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A subadditive property of the gamma function

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Abstract

Let $\Delta = \min_{x \geqslant 0} \Gamma(2x)/\Gamma(x)$ and $\alpha^* = \log 2/\log \Delta = -0.946850\ldots$. We prove that the function $x \mapsto (\Gamma(x))^\alpha$ is subadditive on $(0,\infty)$ if and only if $\alpha^* \leqslant \alpha \leqslant 0$.

1. Introduction

A function $f:(0,\infty) \to \mathbf{R}$ is said to be subadditive, if we have for all positive real numbers x and y

$$f(x+y) \leqslant f(x) + f(y).$$

If the inequality is reversed, then f is called superadditive. These functions play an important role in the theory of differential equations, in the study of semi-groups, and also in the theory of convex bodies; see [24]. The basic properties of sub- and superadditive functions can be found in [6,8–11,15,23–25,27].

Subadditivity problems are also discussed in number theory. We recall the well-known (still unsettled) conjecture due to Hardy and Littlewood, which states that

$$\pi(x+y) \leqslant \pi(x) + \pi(y)$$

for all integers $x, y \geqslant 2$. Here, $\pi(x)$ denotes the number of primes not exceeding x. See [21,22].

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In the recent past, many papers appeared in the literature providing remarkable inequalities involving the classical gamma function,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \quad (x > 0). \tag{1.1}$$

A summary of the most interesting inequalities for the gamma and related functions and a detailed list of references on this subject is given in [13].

Let

$$R_\alpha(x, y) = \frac{(\Gamma(x+y))^\alpha}{(\Gamma(x))^\alpha + (\Gamma(y))^\alpha}.$$

From

$$\lim_{x\to 0} R_1(x, y) = 0, \qquad \lim_{x\to\infty} R_1(x, y) = \infty,$$

$$\lim_{x\to 0} R_{-1}(x, x) = 1, \qquad \frac{d}{dx} R_{-1}(x, x)\bigg|_{x=0} = 0.57721\ldots, \qquad \lim_{x\to\infty} R_{-1}(x, x) = 0,$$

we immediately conclude that $x \mapsto \Gamma(x)$ and $x \mapsto (\Gamma(x))^{-1}$ are neither subadditive nor superadditive on $(0, \infty)$. In view of this result it is natural to ask: do there exist real parameters α such that $x \mapsto (\Gamma(x))^\alpha$ is subadditive or superadditive on $(0, \infty)$? It is the main aim of this paper to answer this question. We establish that $x \mapsto (\Gamma(x))^\alpha$ is subadditive on $(0, \infty)$ if and only if $\alpha^* \leqslant \alpha \leqslant 0$, where $\alpha^* = \log 2/\log \Delta = -0.946850\ldots$. Here, $\Delta = \min_{x\geqslant 0} \Gamma(2x)/\Gamma(x) = 0.480919\ldots$. Further, we show that there does not exist a number β such that $x \mapsto (\Gamma(x))^\beta$ is superadditive on $(0, \infty)$. In order to prove these results we need several lemmas, which we present in the next section.

The numerical values given in Sections 2–4 have been calculated via 'Mathematica 4.0' and 'Maple V Release 5.1.'

2. Lemmas

In what follows, we denote by $\psi = \Gamma'/\Gamma$ the logarithmic derivative of the gamma function and by $c = 1.46163\ldots$ the only positive zero of ψ.

Lemma 2.1. *For all integers $n \geqslant 1$ and all real numbers $x > 0$ we have*

$$(-1)^{n+1}\psi^{(n)}(x) = \int_0^\infty e^{-xt} \frac{t^n}{1 - e^{-t}}\, dt. \tag{2.1}$$

Formula (2.1) and related properties of $\psi^{(n)}$ can be found in [1, p. 260].

Lemma 2.2. *We have*

$$\min_{x\geqslant 0} \frac{\Gamma(2x)}{\Gamma(x)} = 0.4809194\ldots.$$

Proof. Let $x > 0$ and $\delta(x) = \Gamma(2x)/\Gamma(x)$. Since

$$\lim_{x\to 0} \delta(x) = 1/2 \quad \text{and} \quad \lim_{x\to\infty} \delta(x) = \infty,$$

we conclude that there exists a number $\bar{x} \geqslant 0$ such that

$$\delta(x) \geqslant \delta(\bar{x}) \quad \text{for all } x \geqslant 0.$$

We have

$$\delta(0.145034) = 0.48091946\ldots \tag{2.2}$$

and we prove that

$$\delta(x) \geqslant 0.4809194 \quad \text{for all } x \geqslant 0, \tag{2.3}$$

so that (2.2) and (2.3) imply $\delta(\bar{x}) = 0.4809194\ldots$.

