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Journal of Approximation Theory

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Full length article

Rogosinski–Szegö type inequalities for trigonometric sums

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Received 9 September 2013; received in revised form 25 March 2014; accepted 10 April 2014

Communicated by Guillermo López Lagomasino

Abstract

We prove that the inequalities

$$\sum_{k=1}^{n} \frac{\sin(kx)}{k+1} \geq \frac{1}{384}(9-\sqrt{137})\sqrt{110-6\sqrt{137}} = -0.044419686\ldots$$

and

$$\sum_{k=1}^{n} \frac{\sin(kx)+\cos(kx)}{k+1} \geq -\frac{1}{2}$$

are valid for all real numbers $x \in [0, \pi]$ and all positive integers n. The constant lower bounds are sharp. Our theorems complement a classical result of Rogosinski and Szegö, who proved in 1928 that the inequality

$$\sum_{k=1}^{n} \frac{\cos(kx)}{k+1} \geq -\frac{1}{2}$$

holds for all $x \in [0, \pi]$ and $n \geq 1$.

MSC: 26D05; 42A05

Keywords: Trigonometric sums; Inequalities; Sharp bounds

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http://dx.doi.org/10.1016/j.jat.2014.04.007
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1. Introduction

The famous Fejér–Jackson–Gronwall inequality

$$\sum_{k=1}^{n} \frac{\sin(kx)}{k} \geq 0 \quad (0 \leq x \leq \pi;\ n \geq 1) \tag{1.1}$$

is a classical result in the theory of trigonometric polynomials. The validity of (1.1) was conjectured by Fejér in 1910. The first proofs were published by Jackson [12] and Gronwall [11] in 1911 and 1912, respectively. In 1913, Young [21] offered an analogue of (1.1) for cosine sums:

$$\sum_{k=1}^{n} \frac{\cos(kx)}{k} \geq -1 \quad (0 \leq x \leq \pi;\ n \geq 1). \tag{1.2}$$

The constants given in (1.1) and (1.2) are sharp.

The inequalities of Fejér–Jackson–Gronwall and Young attracted the attention of many researchers, who presented not only new proofs but also numerous refinements and related results as well as extensions in various directions. Furthermore, it was shown that inequalities for trigonometric sums and polynomials play an important role in the theory of univalent functions as well as in other fields. For more information on this subject we refer to [5,8], [17, Chapter 4], and the references therein.

The results in this paper have been inspired by an elegant companion of (1.2), which was proved by Rogosinski and Szegö [18] in 1928, namely,

$$C_n(x) = \sum_{k=1}^{n} \frac{\cos(kx)}{k+1} \geq -\frac{1}{2} \quad (0 \leq x \leq \pi;\ n \geq 1). \tag{1.3}$$

Since $C_1(\pi) = -1/2$, it follows that the constant $-1/2$ is sharp. In 2001, Koumandos [14] presented an improvement of (1.3). He demonstrated that the lower bound $-1/2$ can be replaced by $-41/96$ if $n \geq 2$. Moreover, he offered the following converse of (1.3):

$$\sum_{k=1}^{n} \frac{\cos(kx)}{k+1} \leq \mathrm{Ci}(\pi/2) + \sum_{k=1}^{\infty} \frac{\cos(kx)}{k+1} \quad (0 \leq x \leq \pi;\ n \geq 1) \tag{1.4}$$

with

$$\mathrm{Ci}(\pi/2) = -\int_{\pi/2}^{\infty} \frac{\cos(t)}{t} dt = 0.472\ldots.$$

This is the best possible constant in (1.4).

Interesting extensions of (1.3) are given in [6,10,18]. See also [17, Chapter 4.1.3].

In view of (1.1), (1.2), and (1.3) it is natural to look for a sharp lower bound for the sine sum

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k+1},$$

which holds for all $x \in [0, \pi]$ and all $n \geq 1$. In Section 3, we solve this problem. More precisely, we determine the best possible constants c_1 and c_2 such that the inequalities

$$S_n(x) \geq c_1 \quad \text{and} \quad S_n(x) + C_n(x) \geq c_2$$

are valid for all $x \in [0, \pi]$ and all $n \geq 1$.

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In the final part, we present some additional inequalities. Among others, we provide sharp lower bounds for trigonometric sums in two variables.

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2. Lemmas

In order to prove our theorems we need two lemmas. The first one is due to Fejér [9].

