

# Inequalities for Cyclic Functions

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The  $n$ th cyclic function is defined by

$$\varphi_n(z) = \sum_{\nu=0}^{\infty} \frac{z^{n\nu}}{(n\nu)!} \quad (z \in \mathbb{C}, 2 \leq n \in \mathbb{N}).$$

We prove that if  $k$  is an integer with  $1 \leq k \leq n-1$ , then

$$\left( \frac{(n-k)! \varphi_n^{(k)}(x)}{x^{n-k}} \right)^{\alpha} < \varphi_n(x) < \left( \frac{(n-k)! \varphi_n^{(k)}(x)}{x^{n-k}} \right)^{\beta}$$

holds for all positive real numbers  $x$  with the best possible constants

$$\alpha = 1 \quad \text{and} \quad \beta = \binom{2n-k}{n}.$$

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*Key Words:* inequalities; cyclic functions; gamma function.

## 1. INTRODUCTION

The  $n$ th cyclic function  $\varphi_n$  ( $2 \leq n \in \mathbb{N}$ ) is defined as the solution of the differential equation

$$y^{(n)}(z) = y(z), \quad y(0) = 1, \quad y'(0) = \dots = y^{(n-1)}(0) = 0,$$

that is, as the entire function

$$\varphi_n(z) = \sum_{\nu=0}^{\infty} \frac{z^{n\nu}}{(n\nu)!} = \frac{1}{n} \sum_{\nu=0}^{n-1} \exp(\varepsilon_n^\nu z) \quad (\varepsilon_n = e^{2\pi i/n}, \quad z \in \mathbb{C}).$$

For  $n = 2$  we obtain the classical hyperbolic functions

$$\varphi_2(z) = \cosh z \quad \text{and} \quad \varphi_2'(z) = \sinh z.$$

The cyclic functions  $\varphi_n, \varphi_n', \dots, \varphi_n^{(n-1)}$  have found the attention of several authors who established many interesting properties of these functions; we refer to [1] and the references therein.

This paper has been inspired by the known inequalities

$$(1.1) \quad \frac{\sinh x}{x} < \cosh x < \left( \frac{\sinh x}{x} \right)^3 \quad (x > 0).$$

Simple proofs of (1.1) are given in [3] and [4]. Lazarević [3, p. 270] pointed out that in the second inequality of (1.1) the exponent 3 is best possible, that is, this inequality is not true for all  $x > 0$ , if we replace 3 by a smaller constant.

Double-inequality (1.1) leads to inequalities involving the arithmetic, geometric, and logarithmic means of two positive real numbers. Indeed, setting  $x = \frac{1}{2} \log \frac{b}{a}$  ( $b > a > 0$ ) in (1.1), and multiplying by  $\sqrt{ab}$ , we obtain

$$L(a, b) < A(a, b) < (L(a, b))^3 / (G(a, b))^2,$$

where  $A(a, b) = \frac{1}{2}(a+b)$ ,  $G(a, b) = \sqrt{ab}$ , and  $L(a, b) = \frac{a-b}{\log a - \log b}$ . Related results can be found in [2, Chap. VI].

Páles [4, 5] provided other interesting applications of inequalities (1.1). He showed that they play a role in establishing inequalities for differences of powers and for two-parameter mean value families.

Since (1.1) can be written as

$$(1.2) \quad \frac{1}{x} \varphi_2'(x) < \varphi_2(x) < \left( \frac{1}{x} \varphi_2'(x) \right)^3 \quad (x > 0),$$

it is natural to look for an extension of (1.2) involving  $\varphi_n$  and its derivatives. It is the aim of this paper to present such an extension. More precisely, we answer the following question:

*Let  $n$  and  $k$  be integers with  $1 \leq k \leq n-1$ . What is the largest number  $\alpha = \alpha(n, k)$  and what is the smallest number  $\beta = \beta(n, k)$  such that the inequalities*

$$(1.3) \quad \left( \frac{(n-k)! \varphi_n^{(k)}(x)}{x^{n-k}} \right)^\alpha < \varphi_n(x) < \left( \frac{(n-k)! \varphi_n^{(k)}(x)}{x^{n-k}} \right)^\beta$$

*hold for all positive real numbers  $x$ ?*

## 2. A DISCRETE INEQUALITY

In order to determine the best possible values  $\alpha$  and  $\beta$  in (1.3) we need the following discrete inequality which might be of independent interest.