Using the integral formula (2.1) with $n = 1$ we obtain

$$\big(\log \delta(x)\big)'' = 4\psi'(2x) - \psi'(x) = \int_0^\infty e^{-(x+1/2)t} \frac{t}{1 - e^{-t}}\, dt > 0.$$

This implies that δ is strictly convex on $(0, \infty)$. Let $x_0 = 0.145034$. Then we have

$$\delta'(x_0) < 0. \tag{2.4}$$

Further, since $\delta'(0.1450) < 0 < \delta'(0.1451)$ we obtain

$$0.1450 < \bar{x} < 0.1451. \tag{2.5}$$

The convexity of δ and the inequalities (2.4) and (2.5) lead to

$$\delta(\bar{x}) \geqslant (\bar{x} - x_0)\delta'(x_0) + \delta(x_0) \geqslant (0.1451 - x_0)\delta'(x_0) + \delta(x_0) = 0.48091946\ldots.$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *For all integers $n \geqslant 2$ and all real numbers $x > 0$ we have*

$$\frac{n+1}{n} < \frac{\psi^{(n-1)}(x)\psi^{(n+1)}(x)}{(\psi^{(n)}(x))^2} < \frac{n}{n-1}. \tag{2.6}$$

Both bounds are sharp.

A proof of Lemma 2.3 is given in [5].

Lemma 2.4. *Let $u(x) = -x\psi(x)$. There exists a number $x^* = 0.216098\ldots$ such that u' is positive on $(0, x^*)$ and negative on (x^*, ∞). Further, we have*

$$0 < (-1)^{n+1} u^{(n)}(x) \quad \textit{for } n = 2, 3, \ldots \textit{ and } x > 0.$$

Lemma 2.4 is proved in [2] and [17].

Lemma 2.5. *For all positive real numbers $x > 0$ we have*

$$60\psi''(x) + x^3\psi^{(5)}(x) < 0. \tag{2.7}$$

Proof. Using (2.1) and the convolution theorem for Laplace transforms we get for $x > 0$

$$\frac{60}{x^3}\psi''(x) + \psi^{(5)}(x) = -30\int_0^\infty e^{-xt}t^2\,dt \int_0^\infty e^{-xt}\frac{t^2}{1-e^{-t}}\,dt + \int_0^\infty e^{-xt}\frac{t^5}{1-e^{-t}}\,dt$$
$$= \int_0^\infty e^{-xt}\lambda(t)\,dt,$$

where

$$\lambda(t) = -30\int_0^t \frac{s^2(t-s)^2}{1-e^{-s}}\,ds + \frac{t^5}{1-e^{-t}}.$$

We have

$$\lambda(0) = \lambda'(0) = \lambda''(0) = 0$$

and

$$-t^{-3}(1-e^{-t})^4e^{3t}\lambda'''(t) = t^2 + 15t + 60 + e^t[4t^2 - 120] + e^{2t}[t^2 - 15t + 60]$$
$$= \mu(t), \quad \text{say.}$$

Then we get

$$\mu(0) = \mu'(0) = 0, \qquad \mu''(0) = 72,$$

and

$$\frac{1}{4}e^{-t}\mu'''(t) = t^2 + 6t - 24 + 2e^t[t^2 - 12t + 39] = \nu(t), \quad \text{say.}$$

From

$$\nu(0) = 54 \quad \text{and} \quad \nu'(t) = 2t + 6 + 2e^t\left[2 + (t-5)^2\right] > 0 \quad \text{for } t > 0,$$

we conclude that $\lambda'''(t) < 0$ for $t > 0$, which implies that λ is negative on $(0, \infty)$. This proves (2.7) for $x > 0$. □

Lemma 2.6. *The function*

$$v(x) = \frac{6}{x^2} - \frac{\psi'''(x)}{\psi'(x)}$$

is strictly decreasing on $(0, \infty)$ *with* $v(0) = \pi^2$ *and* $v(\infty) = 0$.