Lemma 1. *Let $a_1, a_2, \ldots, a_n$ be a decreasing sequence of nonnegative real numbers satisfying*

$$2a_k \le a_{k-1} + a_{k+1} \quad \text{for } k = 2, 3, \ldots, n-1.$$

Then we have

$$\sum_{k=1}^{n-1} a_k \sin(kx) + \frac{1}{2} a_n \sin(nx) \ge 0 \quad (0 \le x \le \pi).$$

The second lemma was proved by Brown and Hewitt [6].

Lemma 2. *Let $b_1, b_2, \ldots$ be a decreasing sequence of positive real numbers satisfying*

$$b_{2k} \le \frac{2k}{2k+1} b_{2k-1} \quad \text{for } k = 1, 2, \ldots.$$

Then, for $n \ge 1$, we have

$$\sum_{k=1}^{2n-1} b_k \sin(kx) > 0 \quad (0 < x < \pi)$$

and

$$\sum_{k=1}^{2n} b_k \sin(kx) > 0 \quad \big(0 < x \le \pi - \pi/(2n)\big). \tag{2.1}$$

Remark. Let $b_k = 1/(k+1)$ $(k \ge 1)$. Since

$$0 < b_{k+1} < b_k \quad \text{and} \quad b_{2k} = \frac{2k}{2k+1} b_{2k-1} = \frac{1}{2k+1},$$

we conclude from Lemma 2 that

$$S_n(x) > 0 \quad (0 < x < \pi; n = 1, 3, 5, \ldots). \tag{2.2}$$

3. Main results

In the previous section, we pointed out that if $n \ge 1$ is an odd integer, then S_n is positive on $(0, \pi)$. However, if n is even, then S_n attains positive and negative values on $(0, \pi)$. Our first result offers the best possible constant lower bound for $S_n(x)$, which is valid for all $x \in [0, \pi]$ and all $n \ge 1$. We prove the following counterpart of the Rogosinski–Szegö inequality (1.3).

Theorem 1. *For all real numbers $x \in [0, \pi]$ and all integers $n \ge 1$ we have*

$$\sum_{k=1}^{n} \frac{\sin(kx)}{k+1} \ge \frac{1}{384}(9 - \sqrt{137})\sqrt{110 - 6\sqrt{137}} = -0.044419686\ldots. \tag{3.1}$$

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The sign of equality holds in (3.1) if and only if

$$x = \arccos \frac{-3-\sqrt{137}}{16} = 2.736443821\ldots \quad \text{and} \quad n = 2.$$

Proof. With regard to (2.2) it suffices to show that (3.1) is valid for all $x \in [0, \pi]$ and all even integers $n \geq 2$. We have

$$S_2(x) = \frac{\sin(x)}{2} + \frac{\sin(2x)}{3}$$

and

$$S_2'(x) = \frac{4}{3}(\cos(x) - x_1)(\cos(x) - x_2)$$

with

$$x_1 = \frac{-3+\sqrt{137}}{16} = 0.54\ldots \quad \text{and} \quad x_2 = \frac{-3-\sqrt{137}}{16} = -0.91\ldots.$$

This implies

$$S_2'(x) > 0 \quad \text{for } x \in (0, \arccos(x_1)) \cup (\arccos(x_2), \pi)$$

and

$$S_2'(x) < 0 \quad \text{for } x \in (\arccos(x_1), \arccos(x_2)).$$

Since $S_2(0) = S_2(\pi) = 0$, we obtain for all $x \in [0, \pi]$ with $x \neq \arccos(x_2)$:

$$S_2(x) > S_2(\arccos(x_2)) = \frac{1}{384}(9-\sqrt{137})\sqrt{110-6\sqrt{137}} = -0.0444\ldots.$$

Next, we consider the cases $n = 4, 6, 8, 10$. Let

$$S_n^*(x) = S_n(x) - \frac{44}{1000}\sin((n+2)x).$$

Each sum can be expanded using MAPLE into an expression of the form $\sin(x)$ multiplied with a polynomial of $\cos(x)$. For example,

$$S_4^*(x) = \sin(x)\left(\frac{1}{4} - \frac{149}{375}Y + Y^2 + \frac{376}{125}Y^3 - \frac{176}{125}Y^5\right),$$

where $Y = \cos(x)$. We can use the Sturm theory (see [15] and [19, Chapter 79]) and MAPLE to verify that the polynomial has no root in $[-1, 1]$. Here, we offer the details for the case $n = 4$. Let

$$f(x) = \frac{1}{4} - \frac{149}{375}x + x^2 + \frac{376}{125}x^3 - \frac{176}{125}x^5.$$