**THEOREM 1.** *Let  $u, v, x,$  and  $y$  be nonnegative integers such that  $(u, v) \neq (0, 0)$  and  $1 \leq y \leq x-1$ , and let*

$$(2.1) \quad F(u, v, x, y) = \frac{(x+y)!}{(u+x+y)! v!} - \frac{x! y!}{(u+x)! (v+y)!}.$$

*Then we have*

$$(2.2) \quad (u+x) F(u, v, x, y) + (v+x) F(v, u, x, y) > 0.$$

Using Euler's gamma function to define

$$(2.3) \quad a! = \Gamma(a+1)$$

for nonnegative real numbers  $a$ , it is likely that inequality (2.2) holds for all real numbers  $u, v, y \geq 0$  and  $x \geq 1$ . In fact, a detailed analysis of our proof of Theorem 1 reveals that (2.2) is valid at least for large parts of this latter range.

*Proof.* In what follows we use the abbreviations

$$H(u, v, x, y) = (u+x) F(u, v, x, y) + (v+x) F(v, u, x, y),$$

$$\gamma(s) = \Gamma(s+1), \quad \text{and} \quad \psi(s) = \Psi(s+1) = \frac{\Gamma'(s+1)}{\Gamma(s+1)} \quad (0 \leq s \in \mathbb{R}),$$

and we employ freely the extension of  $F(u, v, x, y)$  to nonnegative real numbers via (2.3). We have to show that

$$(2.4) \quad H(u, v, x, y) > 0.$$

Since  $H(u, v, x, y)$  is symmetric in  $u$  and  $v$ , it suffices to establish (2.4) for  $v \geq u$ . We consider three cases.

*Case 1:  $u = v$ .* Since  $\psi'$  is strictly decreasing on  $[0, \infty)$ , we conclude from the integral representation

$$H(u, u, x, y) = \frac{2(u+x)\gamma(x)\gamma(x+y)}{\gamma(u+x)\gamma(u+x+y)} \int_0^y \frac{\gamma(s)\gamma(u+x+s)}{\gamma(u+s)\gamma(x+s)} \int_s^{u+s} [\psi'(t) - \psi'(t+x)] dt ds$$

that

$$(2.5) \quad H(u, u, x, y) \geq 0,$$

with equality holding if and only if  $u = 0$ .

*Case 2:  $v \geq 2u + 1$ .* Let

$$S(u, x, y) = \frac{(u+x-1)(x+y)!}{(u+x+y)!} + \frac{u! x! y!}{(u+x)! (u+y)!}.$$

Then we have

$$(2.6) \quad v! H(u, v, x, y) - u! F(u, u, x, y) > S(u, x, y) - \frac{v! x! y!}{(v+y)!} \left[ \frac{1}{(u+x-1)!} + \frac{(v+y)!}{(u+y)! (v+x-1)!} \right].$$

A simple calculation yields

$$(2.7) \quad S(u, x, y) \geq \frac{u! x! y!}{(u+x-1)! (u+y)!},$$

so that (2.6) and (2.7) lead to

$$(2.8) \quad v! H(u, v, x, y) - u! F(u, u, x, y) > x! y! \left[ \frac{u!}{(u+x-1)! (u+y)!} - \frac{v!}{(v+y)!} \left( \frac{1}{(u+x-1)!} + \frac{(v+y)!}{(u+y)! (v+x-1)!} \right) \right].$$

