD_2 + bD_3] + S(x,1), \qquad (2.6)$$

where $b = v - 1 > 0$, and

$$D_1 = \frac{4x(a^*-x)b^2 + 2b(a^*+x)(1-4x^2) + (1+2x)^2(1-4a^*x)}{(a^*+2a^*x+bx)(b+1+2x)(2x+2a^*b+1)[2(b+1)x+1]} \ge 0, \qquad (2.7)$$

$$D_2 = \frac{2(1-x)(a^*-x)b^2 + (2-a^*-x)(3-2x)(1-2x)b + (3-2x)^2[2(1-x)(1-a^*)-\frac{1}{2}]}{[\frac{3}{2}-x+b(1-a^*)][2(b+1)(1-x)+1][b(1-x)+(1-a^*)(3-2x)](b+3-2x)} \ge 0, \qquad (2.8)$$

$$D_3 = \frac{(1-2x)(8a^*x+3b+b^2)+(1-a^*-x)[(4x^2-4x+3)b+(3-2x)^2]}{(a^*+2a^*x+bx)(b+1+2x)[b(1-x)+(1-a^*)(3-2x)](b+3-2x)} \geq 0. \tag{2.9}$$

From Lemma 1 we get $S(x,1) = \Gamma(v,1,x,\frac{1}{2}) > 1$, so that (2.6)–(2.9) lead to $S(a^*,1) > 1$. □

Lemma 3. *Let x, a, u, β, and s be real numbers with $0 < x \leq a \leq 1/2$, $x < 1/2$, $u \geq 1$, $u + \beta > 2$, and $0 \leq s \leq \beta - 1$. Then we have*

$$\begin{aligned} &\frac{(u+\beta-1)x}{u+s\frac{1-x}{1-a}+2(\beta-s)(1-x)} + \frac{(u+\beta-1)(1-x)}{u+s\frac{x}{a}+2(\beta-s)x} \\ &\quad + \frac{1-x}{u+s\frac{a}{x}+\frac{\beta-s}{2x}} + \frac{x}{u+s\frac{1-a}{1-x}+\frac{\beta-s}{2(1-x)}} > 1. \end{aligned} \tag{2.10}$$

Proof. If $\beta = 1$, then the expression on the left of (2.10) equals $\Gamma(u,1,x,\frac{1}{2})$ and we conclude from Lemma 1 that (2.10) is valid. Next, let $\beta > 1$, $u \geq 1$, $x \in (0,1/2)$ be fixed numbers, and let $M_1 = \{(s,a) \in \mathbb{R}^2 | \, 0 \leq s \leq \beta - 1, \, x \leq a \leq 1/2\}$. Further, we define $H(s,a)$ as the expression on the left-hand side of (2.10), where $(s,a) \in M_1$. We denote the absolute minimum of H by $\tilde{\alpha} = (\tilde{s},\tilde{a})$. If $\tilde{\alpha}$ is an interior point of M_1, then we obtain

$$\begin{aligned} 0 &= \left(\frac{1}{s}\frac{\partial H(s,a)}{\partial a} - \frac{1}{a-1/2}\frac{\partial H(s,a)}{\partial s}\right)\Big|_{(s,a)=(\tilde{s},\tilde{a})} \\ &= x(1-x)(u+\beta-1)(1-2\tilde{a})\Big[\Big(\tilde{s}x+\tilde{a}[u+2x(\beta-\tilde{s})]\Big)^{-2} \\ &\qquad + \Big((1-x)[(1-2\tilde{a})(\beta-1-\tilde{s})+\beta-1]+(1-\tilde{a})[u+2(1-x)]\Big)^{-2}\Big] \\ &> 0. \end{aligned}$$

Hence, $\tilde{\alpha}$ is a boundary point of M_1, so that we get

$$\tilde{\alpha} \in \{(\tilde{s},x),(\tilde{s},1/2),(0,\tilde{a}),(\beta-1,\tilde{a})\}.$$

Using Lemma 1 we obtain

$$\begin{aligned} H(\tilde{s},x) &= \Gamma\Big(u+\tilde{s},\beta-\tilde{s},x,\frac{1}{2}\Big) > 1, \\ H\Big(\tilde{s},\frac{1}{2}\Big) &= H(0,\tilde{a}) = \Gamma\Big(u,\beta,x,\frac{1}{2}\Big) > 1, \end{aligned}$$

and Lemma 2 yields

$$H(\beta-1,\tilde{a}) = \Sigma(u,u+\beta-1,x,\tilde{a}) > 1.$$

Thus, we get: if $(s,a) \in M_1$, then $H(s,a) > 1$. □

3. Main result

We are now in a position to prove our main result. The following refinement of inequality (1.2) is valid.

