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Note

# A converse of Minkowski's inequality

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**Abstract**

The following converse of the classical Minkowski inequality was proved by H. Tôyama in 1948.

Let $r,s$ be real numbers with $0<r<s$. Then we have for all real numbers $A_{ij}\geqslant 0$ ($i=1,\ldots,m$; $j=1,\ldots,n$), which are not all equal to 0:

$$\frac{[\sum_{j=1}^{n}(\sum_{i=1}^{m} A_{ij}^{s})^{r/s}]^{1/r}}{[\sum_{i=1}^{m}(\sum_{j=1}^{n} A_{ij}^{r})^{s/r}]^{1/s}} \leqslant (\min(m,n))^{1/r-1/s}. \qquad (*)$$

The bound is sharp.

In this note we give a short and simple proof for a weighted version of $(*)$. 

*Keywords:* Minkowski's inequality; Tôyama's inequality; Converse inequality

The classical inequality

$$\left(\sum_{i=1}^{m}(a_i+b_i)^s\right)^{1/s} \leqslant \left(\sum_{i=1}^{m} a_i^s\right)^{1/s} + \left(\sum_{i=1}^{m} b_i^s\right)^{1/s}, \qquad (1)$$

which holds for all real numbers $s>1$, $a_i\geqslant 0$, and $b_i\geqslant 0$ ($i=1,\ldots,m$), was published by Minkowski [3, pp. 115–117] in his famous book 'Geometrie der Zahlen'. A proof of Eq. (1) as well as several extensions, related results, and interesting geometrical interpretations can be found in [1, pp. 147–159; 2, pp. 30–39].

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The following extension of Minkowski's inequality is due to A.E. Ingham and B. Jessen:

Let $r$, $s$ be real numbers with $0<r<s$, and let $u_i$ $(i=1,\ldots,m)$, $v_j$ $(j=1,\ldots,n)$ be positive real numbers. Then we have for all real numbers $a_{ij}\geqslant 0$ $(i=1,\ldots,m;$ $j=1,\ldots,n)$, which are not all equal to 0:

$$1\leqslant\frac{[\sum_{j=1}^{n} v_j(\sum_{i=1}^{m} u_i a_{ij}^s)^{r/s}]^{1/r}}{[\sum_{i=1}^{m} u_i(\sum_{j=1}^{n} v_j a_{ij}^r)^{s/r}]^{1/s}}. \tag{2}$$

An elegant proof of (2), which is based on Hölder's inequality, is given in [2, pp. 31–32]. A simple calculation reveals that (2) with $n=2$, $r=1$, $u_i=1$ $(i=1,\ldots,m)$ and $v_1=v_2=1$ leads to (1). If we set $a_{ij}=1$ $(i=1,\ldots,m; j=1,\ldots,n)$, then equality holds in (2). This implies that the constant 1 is the best possible lower bound.

We note that the weights $u_i$ and $v_j$ can be eliminated by a simple transformation. Indeed, if we define

$$A_{ij}=u_i^{1/s}v_j^{1/r}a_{ij} \qquad (i=1,\ldots,m;\ j=1,\ldots,n), \tag{3}$$

then inequality (2) is equivalent to the special case

$$1\leqslant\frac{[\sum_{j=1}^{n}(\sum_{i=1}^{m} A_{ij}^s)^{r/s}]^{1/r}}{[\sum_{i=1}^{m}(\sum_{j=1}^{n} A_{ij}^r)^{s/r}]^{1/s}}. \tag{4}$$

In 1948, Tôyama [4] published the following remarkable converse of (4):

$$\frac{[\sum_{j=1}^{n}(\sum_{i=1}^{m} A_{ij}^s)^{r/s}]^{1/r}}{[\sum_{i=1}^{m}(\sum_{j=1}^{n} A_{ij}^r)^{s/r}]^{1/s}}\leqslant(\min(m,n))^{1/r-1/s}. \tag{5}$$

Inequality (5) is valid for all positive real numbers $r$ and $s$ with $r<s$ and for all real numbers $A_{ij}\geqslant 0$ $(i=1,\ldots,m;\ j=1,\ldots,n)$, which are not all equal to 0. Equality holds in (5) if $A_{ii}=1$ $(i=1,\ldots,\min(m,n))$ and all other $A_{ij}$ are equal to 0.