Since  $y \leq x-1$  and  $2u+1 \leq v$ , we get

$$(2.9) \quad \frac{(v+y)!}{(u+y)!(v+x-1)!} \leq \frac{1}{(u+x-1)!} \quad \text{and} \quad \frac{2v!}{(v+y)!} \leq \frac{u!}{(u+y)!}.$$

Using (2.9) we obtain

$$(2.10) \quad \frac{v!}{(v+y)!} \left( \frac{1}{(u+x-1)!} + \frac{(v+y)!}{(u+y)!(v+x-1)!} \right) \leq \frac{v!}{(v+y)!} \frac{2}{(u+x-1)!} \\ \leq \frac{u!}{(u+x-1)!(u+y)!}.$$

From (2.8), (2.10), and (2.5) we conclude

$$H(u, v, x, y) > \frac{u!}{v!} F(u, u, x, y) = \frac{u!}{2v!} \frac{H(u, u, x, y)}{u+x} \geq 0.$$

*Case 3:*  $0 < u < v \leq 2u$ . We define

$$G(t) = G_{u, v, x, y}(t) = \frac{\gamma(x+t) \gamma(y-t)}{\gamma(u+x+t) \gamma(v+y-t)} \quad (-x \leq t \leq y).$$

Since

$$G_{v, u, x, y}(t) = G_{u, v, x, y}(y-x-t),$$

we have the representation

$$(2.11) \quad H(u, v, x, y) = [(u+x) G_{u, v, x, y}(t) + (v+x) G_{v, u, x, y}(t)]_{t=0}^{t=y} \\ = [(u+x) G_{u, v, x, y}(t) - (v+x) G_{u, v, x, y}(t-x)]_{t=0}^{t=y} \\ = \left[ \left[ (u+v+x-s) G_{u, v, x, y} \left( t - \frac{v-s}{v-u} x \right) \right]_{s=u}^{s=v} \right]_{t=0}^{t=y} \\ = \int_0^y \int_u^v \left\{ -G'(\tau) + \left( \frac{x(u+x)}{v-u} + \frac{x(v-s)}{v-u} \right) G''(\tau) \right\} ds dt,$$

where

$$\tau = t - \frac{(v-s)x}{v-u} \in [-x, y].$$

Direct computation yields

$$(2.12) \quad G''(t) = (X^2 + Y) G(t),$$

where

$$(2.13) \quad X = \frac{G'(t)}{G(t)}$$

and

$$(2.14) \quad Y = \left( \frac{G'(t)}{G(t)} \right)' = \psi'(x+t) - \psi'(x+t+u) + \psi'(y-t) - \psi'(y-t+v).$$

Since  $\psi'$  is decreasing, we conclude from (2.14) that  $Y \geq 0$ , so that (2.12) leads to

$$(2.15) \quad G''(t) \geq 0.$$

From (2.11) and (2.15) we get the estimate

$$H(u, v, x, y) \geq \int_0^y \int_u^v \left\{ -G'(\tau) + \frac{x(u+x)}{v-u} G''(\tau) \right\} ds dt.$$

Thus, to prove (2.4) it suffices to show that for  $\tau \in [-x, y]$ :

$$(2.16) \quad 0 < -G'(\tau) + \frac{x(u+x)}{v-u} G''(\tau) = m(\tau), \text{ say.}$$

Let  $A = \frac{x(u+x)}{v-u}$ , and let  $X$  and  $Y$  be as given in (2.13) and (2.14) (with  $\tau$  instead of  $t$ ). Then we have

$$(2.17) \quad \begin{aligned} m(\tau) & \frac{A}{G(\tau)} + \frac{1}{4} \\ & = \left( AX - \frac{1}{2} \right)^2 + A^2 Y \\ & \geq A^2 Y \\ & = \left( \frac{x(u+x)}{v-u} \right)^2 [\psi'(x+\tau) - \psi'(x+\tau+u) + \psi'(y-\tau) - \psi'(y-\tau+v)] \\ & = P_{u, v, x, y}(\tau), \text{ say.} \end{aligned}$$