Theorem. *For all real numbers $x_j \in (0, 1/2]$ $(j = 1, \ldots, n)$ we have*

$$\left(\frac{H_n}{H'_n}\right)^{n-1} \frac{A_n}{A'_n} \le \left(\frac{G_n}{G'_n}\right)^n. \tag{3.1}$$

If $n = 1, 2$, then the sign of equality holds in (3.1). *If $n \ge 3$, then equality is valid if and only if $x_1 = \cdots = x_n$.*

Proof. Let $n \ge 3$ and $0 < x_1 \le x_2 \le \ldots \le x_n \le 1/2$, $x_1 < x_n$. We define

$$f(x_1, \ldots, x_n) = \sum_{j=1}^n \log \frac{x_j}{1 - x_j} - (n-1) \log \sum_{j=1}^n \frac{1}{1 - x_j} + (n-1) \log \sum_{j=1}^n \frac{1}{x_j} - \log \sum_{j=1}^n x_j + \log \sum_{j=1}^n (1 - x_j).$$

In what follows we show that f attains only positive values, which implies (3.1) with "$<$" instead of "$\le$".

Let $q \in \{1, \ldots, n-1\}$ and $f_q(x) = f(x, \ldots, x, x_{q+1}, \ldots, x_n)$. We prove that f_q is strictly decreasing on $(0, x_{q+1}]$, so that we obtain

$$\begin{aligned} f(x_1, \ldots, x_n) = f_1(x_1) \ge f_1(x_2) = f_2(x_2) \ge f_2(x_3) \ge \ldots \\ \ge f_{n-1}(x_{n-1}) \ge f_{n-1}(x_n) = 0. \end{aligned} \tag{3.2}$$

Since f_q is strictly monotonic, we conclude from $x_1 < x_n$ that at least one of the inequalities in (3.2) is strict. Hence, it remains to establish that

$$f'_q(x) < 0 \quad \text{for} \quad 0 < x < x_{q+1} \le \ldots \le x_n \le 1/2. \tag{3.3}$$

Let $x \in (0, 1/2)$ be a fixed real number. A simple calculation reveals that $f'_q(x) < 0$ is equivalent to

$$\begin{aligned} F(x_{q+1}, \ldots, x_n) := {} & \frac{(n-1)x}{q + \sum_{j=q+1}^n (1-x)/(1-x_j)} + \frac{(n-1)(1-x)}{q + \sum_{j=q+1}^n x/x_j} \\ & + \frac{1-x}{q + \sum_{j=q+1}^n x_j/x} + \frac{x}{q + \sum_{j=q+1}^n (1-x_j)/(1-x)} \\ > {} & 1. \end{aligned}$$

Let $M_2 = \{(x_{q+1}, \ldots, x_n) \in \mathbb{R}^{n-q} \,|\, 0 < x \le x_{q+1} \le \ldots \le x_n \le 1/2\}$. We prove that $F : M_2 \to \mathbb{R}$ attains its absolute minimum only at $(x, \ldots, x)$. This implies: if $0 < x < x_{q+1} \le \ldots \le x_n \le 1/2$, then

$$F(x_{q+1}, \ldots, x_n) > F(x, \ldots, x) = 1,$$

that is, (3.3) is valid.

We denote the absolute minimum of F by $\alpha = (a_{q+1}, \dots, a_n)$ and assume that $\alpha \neq (x, \dots, x)$. Next, we show that

$$\alpha = (x, \dots, x, a, \dots, a, 1/2, \dots, 1/2) \quad \text{with} \quad x < a < 1/2, \tag{3.4}$$

where the numbers x, a, and $1/2$ appear r, s, and t times, respectively, with r, s, $t \geq 0$, $r + s + t = n - q$, and $s + t \geq 1$.

If $q = n-1$, then obviously we have (3.4). Now, let $1 \leq q \leq n-2$. We suppose that the representation (3.4) does not hold. This implies that two components of α have different values and are interior points of M_2. We denote these values by a_k and a_l. Partial differentiation leads to

$$\frac{A}{(1-a_k)^2} + \frac{B}{a_k^2} + C = 0, \quad \frac{A}{(1-a_l)^2} + \frac{B}{a_l^2} + C = 0, \tag{3.5}$$

where

$$A = -(n-1)x(1-x)\Big(q + \sum_{j=q+1}^{n} \frac{1-x}{1-a_j}\Big)^{-2} < 0,$$

$$B = (n-1)x(1-x)\Big(q + \sum_{j=q+1}^{n} \frac{x}{a_j}\Big)^{-2} > 0,$$

$$C = \frac{x}{1-x}\Big(q + \sum_{j=q+1}^{n} \frac{1-a_j}{1-x}\Big)^{-2} - \frac{1-x}{x}\Big(q + \sum_{j=q+1}^{n} \frac{a_j}{x}\Big)^{-2}.$$