Proof. From

$$\psi'(x) = \psi'(x+1) + \frac{1}{x^2} = \frac{1}{x} + \frac{1}{2x^2} + O\left(\frac{1}{x^2}\right) \quad (x \to \infty)$$

and

$$\psi'''(x) = \psi'''(x+1) + \frac{6}{x^4} = \frac{2}{x^3} + O\left(\frac{1}{x^3}\right) \quad (x \to \infty)$$

(see [1, p. 260]), we obtain

$$v(x) = \frac{6\psi'(x+1) - x^2\psi'''(x+1)}{x^2\psi'(x+1)+1} = \frac{6}{x^2} - \frac{2/x^2 + O(1/x^2)}{1+O(1/x)}.$$

This leads to

$$\lim_{x\to 0} v(x) = 6\psi'(1) = \pi^2 \quad \text{and} \quad \lim_{x\to\infty} v(x) = 0.$$

Let $x > 0$. An application of the arithmetic mean–geometric mean inequality gives

$$\begin{aligned}-v'(x)\big(\psi'(x)\big)^2 - \psi'(x)\psi^{(4)}(x) &= \frac{12}{x^3}\big(\psi'(x)\big)^2 - \psi''(x)\psi'''(x)\\ &\geqslant 2\psi'(x)\sqrt{-\frac{12}{x^3}\psi''(x)\psi'''(x)}.\end{aligned} \tag{2.8}$$

From Lemmas 2.3 and 2.5 we obtain

$$\big(\psi^{(4)}(x)\big)^2 < \frac{4}{5}\psi'''(x)\psi^{(5)}(x) < -\frac{48}{x^3}\psi''(x)\psi'''(x).$$

This leads to

$$-\psi'(x)\psi^{(4)}(x) < 2\psi'(x)\sqrt{-\frac{12}{x^3}\psi''(x)\psi'''(x)}, \tag{2.9}$$

so that (2.8) and (2.9) yield $v'(x) < 0$. □

Lemma 2.7. *The function $q(x) = 4x^3\psi'(x)\psi''(x) + 2x^4(\psi''(x))^2$ is strictly increasing on $(0,\infty)$.*

Proof. Let $x > 0$. Differentiation gives

$$q'(x) = 4x^2\sigma(x)\tau(x),$$

where

$$\sigma(x) = \psi'(x) + x\psi''(x) \quad \text{and} \quad \tau(x) = 3\psi''(x) + x\psi'''(x).$$

We prove that σ and τ are negative on $(0,\infty)$. Applying (2.1) we get

$$\sigma(x) = x\int_0^\infty e^{-xt}\phi(t)\,dt \quad \text{and} \quad \tau(x) = x\int_0^\infty e^{-xt}\chi(t)\,dt,$$

where

$$\phi(t) = \int_0^t \frac{s}{1-e^{-s}}\,ds - \frac{t^2}{1-e^{-t}} \quad \text{and} \quad \chi(t) = -3\int_0^t \frac{s^2}{1-e^{-s}}\,ds + \frac{t^3}{1-e^{-t}}.$$

We have

$$\phi(0) = 0, \qquad t^{-1}e^t(1-e^{-t})^2\phi'(t) = 1 + t - e^t < 0 \quad \text{for } t > 0,$$

and

$$\chi(0) = 0, \qquad \chi'(t) = -\frac{t^3 e^{-t}}{(1-e^{-t})^2} < 0 \quad \text{for } t > 0.$$

This implies that ϕ and χ are negative on $(0, \infty)$, which leads to $\sigma(x) < 0$ and $\tau(x) < 0$ for $x > 0$. □

Lemma 2.8. *Let $n \geqslant 1$ be an integer. The function $x \mapsto x^{n+1}|\psi^{(n)}(x)|$ is strictly increasing on $(0, \infty)$.*

An extended version of Lemma 2.8 is proved in [3].

Lemma 2.9. *Let $n \geqslant 1$ be an integer. Then we have for all $x > 0$*

$$n < -x\frac{\psi^{(n+1)}(x)}{\psi^{(n)}(x)} < n+1.$$

Both bounds are sharp.

A proof is given in [3]. The following lemma, which might be of independent interest, plays a crucial role in the proof of our main result. It provides a concavity/convexity property of $x \mapsto x\psi^{(n+1)}(x)/\psi^{(n)}(x)$ with $n = 0$. The monotonicity behavior of this function with $n \geqslant 1$ has been studied in [3].

Lemma 2.10. *The function $h(x) = x\psi'(x)/\psi(x)$ is strictly concave on $(0, c)$ and strictly convex on (c, ∞).*

Proof. Let u and x^* be defined as in Lemma 2.4. To show that $h''(x)$ is negative for $x \in (0, c)$ we distinguish two cases.

Case 1. $x \in (x^*, c)$.