We have  $u < v \leq 2u$  and  $y < x$ . Since  $\psi'$  is decreasing and  $\psi''$  is increasing on  $[0, \infty)$ , we get

$$(2.18) \quad \begin{aligned} P_{u,v,x,y}(\tau) &\geq \left(\frac{x(u+x)}{u}\right)^2 [\psi'(x+\tau) - \psi'(x+\tau+u) + \psi'(x-\tau) - \psi'(x-\tau+u)] \\ &= Q_{u,x}(\tau), \text{ say.} \end{aligned}$$

The function  $\tau \mapsto Q_{u,x}(\tau)$  is even on  $[-x, x]$ , and since  $\psi'''$  is decreasing on  $[0, \infty)$ , we verify that it is also convex. Hence, it attains its minimum at  $\tau = 0$ . This leads to

$$(2.19) \quad \begin{aligned} Q_{u,x}(\tau) &\geq Q_{u,x}(0) \\ &= 2 \left(\frac{x(u+x)}{u}\right)^2 [\psi'(x) - \psi'(x+u)] \\ &= 2 \left(\frac{x(u+x)}{u}\right)^2 \sum_{k=1}^{\infty} \left(\frac{1}{(x+k)^2} - \frac{1}{(x+k+u)^2}\right) \\ &\geq 2 \left(\frac{x(u+x)}{u}\right)^2 \left(\frac{1}{(x+1)^2} - \frac{1}{(x+1+u)^2}\right) \\ &\geq 2 \frac{x^2}{(x+1)^2} \\ &> \frac{1}{4}. \end{aligned}$$

From (2.17)–(2.19) we conclude the validity of (2.16) for  $\tau \in [-x, y]$ . This completes the proof of Theorem 1. ■

### 3. MAIN RESULT

We are now in a position to prove the following extension of inequality (1.2).

**THEOREM 2.** *Let  $n$  and  $k$  be integers with  $1 \leq k \leq n-1$ . Then we have for all positive real numbers  $x$ :*

$$(3.1) \quad \left(\frac{(n-k)! \varphi_n^{(k)}(x)}{x^{n-k}}\right)^\alpha < \varphi_n(x) < \left(\frac{(n-k)! \varphi_n^{(k)}(x)}{x^{n-k}}\right)^\beta,$$

with the best possible constants

$$\alpha = 1 \quad \text{and} \quad \beta = \binom{2n-k}{n}.$$

*Proof.* The left-hand inequality of (3.1) with  $\alpha = 1$  is an immediate consequence of the identity

$$\varphi_n(x) - \frac{(n-k)! \varphi_n^{(k)}(x)}{x^{n-k}} = (n-k)! \sum_{\nu=1}^{\infty} \left( \prod_{\mu=1}^{n-k} \left( 1 + \frac{n\nu}{\mu} \right) - 1 \right) \frac{x^{n\nu}}{(n(\nu+1)-k)!}.$$

To prove the second inequality of (3.1) with  $\beta = \binom{2n-k}{n}$ , we define

$$f(x) = \beta \log(n-k)! + \beta \log \varphi_n^{(k)}(x) - \beta(n-k) \log x - \log \varphi_n(x).$$

We have to show that  $f(x) > 0$  for  $x > 0$ . Since  $\lim_{t \rightarrow 0} f(t) = 0$ , it is sufficient to prove for  $x > 0$ :

$$f'(x) = \beta \frac{\varphi_n^{(k+1)}(x)}{\varphi_n^{(k)}(x)} - \frac{\beta(n-k)}{x} - \frac{\varphi_n'(x)}{\varphi_n(x)} > 0,$$

or, equivalently,

$$(3.2) \quad \beta x \varphi_n^{(n)}(x) \varphi_n^{(k+1)}(x) - \beta(n-k) \varphi_n^{(n)}(x) \varphi_n^{(k)}(x) - x \varphi_n'(x) \varphi_n^{(k)}(x) > 0.$$