Since $z \mapsto A(1-z)^{-2} + Bz^{-2}$ is strictly monotonic on $(0,1)$, (3.5) yields $a_k = a_l$. This contradicts our assumption that $a_k \neq a_l$. Thus, (3.4) is valid and we get

$$\begin{aligned} F(a_{q+1}, \dots, a_n) &= \frac{(u+s+t-1)x}{u+s\frac{1-x}{1-a} + 2t(1-x)} + \frac{(u+s+t-1)(1-x)}{u+s\frac{x}{a} + 2tx} + \frac{1-x}{u+s\frac{a}{x} + \frac{t}{2x}} \\ &\quad + \frac{x}{u + s\frac{1-a}{1-x} + \frac{t}{2(1-x)}} \\ &= G(u,s,t,x,a), \quad \text{say,} \end{aligned}$$

where $0 < x < a < 1/2$, $u = q + r \geq 1$, $s \geq 0$, $t \geq 0$, and $s + t \geq 1$. We distinguish two cases.

Case 1. $t = 0$.
Then we have $s \geq 1$ and $u + s = n \geq 3$. Since $G(u,s,0,x,a)$ is equal to the sum on the left-hand side of (2.1), we obtain from Lemma 1

$$F(a_{q+1}, \dots, a_n) = G(u,s,0,x,a) = \Gamma(u,s,x,a) > 1. \tag{3.6}$$

Case 2. $t \geq 1$.
We set $\beta = s + t$. Then we have $u + \beta = n \geq 3$, $0 \leq s \leq \beta - 1$, and $G(u,s,t,x,a)$ is equal to the expression on the left-hand side of (2.10). From Lemma 3 we get

$$F(a_{q+1}, \dots, a_n) = G(u,s,t,x,a) > 1. \tag{3.7}$$

Since $F(x,\dots,x)=1$, we conclude from (3.6) and (3.7) that F attains its absolute minimum only at $(x,\dots,x)$. This completes the proof of the Theorem. □

From (1.3) and (3.1) we obtain upper and lower bounds for G_n/G_n' in terms of A_n/A_n' and H_n/H_n'. We show that these inequalities are best possible in a certain sense.

Corollary. *Let $n\ge 2$ be an integer, and let a, b be real numbers. The inequalities*

$$\left(\frac{A_n}{A_n'}\right)^a\left(\frac{H_n}{H_n'}\right)^{1-a}\le\frac{G_n}{G_n'}\le\left(\frac{A_n}{A_n'}\right)^b\left(\frac{H_n}{H_n'}\right)^{1-b} \tag{3.8}$$

hold for all real numbers $x_j\in(0,1/2]$ $(j=1,\dots,n)$ if and only if

$$a\le 1/n \quad \textit{and} \quad b\ge(n-1)/n. \tag{3.9}$$

Proof. From (1.1) and (1.2) we obtain that $t\mapsto(A_n/A_n')^t(H_n/H_n')^{1-t}$ is increasing on $\mathbb{R}$, so that (1.3) and (3.1) imply: if $a\le 1/n$ and $b\ge(n-1)/n$, then (3.8) is valid for all $x_j\in(0,1/2]$ $(j=1,\dots,n)$.

Next, we assume that (3.8) holds for all $x_j\in(0,1/2]$ $(j=1,\dots,n)$. If the x_j's are not all equal, then (3.8) is equivalent to

$$a\le Q(x_1,\dots,x_n)\le b, \tag{3.10}$$

where

$$Q(x_1,\dots,x_n)=\log\frac{G_n/G_n'}{H_n/H_n'}\Big/\log\frac{A_n/A_n'}{H_n/H_n'}.$$