Let

$$\eta(x) = x^2\psi'(x) \quad \text{and} \quad \theta(x) = \frac{1}{u(x)} = -\frac{1}{x\psi(x)}.$$

We prove that

$$\eta > 0, \quad \eta' \geqslant 0, \quad \eta'' > 0 \quad \text{and} \quad \theta > 0, \quad \theta' > 0, \quad \theta'' > 0,$$

which leads to

$$-h'' = (\eta\theta)'' = \eta''\theta + 2\eta'\theta' + \eta\theta'' > 0.$$

From Lemmas 2.1 and 2.8 we conclude that $\eta > 0$ and $\eta' \geqslant 0$. Further, we get

$$\frac{1}{x^2}\eta''(x) = \frac{2}{x^2}\psi'(x) + \frac{4}{x}\psi''(x) + \psi'''(x) = \int_0^\infty e^{-xt}\omega(t)\,dt,$$

where

$$\omega(t) = 2t\int_0^t \frac{s}{1-e^{-s}}\,ds - 6\int_0^t \frac{s^2}{1-e^{-s}}\,ds + \frac{t^3}{1-e^{-t}}.$$

We have

$$\omega(0) = \omega'(0) = 0 \quad \text{and}$$

$$t^{-2}e^{2t}(1-e^{-t})^3\omega''(t) = 2t + (t-2)(e^t-1) = \sum_{k=3}^{\infty}(k-2)\frac{t^k}{k!}.$$

This implies that ω and η'' are positive on $(0,\infty)$.

The functions u and θ are positive on $(0,c)$. From Lemma 2.4 we obtain that $u''(x) < 0$ for $x > 0$. Hence, we get for $x \in (x^*, c)$

$$u'(x) < u'(x^*) = 0, \qquad \theta'(x) = -\frac{u'(x)}{(u(x))^2} > 0, \quad \text{and}$$

$$\theta''(x) = \frac{1}{(u(x))^3}\bigl[2\bigl(u'(x)\bigr)^2 - u(x)u''(x)\bigr] > 0.$$

Case 2. $x \in (0, x^*]$.

We have $h(x) = -1/(u(x)w(x))$, where $w(x) = 1/\eta(x)$. From Lemma 2.4 we obtain

$$u > 0, \quad u' \geqslant 0, \quad u'' < 0 \quad \text{on } (0, x^*], \tag{2.10}$$

and we prove that

$$w > 0, \quad w' \leqslant 0, \quad w'' < 0 \quad \text{on } (0, x^*], \tag{2.11}$$

so that (2.10) and (2.11) lead to

$$h'' = (uw)^{-3}\bigl[-2\bigl((uw)'\bigr)^2 + uw(u''w + 2u'w' + uw'')\bigr] < 0.$$

The Lemmas 2.1 and 2.8 imply that $w > 0$ and $w' \leqslant 0$. Let v and q be defined as in Lemmas 2.6 and 2.7, and let

$$p(x) = -6x^2\bigl(\psi'(x)\bigr)^2 + x^4\psi'(x)\psi'''(x) = -v(x)\bigl(\eta(x)\bigr)^2.$$

Then we get

$$-\bigl(\eta(x)\bigr)^3 w''(x) = p(x) - q(x). \tag{2.12}$$

Let $0 \leqslant a < x \leqslant b \leqslant x^*$. Applying Lemmas 2.6, 2.7, and 2.8 we obtain

$$p(x) - q(x) \geqslant -v(a)\bigl(\eta(b)\bigr)^2 - q(b) = G(a,b), \quad \text{say.}$$

Since

$$G(0, 0.1) = 0.577\ldots, \qquad G(0.1, 0.15) = 0.847\ldots, \qquad G(0.15, x^*) = 0.101\ldots,$$

we conclude that $p - q$ is positive on $(0, x^*]$. Thus, (2.12) implies that $w''(x) < 0$ for $x \in (0, x^*]$. It remains to show that $h''(x) > 0$ for $x > c$. We have the representation

$$\big(\psi(x)\big)^3 h''(x) = 2x\big(\psi'(x)\big)^3 + \big(\psi(x)\big)^2 \psi''(x)\left[2 + x\frac{\psi'''(x)}{\psi''(x)}\right] - \psi(x)\big(\psi'(x)\big)^2\left[2 + 3x\frac{\psi''(x)}{\psi'(x)}\right]. \tag{2.13}$$

From Lemma 2.9 we obtain for $x > 0$

$$2 + x\frac{\psi'''(x)}{\psi''(x)} < 0 \quad \text{and} \quad 2 + 3x\frac{\psi''(x)}{\psi'(x)} < -1,$$

so that (2.13) yields $(\psi(x))^3 h''(x) > 0$ for $x > c$. This completes the proof of Lemma 2.10. □

3. Main result

We are now in a position to prove our main result.