Expanding the expression on the left-hand side into a power series around 0, inequality (3.2) can be written as

$$(3.3) \quad \sum_{m=3}^{\infty} \sum_{\nu=1}^{m-1} \frac{a_{\nu} + a_{m-\nu}}{2} x^{m-k} > 0,$$

where

$$a_{\nu} = a_{\nu}(m, n, k) = \frac{n\nu\beta}{(n(m-\nu-1))! (n(\nu+1)-k)!} - \frac{1}{(n\nu-k)! (n(m-\nu)-1)!}.$$

If we set

$$u = n(\nu-1), \quad v = n(m-\nu-1), \quad \text{and} \quad y = n-k,$$

then we obtain

$$(3.4) \quad (u+n) F(u, v, n, y) + (v+n) F(v, u, n, y) = (a_{\nu} + a_{m-\nu}) n!(n-k),$$



where  $F$  is given in (2.1). Since  $m \geq 3$ ,  $1 \leq v \leq m-1$ , and  $1 \leq k \leq n-1$  imply  $u \geq 0$ ,  $v \geq 0$ ,  $(u, v) \neq (0, 0)$ , and  $1 \leq y \leq n-1$ , we conclude from Theorem 1 and (3.4) that

$$a_v + a_{m-v} > 0 \quad (v = 1, \dots, m-1)$$

which implies (3.3).

It remains to show that the constants  $\alpha = 1$  and  $\beta = \binom{2n-k}{n}$  are best possible in (3.1). Let

$$(3.5) \quad g(x) = \frac{\log \varphi_n(x)}{\log \frac{(n-k)! \varphi_n^{(k)}(x)}{x^{n-k}}};$$

then double-inequality (3.1) is equivalent to

$$(3.6) \quad \alpha < g(x) < \beta \quad (x > 0).$$

We have

$$g(x) = \frac{\log(1+s(x))}{\log(1+t(x))},$$

with

$$s(x) = \sum_{v=1}^{\infty} \frac{x^{nv}}{(nv)!} \quad \text{and} \quad t(x) = (n-k)! \sum_{v=1}^{\infty} \frac{x^{nv}}{(nv+n-k)!}.$$

Hence, we obtain

$$(3.7) \quad \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} \frac{s'(x)}{t'(x)} = \binom{2n-k}{n}.$$

If  $1 \leq v \leq n-1$ , then we have  $\lim_{x \rightarrow \infty} \exp((\varepsilon_n^v - 1)x) = 0$ . This implies

$$\lim_{x \rightarrow \infty} \frac{\varphi_n^{(r)}(x)}{\varphi_n^{(r-1)}(x)} = \lim_{x \rightarrow \infty} \frac{1 + \sum_{v=1}^{n-1} \exp((\varepsilon_n^v - 1)x) \varepsilon_n^{vr}}{1 + \sum_{v=1}^{n-1} \exp((\varepsilon_n^v - 1)x) \varepsilon_n^{v(r-1)}} = 1 \quad (r = 1, \dots, n),$$

which leads to

$$(3.8) \quad \lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} \frac{\varphi_n'(x)}{\varphi_n(x)} \frac{1}{\frac{\varphi_n^{(k+1)}(x)}{\varphi_n^{(k)}(x)} \frac{n-k}{x}} = 1.$$

From (3.6)–(3.8) we conclude that in (3.1) the constants  $\alpha = 1$  and  $\beta = \binom{2n-k}{n}$  are sharp, which completes the proof of Theorem 2. ■

*Remark.* Numerous computer calculations have led to the conjecture that the function  $g$ , as defined in (3.5), is strictly decreasing on  $[0, \infty)$ .

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