Let $0<x\ne y\le 1/2$. Since

$$\lim_{x\to 0}Q(x,y,\dots,y)=(n-1)/n \quad\text{and}\quad \lim_{y\to 0}Q(x,y,\dots,y)=1/n,$$

we conclude from (3.10) that (3.9) holds. □

4. Concluding remarks

1. Let $x=(x_1,\dots,x_n)$ be an n-tuple of positive real numbers, and let $r\in\{1,\dots,n\}$. The rth symmetric function of x is defined by

$$E_r(x)=\sum_{1\le i_1<\dots<i_r\le n}\prod_{j=1}^{r}x_{i_j}.$$

The following counterpart of Maclaurin's classical inequality for symmetric functions is due to Wang and Wang [17]:

$$\left(\frac{E_{k+1}(x)}{E_{k+1}(1-x)}\right)^{1/(k+1)}\le\left(\frac{E_k(x)}{E_k(1-x)}\right)^{1/k}, \tag{4.1}$$

where $k \in \{1, \ldots, n-1\}$, $1 - x = (1 - x_1, \ldots, 1 - x_n)$, and $x_j \in (0, 1/2]$ $(j = 1, \ldots, n)$. We have

$$E_1(x)/E_1(1-x) = A_n/A'_n \quad \text{and} \quad E_n(x)/E_n(1-x) = (G_n/G'_n)^n,$$

so that (4.1) leads to a chain of inequalities which refines (1.1). Moreover, since

$$E_{n-1}(x)/E_{n-1}(1-x) = (G_n/G'_n)^n (H_n/H'_n)^{-1},$$

we get that (1.3) follows from (4.1). It is tempting to conjecture that the following companion of (4.1) is also valid for all $x_j \in (0, 1/2]$ $(j = 1, \ldots, n)$:

$$\left(\frac{E_{k+1}(1/x)}{E_{k+1}(1/(1-x))}\right)^{1/(k+1)} \le \left(\frac{E_k(1/x)}{E_k(1/(1-x))}\right)^{1/k}, \tag{4.2}$$

where $k \in \{1, \ldots, n-1\}$, $1/x = (1/x_1, \ldots, 1/x_n)$, and $1/(1-x) = (1/(1-x_1), \ldots, 1/(1-x_n))$. Since

$$E_1(1/x)/E_1(1/(1-x)) = (H_n/H'_n)^{-1},$$

$$E_{n-1}(1/x)/E_{n-1}(1/(1-x)) = (A_n/A'_n)(G_n/G'_n)^{-n},$$

and

$$E_n(1/x)/E_n(1/(1-x)) = (G_n/G'_n)^{-n},$$

we conclude that (1.2) and (1.5) follow from (4.2).

In a recently published article, H.-T. Ku, M.-C. Ku, and X.-M. Zhang [10] claimed that (4.2) has been established in [17]. However, in [17] only a proof for (4.1), but not for (4.2), is given. The problem to prove or disprove (4.2) is open!

2. The following additive analogue of Ky Fan's inequality holds for all $x_j \in (0, 1/2]$ $(j = 1, \ldots, n)$:

$$G_n - G'_n \le A_n - A'_n. \tag{4.3}$$

Proofs and refinements of (4.3) are given in [2, 4, 5, 6, 7, 12]. It is easy to show that (1.1) and (4.3) are included in a chain of five inequalities, which in particular provides refinements of Ky Fan's inequality and its additive counterpart:

$$2-(A_n+A'_n) \le \frac{A'_n}{G'_n} \le 1 + 2(A'_n - G'_n) \le 2 - (G_n + G'_n) \le 1 + 2(A_n - G_n) \le \frac{A_n}{G_n}.$$

In 1993, it was conjectured that an additive companion of (1.3) is also true (see [3]):

$$n(G_n - G'_n) \le (n-1)(A_n - A'_n) + H_n - H'_n. \tag{4.4}$$

Since $H_n - H'_n \le A_n - A'_n$ (see [3]), we conclude that (4.4) sharpens inequality (4.3). Numerous computer calculations support the validity of (4.4), but a proof (or a counterexample) is still missing. In view of (3.8) it is natural to look for the best possible bounds for $G_n - G'_n$ in terms of $A_n - A'_n$ and $H_n - H'_n$. More precisely, we ask: what is the largest number $\alpha = \alpha(n)$ and what is the smallest number $\beta = \beta(n)$ such that

$$\alpha(A_n - A_n') + (1-\alpha)(H_n - H_n') \leq G_n - G_n' \leq \beta(A_n - A_n') + (1-\beta)(H_n - H_n')$$

for all $x_j \in (0, 1/2]$ $(j = 1, \ldots, n)$?

3. In 1998, A. McD. Mercer [14] proved an interesting inequality for the weighted means

$$A_n = \sum_{j=1}^n p_j x_j, \quad G_n = \prod_{j=1}^n x_j^{p_j}, \quad A_n' = \sum_{j=1}^n p_j(1-x_j), \quad G_n' = \prod_{j=1}^n (1-x_j)^{p_j},$$

where $x_j \in (0, 1/2]$ and $p_j > 0$ $(j = 1, \ldots, n)$ with $\sum_{j=1}^n p_j = 1$:

$$\frac{A_n G_n' - A_n' G_n}{G_n + G_n'} = \gamma \sum_{1 \leq i < j \leq n} p_i p_j \left[\log \frac{x_i(1-x_j)}{x_j(1-x_i)}\right]^2, \tag{4.5}$$

where

$$0 \leq \gamma \leq \sqrt{3}/36 = 0.0481\ldots. \tag{4.6}$$

We note that (4.5) yields a new proof for the weighted version of Ky Fan's inequality.