Theorem. *Let α be a real number and $\Delta = \min_{x\geqslant 0}(\Gamma(2x)/\Gamma(x)) = 0.4809194\ldots$. The inequality*

$$\big(\Gamma(x+y)\big)^\alpha \leqslant \big(\Gamma(x)\big)^\alpha + \big(\Gamma(y)\big)^\alpha \tag{3.1}$$

holds for all positive real numbers x and y if and only if

$$-0.946850\ldots = \frac{\log 2}{\log \Delta} \leqslant \alpha \leqslant 0.$$

Proof. First, we assume that (3.1) is valid for all $x, y > 0$. If $\alpha > 0$, then we obtain

$$\Gamma(2x)/\Gamma(x) \leqslant 2^{1/\alpha} \quad (x > 0).$$

This contradicts the limit relation

$$\lim_{x\to\infty} \Gamma(2x)/\Gamma(x) = \infty.$$

Hence, we have $\alpha \leqslant 0$. If α is negative, then (3.1) yields

$$2^{1/\alpha} \leqslant \min_{x\geqslant 0}\big(\Gamma(2x)/\Gamma(x)\big) = \Delta,$$

which is equivalent to

$$\log 2/\log \Delta \leqslant \alpha.$$

Next, we suppose that $\log 2/\log \Delta \leqslant \alpha < 0$. It is known that the function

$$t \mapsto (u^t + v^t)^{1/t} \quad (u, v > 0)$$

is decreasing on $(-\infty, 0)$; see [7, p. 18]. This implies that it suffices to prove (3.1) for $\alpha^* = \log 2/\log \Delta$. We consider two cases.

Case 1. $y > c$.

Since Γ is increasing on $[c, \infty)$ we get

$$\big(\Gamma(x+y)\big)^{\alpha^*} < \big(\Gamma(y)\big)^{\alpha^*} < \big(\Gamma(x)\big)^{\alpha^*} + \big(\Gamma(y)\big)^{\alpha^*}.$$

Case 2. $y \leqslant c$.

Let $Q = \{(x, y) \in \mathbf{R}^2 \mid 0 \leqslant x \leqslant y \leqslant c\}$ and

$$f(x, y) = \big(\Gamma(x)\big)^{\alpha^*} + \big(\Gamma(y)\big)^{\alpha^*} - \big(\Gamma(x+y)\big)^{\alpha^*},$$

where $(x, y) \in Q$. Since f is continuous on Q there exist real numbers $\tilde{x}$ and $\tilde{y}$ such that $(\tilde{x}, \tilde{y}) \in Q$ and $f(x, y) \geqslant f(\tilde{x}, \tilde{y})$ for all $(x, y) \in Q$. We have to show that $f(\tilde{x}, \tilde{y}) \geqslant 0$. If $(\tilde{x}, \tilde{y})$ is an interior point of Q, then we get

$$\frac{\partial f(x, y)}{\partial x}\bigg|_{(x,y)=(\tilde{x},\tilde{y})} = \alpha^*\Big[\big(\Gamma(\tilde{x})\big)^{\alpha^*}\psi(\tilde{x}) - \big(\Gamma(\tilde{x}+\tilde{y})\big)^{\alpha^*}\psi(\tilde{x}+\tilde{y})\Big] = 0$$

and

$$\frac{\partial f(x, y)}{\partial y}\bigg|_{(x,y)=(\tilde{x},\tilde{y})} = \alpha^*\Big[\big(\Gamma(\tilde{y})\big)^{\alpha^*}\psi(\tilde{y}) - \big(\Gamma(\tilde{x}+\tilde{y})\big)^{\alpha^*}\psi(\tilde{x}+\tilde{y})\Big] = 0.$$

Hence,

$$F(\tilde{x}) = F(\tilde{y}) = F(\tilde{x}+\tilde{y}), \tag{3.2}$$

where

$$F(x) = \alpha^*\big(\Gamma(x)\big)^{\alpha^*}\psi(x) \quad \text{and} \quad 0 < \tilde{x} < \tilde{y} < c.$$

If $\tilde{x}+\tilde{y} \geqslant c$, then we have $F(\tilde{x}+\tilde{y}) \leqslant 0 < F(\tilde{x})$. Therefore, we may assume that $\tilde{x}+\tilde{y} < c$.