Tôyama used analytical and combinatorial ideas in his quite complicated proof of inequality (5). It is the only one we could locate in the literature. In this note we present a short and simple proof for a weighted version, which is only based on the following well-known monotonicity properties of power means and power sums: if $0<r<s$ and $x_i\geqslant 0$ $(i=1,\ldots,N)$, then

$$\left(\frac{1}{N}\sum_{i=1}^{N} x_i^r\right)^{1/r}\leqslant\left(\frac{1}{N}\sum_{i=1}^{N} x_i^s\right)^{1/s} \tag{6}$$

and

$$\left(\sum_{i=1}^{N} x_i^s\right)^{1/s}\leqslant\left(\sum_{i=1}^{N} x_i^r\right)^{1/r}. \tag{7}$$

Elementary short proofs for (6) and (7) are given in [1, p. 143, pp. 159–167; 2, pp. 26–30].

A combination of (2) and a weighted version of (5) can be presented in a symmetrical form as follows.

**Theorem 1.** *Let* $r$, $s$ *and* $u_i$ $(i=1,\dots,m)$, $v_j$ $(j=1,\dots,n)$ *be positive real numbers. Then we have for all real numbers* $a_{ij}\geqslant 0$ $(i=1,\dots,m;\ j=1,\dots,n)$, *which are not all equal to* 0:

$$\min(1,k^{1/r-1/s})\leqslant\frac{[\sum_{j=1}^{n}v_j(\sum_{i=1}^{m}u_i a_{ij}^{s})^{r/s}]^{1/r}}{[\sum_{i=1}^{m}u_i(\sum_{j=1}^{n}v_j a_{ij}^{r})^{s/r}]^{1/s}}\leqslant\max(1,k^{1/r-1/s}),\tag{8}$$

*where* $k=\min(m,n)$. *Both bounds are sharp.*

**Proof.** In view of (2) it suffices to establish the right-hand side of (8) for $0<r<s$. We assume that $m\leqslant n$; the proof for $m>n$ is similar. We denote the ratio in (8) by $R$. Using transformation (3) we obtain

$$R=\frac{[\sum_{j=1}^{n}(\sum_{i=1}^{m}A_{ij}^{s})^{r/s}]^{1/r}}{[\sum_{i=1}^{m}(\sum_{j=1}^{n}A_{ij}^{r})^{s/r}]^{1/s}}.$$

Let

$$x_j=\left(\sum_{i=1}^{m}A_{ij}^{s}\right)^{1/s}\qquad(j=1,\dots,n)$$

and

$$y_i=\left(\sum_{j=1}^{n}A_{ij}^{r}\right)^{1/r}\qquad(i=1,\dots,m).$$

From (7) we get

$$x_j^r\leqslant\sum_{i=1}^{m}A_{ij}^{r}\quad(j=1,\dots,n),\tag{9}$$

so that (9) and (6) imply

$$\begin{aligned}\left(\sum_{j=1}^{n}x_j^r\right)^{1/r}\leqslant\left(\sum_{j=1}^{n}\sum_{i=1}^{m}A_{ij}^{r}\right)^{1/r}&=\left(\frac{1}{m}\sum_{i=1}^{m}y_i^r\right)^{1/r}m^{1/r}\\&\leqslant\left(\frac{1}{m}\sum_{i=1}^{m}y_i^s\right)^{1/s}m^{1/r}\\&=\left(\sum_{i=1}^{m}y_i^s\right)^{1/s}m^{1/r-1/s}.\end{aligned}$$

This leads to

$$R=\frac{(\sum_{j=1}^{n}x_j^r)^{1/r}}{(\sum_{i=1}^{m}y_i^s)^{1/s}}\leqslant(\min(m,n))^{1/r-1/s},$$

which we had to show. If we set

$$a_{ii} = u_i^{-1/s} v_i^{-1/r} \quad (i = 1, \ldots, \min(m,n))$$

and all other $a_{ij}$ equal to 0, then equality holds on the right-hand side of (8).

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