Mercer remarked that the upper bound for γ is "likely best-possible" [14, p. 777]. We show that this conjecture is true. Let

$$W(x_1, \ldots, x_n) = \frac{A_n G_n' - A_n' G_n}{G_n + G_n'} \left(\sum_{1 \leq i < j \leq n} p_i p_j \left[\log \frac{x_i(1-x_j)}{x_j(1-x_i)}\right]^2 \right)^{-1}$$

and $0 < x \neq y \leq 1/2$. Applying l'Hospital's rule we get

$$\lim_{x \to y} W(x, y, \ldots, y) = \frac{1}{2} y(1-y)(1-2y) = h(y), \quad \text{say.}$$

Since $h(1/2 - 1/\sqrt{12}) = \sqrt{3}/36$ and $h(1/2) = 0$, we conclude that the bounds for γ, given in (4.6), are sharp.

References

[1] H. Alzer, *Verschärfung einer Ungleichung von Ky Fan*, Aequationes Math. *36* (1988), 246–250.

[2] H. Alzer, *Ungleichungen für geometrische und arithmetische Mittelwerte*, Indag. Math. (N.S.) *50* (1988), 365–374.

[3] H. Alzer, *An inequality for arithmetic and harmonic means*, Aequationes Math. *46* (1993), 257–263.

[4] H. Alzer, *The inequality of Ky Fan and related results*, Acta Appl. Math. *38* (1995), 305–354.

[5] H. Alzer, *On Ky Fan's inequality and its additive analogue*, J. Math. Anal. Appl. *204* (1995), 291–297.
[6] H. Alzer, *On an additive analogue of Ky Fan's inequality*, Indag. Math. (N.S.) *8* (1997), 1–6.
[7] H. Alzer, *Some inequalities for arithmetic and geometric means*, Proc. Roy. Soc. Edinburgh Sect. A *129* (1999), 221–228.
[8] E. F. Beckenbach and R. Bellman, *Inequalities*, Springer, Berlin, 1961.
[9] S. S. Dragomir, *Some refinements of Ky Fan's inequality*, J. Math. Anal. Appl. *163* (1992), 317–321.
[10] H.-T. Ku, M.-C. Ku and X.-M. Zhang, *Generalized power means and interpolating inequalities*, Proc. Amer. Math. Soc. *127* (1999), 145–154.
[11] M. T. McGregor, *On some inequalities of Ky Fan and Wang-Wang*, J. Math. Anal. Appl. *180* (1993), 182–188.
[12] M. T. McGregor, *On an inequality of Horst Alzer*, Indag. Math. (N.S.) *7* (1996), 161–164.
[13] A. McD. Mercer, *A short proof of Ky Fan's arithmetic-geometric inequality*, J. Math. Anal. Appl. *204* (1996), 940–942.
[14] A. McD. Mercer, *An 'error term' for the Ky Fan inequality*, J. Math. Anal. Appl. *220* (1998), 774–777.
[15] W. Sierpiński, *On an inequality for arithmetic, geometric, and harmonic means*, Warsch. Sitzungsber. *2* (1909), 354–358 (in Polish).
[16] W.-L. Wang, *Some inequalities involving means and their converses*, J. Math. Anal. Appl. *238* (1999), 567–579.
[17] W.-L. Wang and P.-F. Wang, *A class of inequalities for the symmetric functions*, Acta Math. Sinica *27* (1984), 485–497 (in Chinese).
[18] G.-S. Yang and C.-S. Wang, *Refinements of an inequality of Ky Fan*, J. Math. Anal. Appl. *201* (1996), 955–965.

H. Alzer
Morsbacher Str. 10
D–51545 Waldbröl
Germany
e-mail: alzer@wmax03.mathematik.uni-wuerzburg.de

S. Ruscheweyh
Department of Mathematics
University of Würzburg
D–97074 Würzburg
Germany
e-mail: ruscheweyh@mathematik.uni-wuerzburg.de

L. Salinas
Departamento de Informática
Universidad Técnica F. Santa María
Valparaíso
Chile
e-mail: lsalinas@inf.utfsm.cl

Manuscript received: March 14, 2000.