Let

$$g(x) = \alpha^* x\psi(x) + x\psi'(x)/\psi(x).$$

From Lemmas 2.4 and 2.10 we conclude that g is strictly concave on $(0, c)$. We have

$$g(0.06) = -0.0028\ldots, \qquad g(0.07) = 0.0024\ldots, \qquad g(0.22) = 0.00007\ldots,$$
$$g(0.23) = -0.0048\ldots.$$

This implies that g has precisely two zeros on $(0, c)$, which we denote by $x_1 \in (0.06, 0.07)$ and $x_2 \in (0.22, 0.23)$. Thus, g is negative on $(0, x_1) \cup (x_2, c)$ and positive on (x_1, x_2). Moreover, since $g'(0.14) = 0.0010\ldots$ and $g'(0.15) = -0.061\ldots$, it follows that there exists a number $x_3 \in (0.14, 0.15)$ such that g is strictly increasing on $(0, x_3]$ and strictly decreasing on $[x_3, c)$.

The representation $F'(x) = F(x)g(x)/x$ implies that F is strictly decreasing on $(0, x_1]$, strictly increasing on $[x_1, x_2]$, and strictly decreasing on $[x_2, c)$. From (3.2) and $0 < \tilde{x} < \tilde{y} < \tilde{x}+\tilde{y} < c$ we obtain

$$0 < \tilde{x} \leqslant x_1 \leqslant \tilde{y} \leqslant x_2 \leqslant \tilde{x}+\tilde{y} < c.$$

Hence, we have

$$0 < \tilde{x} < 0.07 \quad \text{and} \quad 0.22 < \tilde{x}+\tilde{y},$$

which implies

$$\tilde{y} = (\tilde{x}+\tilde{y}) - \tilde{x} > 0.15 \quad \text{and} \quad F(\tilde{y}) > F(0.15) = 1.1777\ldots.$$

Since

$$F(0.03) = 1.1772\ldots < F(\tilde{y}) = F(\tilde{x})$$

we get

$$\tilde{x} < 0.03 \quad \text{and} \quad \tilde{x} + \tilde{y} < 0.26.$$

The strict monotonicity of g gives

$$g(\tilde{x}) < g(0.03) = -0.0239\ldots \quad \text{and} \quad g(\tilde{x} + \tilde{y}) > g(0.26) = -0.0228\ldots.$$

This leads to

$$\begin{aligned}(\tilde{x} + \tilde{y})g(\tilde{x}) - \tilde{x}g(\tilde{x} + \tilde{y}) &< (\tilde{x} + \tilde{y})g(0.03) - \tilde{x}g(0.26)\\ &< \tilde{x}\bigl[g(0.03) - g(0.26)\bigr] < 0,\end{aligned}$$

so that we obtain

$$\begin{aligned}\frac{\partial^2 f(x,y)}{\partial x^2}\bigg|_{(x,y)=(\tilde{x},\tilde{y})} &= F(\tilde{x})\frac{g(\tilde{x})}{\tilde{x}} - F(\tilde{x} + \tilde{y})\frac{g(\tilde{x} + \tilde{y})}{\tilde{x} + \tilde{y}}\\ &= \frac{F(\tilde{x})}{\tilde{x}(\tilde{x} + \tilde{y})}\bigl[(\tilde{x} + \tilde{y})g(\tilde{x}) - \tilde{x}g(\tilde{x} + \tilde{y})\bigr] < 0,\end{aligned}$$

which implies that f does not attain its minimum at $(\tilde{x}, \tilde{y})$. Thus, $(\tilde{x}, \tilde{y})$ is a boundary point of Q.

If $\tilde{x} = 0$, then we get $f(\tilde{x}, \tilde{y}) = 0$. Let $0 < x \leqslant c$; then we have $\psi(x) \leqslant 0 < \psi(x + c)$, which leads to

$$\frac{\partial f(x,c)}{\partial x} = \alpha^*\bigl[\bigl(\Gamma(x)\bigr)^{\alpha^*}\psi(x) - \bigl(\Gamma(x + c)\bigr)^{\alpha^*}\psi(x + c)\bigr] > 0.$$

Thus, if $\tilde{y} = c$, then

$$f(\tilde{x}, \tilde{y}) \geqslant f(0, c) = 0.$$

And, if $\tilde{x} = \tilde{y}$, then we obtain

$$f(\tilde{x}, \tilde{y}) = \bigl(\Gamma(\tilde{x})\bigr)^{\alpha^*}\bigl[2 - \bigl(\Gamma(2\tilde{x})/\Gamma(\tilde{x})\bigr)^{\alpha^*}\bigr] \geqslant \bigl(\Gamma(\tilde{x})\bigr)^{\alpha^*}[2 - \Delta^{\alpha^*}] = 0.$$

This completes the proof of the Theorem. □

4. Concluding remarks

(1) There does not exist a real number β such that $x \mapsto (\Gamma(x))^\beta$ is superadditive on $(0, \infty)$. Otherwise, we have

$$\bigl(\Gamma(x)\bigr)^\beta + \bigl(\Gamma(y)\bigr)^\beta \leqslant \bigl(\Gamma(x + y)\bigr)^\beta \quad (x, y > 0).$$

This implies for $\beta > 0$

$$\lim_{x\to 0}\bigl[\bigl(\Gamma(x)\bigr)^\beta + \bigl(\Gamma(y)\bigr)^\beta\bigr] = \infty \leqslant \bigl(\Gamma(y)\bigr)^\beta = \lim_{x\to 0}\bigl(\Gamma(x + y)\bigr)^\beta.$$

And, if $\beta < 0$, then we obtain

$$\lim_{x\to\infty}\left[\left(\Gamma(x)\right)^\beta + \left(\Gamma(y)\right)^\beta\right] = \left(\Gamma(y)\right)^\beta \leqslant 0 = \lim_{x\to\infty}\left(\Gamma(x+y)\right)^\beta.$$

(2) A function $f:(0,\infty)\to\mathbf{R}$ is called star-shaped if

$$f(ax) \leqslant af(x) \quad \text{for all } x>0 \text{ and for all } a\in(0,1).$$

Interesting properties of these functions can be found in [6,11]. Since a star-shaped function is also superadditive (see [19, p. 453]), it follows that for all real β the function $x \mapsto (\Gamma(x))^\beta$ is not star-shaped on $(0,\infty)$.

(3) If $f:(0,\infty)\to\mathbf{R}$ satisfies

$$f(xy) \leqslant f(x)f(y) \quad \text{for all } x,y>0, \tag{4.1}$$

then f is said to be submultiplicative. And, f is called supermultiplicative, if (4.1) holds with "$\geqslant$" instead of "$\leqslant$." These functions have applications in interpolation theory, functional analysis, and semi-group theory; see [12,14–16,18]. A submultiplicative property of the logarithmic derivative of the gamma function is proved in [4]. Since $\Gamma(4) > \Gamma(2)\Gamma(2)$ and $\Gamma(1) < \Gamma(2)\Gamma(1/2)$, we conclude that $x\mapsto(\Gamma(x))^\beta$ (with $\beta\neq 0$) is neither sub- nor supermultiplicative on $(0,\infty)$.

(4) The following theorem was recently proved in [20]:

Let h and k be positive real numbers. The inequality

$$\psi(x)\psi(x+h+k) < \psi(x+h)\psi(x+k)$$

holds for all $x>0$ if and only if $h+k\geqslant c$, where c is the only positive zero of ψ.

This is a counterpart of

$$\Gamma(x+h)\Gamma(x+k) < \Gamma(x)\Gamma(x+h+k) \quad (x,h,k>0), \tag{4.2}$$

which is a consequence of the fact that $x\mapsto \Gamma(x)\Gamma(x+h+k)/(\Gamma(x+h)\Gamma(x+k))$ is strictly decreasing on $(0,\infty)$ and converges to 1 if x tends to ∞; see [26]. An extension of this result is given in [2]. There exists an additive companion of (4.2).

The inequality

$$\left(\Gamma(x+y)\right)^\alpha + \left(\Gamma(x+z)\right)^\alpha \leqslant \left(\Gamma(x)\right)^\alpha + \left(\Gamma(x+y+z)\right)^\alpha \tag{4.3}$$

is valid for all positive real numbers x,y,z if and only if $\alpha\geqslant 0$. Moreover, if $\alpha>0$, then (4.3) *is strict.*

The proof is surprisingly simple. First, we assume that there exists a number $\alpha<0$ such that (4.3) holds. Then we let x tend to 0 and obtain

$$\left(\Gamma(y)\right)^\alpha + \left(\Gamma(z)\right)^\alpha \leqslant \left(\Gamma(y+z)\right)^\alpha. \tag{4.4}$$

And, if y tends to ∞, then (4.4) yields $(\Gamma(z))^\alpha \leqslant 0$. Conversely, let $\alpha>0$ and

$$f_\alpha(x,y,z) = \left(\Gamma(x)\right)^\alpha + \left(\Gamma(x+y+z)\right)^\alpha - \left(\Gamma(x+y)\right)^\alpha - \left(\Gamma(x+z)\right)^\alpha.$$

Then we get

$$\frac{\partial f_\alpha(x,y,z)}{\partial z} = \alpha\big[g_\alpha(x+y+z) - g_\alpha(x+z)\big], \tag{4.5}$$

where $g_\alpha(t) = (\Gamma(t))^\alpha \psi(t)$. Since

$$\frac{\partial g_\alpha(t)}{\partial t} = \big(\Gamma(t)\big)^\alpha\big[\alpha\big(\psi(t)\big)^2 + \psi'(t)\big] > 0 \quad \text{for } t > 0,$$

we conclude from (4.5) that $\partial f_\alpha(x,y,z)/\partial z > 0$. Hence, $f_\alpha(x,y,z) > f_\alpha(x,y,0) = 0$.

There does not exist a real number $\beta \neq 0$ such that the inequality

$$\big(\Gamma(x)\big)^\beta + \big(\Gamma(x+y+z)\big)^\beta \leqslant \big(\Gamma(x+y)\big)^\beta + \big(\Gamma(x+z)\big)^\beta$$

is valid for all positive real numbers x, y, z. Otherwise, we have $\beta < 0$ and $f_\beta(x,y,z) \leqslant 0 = f_\beta(x,y,0)$. This implies

$$\left.\frac{\partial f_\beta(x,y,z)}{\partial z}\right|_{z=0} \leqslant 0.$$

Hence, we get

$$g_\beta(x) \leqslant g_\beta(x+y) \quad (x, y > 0). \tag{4.6}$$

Since $\lim_{t\to\infty} g_\beta(t) = 0$, we conclude from (4.6) that $g_\beta(x) \leqslant 0$, which is false for $x > c$.

(5) We conclude this paper with a refinement of the well-known inequality

$$\Gamma\left(\sum_{i=1}^n p_i x_i\right) \leqslant \sum_{i=1}^n p_i \Gamma(x_i),$$

where $x_i, p_i > 0$ $(i = 1, \ldots, n)$, $\sum_{i=1}^n p_i = 1$.

Let p_i $(i = 1, \ldots, n; n \geqslant 2)$ be positive real numbers with $\sum_{i=1}^n p_i = 1$. Then we have for all $x_i > 0$ $(i = 1, \ldots, n)$

$$a \sum_{1\leqslant i<j\leqslant n} p_i p_j (x_i - x_j)^2 \leqslant \sum_{i=1}^n p_i \Gamma(x_i) - \Gamma\left(\sum_{i=1}^n p_i x_i\right) \tag{4.7}$$

with the best possible constant factor

$$a = \frac{1}{2}\min_{x>0} \Gamma''(x) = 0.37922\ldots.$$

From (1.1) we obtain for $x > 0$

$$\Gamma^{(4)}(x) = \int_0^\infty e^{-t} t^{x-1} (\log t)^4 \, dt > 0,$$

which implies that Γ''' is strictly increasing on $(0, \infty)$. Since

$$\Gamma'''(1.7410) < 0 < \Gamma'''(1.7411),$$

there exists a number $\hat{x} \in (1.7410, 1.7411)$ such that

$$\Gamma''(x) \geqslant \Gamma''(\hat{x}) \quad \text{for all } x > 0.$$

Let $\hat{y} = 1.741$. The convexity of Γ'' and $\Gamma'''(\hat{y}) < 0$ imply

$$\Gamma''(\hat{x}) \geqslant (\hat{x} - \hat{y})\Gamma'''(\hat{y}) + \Gamma''(\hat{y}) \geqslant (1.7411 - \hat{y})\Gamma'''(\hat{y}) + \Gamma''(\hat{y}) = 0.758458\ldots.$$

From

$$0.758458 \leqslant \Gamma''(\hat{x}) \leqslant \Gamma''(\hat{y}) = 0.7584584\ldots$$

we obtain $(1/2)\min_{x>0} \Gamma''(x) = 0.37922\ldots.$

If we set $a = (1/2)\min_{x>0} \Gamma''(x)$, then $x \mapsto \Gamma(x) - ax^2$ is convex on $(0, \infty)$. Applying Jensen's inequality we obtain (4.7).

Furthermore, from (4.7) with $x_1 = \cdots = x_{n-1} = x$, $x_n = y$ $(x \neq y)$, and $p_n = p$ we get

$$a \leqslant \frac{(1-p)\Gamma(x) + p\Gamma(y) - \Gamma((1-p)x + py)}{p(1-p)(x-y)^2}.$$

We let x tend to y and obtain

$$a \leqslant \frac{1}{2}\Gamma''(y) \quad \text{for } y > 0.$$

This implies that the largest constant factor in (4.7) is given by $a = (1/2)\min_{x>0} \Gamma''(x)$.

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