Then, $f(-1) = 0.04\ldots$ and $f(1) = 2.45\ldots$. We set

$$X_0 = f(x), \qquad X_1 = f'(x) = -\frac{149}{375} + 2x + \frac{1128}{125}x^2 - \frac{176}{25}x^4.$$

Using Euclid's algorithm we conclude that there exist polynomials Q_j and X_j such that

$$X_0 = Q_1 X_1 - X_2,$$
$$X_1 = Q_2 X_2 - X_3,$$

$$X_2 = Q_3 X_3 - X_4,$$
$$X_3 = Q_4 X_4 - X_5,$$
$$X_4 = Q_5 X_5.$$

These polynomials are

$$Q_1(x) = \frac{1}{5}x, \qquad Q_2(x) = -\frac{103125}{35344} + \frac{275}{47}x,$$
$$Q_3(x) = -\frac{357348449970000}{5148121478668921} + \frac{79736064}{358752055}x,$$
$$Q_4(x) = -\frac{1042367548255476205511239110431208 75}{5247212994057230886768934625050854 4} - \frac{3693798319724228146765 31}{1619235590124410613094 4}x,$$
$$Q_5(x) = -\frac{19694428658883890258351401886060998416031964160 00}{16830136821903979543287956883081821221407460291091} - \frac{27206165055092534911385221088255142684271348183 04}{16830136821903979543287956883081821221407460291091}x,$$
$$X_2(x) = -\frac{1}{4} + \frac{596}{1875}x - \frac{3}{5}x^2 - \frac{752}{625}x^3,$$
$$X_3(x) = \frac{59736899}{53016000} - \frac{9698}{2209}x - \frac{71750411}{13254000}x^2,$$
$$X4(x) = \frac{884380447687744}{5148121478668921} + \frac{458135918437191776}{1930545554500845375}x,$$
$$X_5(x) = -\frac{32691802032331833511876218089937 71}{22269181448296305823204509782640 00} = -1.46\ldots.$$

We denote by $\omega(a)$ the number of sign changes in the sequence

$$X_0(a),\ X_1(a),\ X_2(a),\ X_3(a),\ X_4(a),\ X_5(a).$$

Setting $a = -1$ and $a = 1$, respectively, we obtain the sequences

$$0.04\ldots,\ -0.41\ldots,\ 0.03\ldots,\ 0.10\ldots,\ -0.06\ldots,\ -1.46\ldots$$

and

$$2.45\ldots,\ 3.58\ldots,\ -1.73\ldots,\ -8.67\ldots,\ 0.40\ldots,\ -1.46\ldots.$$

This yields $\omega(-1) = \omega(1) = 3$. It follows that the number of distinct roots of f in $[-1, 1]$ is equal to $\omega(-1) - \omega(1) = 0$.

Hence, $S_4^*(x) \geq 0$ implying that

$$S_4(x) \geq \frac{44}{1000}\sin(6x) \geq -\frac{44}{1000} = -0.044.$$

This settles (3.1) for $n = 4$. The same method also works for $n = 6, 8, 10$.

Let $n \geq 12$. Applying Lemma 1 with $a_k = 1/(k+1)$ leads to

$$S_n(x) = \sum_{k=1}^{n-1}\frac{\sin(kx)}{k+1} + \frac{\sin(nx)}{2(n+1)} + \frac{\sin(nx)}{2(n+1)} \geq \frac{\sin(nx)}{2(n+1)} \geq -\frac{1}{26} = -0.038\ldots.$$

The proof of Theorem 1 is complete. □

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From (1.3) and (3.1) we conclude that the function $T_n(x) = S_n(x) + C_n(x)$ satisfies

$$T_n(x) \geq -0.5444\ldots \quad (0 \leq x \leq \pi;\ n \geq 1). \tag{3.2}$$

Is the given constant lower bound sharp? Our second theorem reveals that the answer is "no". Indeed, we show that the number given in (3.2) can be replaced by $-1/2$, which is the best possible constant.

Theorem 2. *For all real numbers $x \in [0, \pi]$ and all integers $n \geq 1$ we have*

$$\sum_{k=1}^{n} \frac{\sin(kx) + \cos(kx)}{k+1} \geq -\frac{1}{2}. \tag{3.3}$$

The sign of equality holds in (3.3) if and only if $x = \pi$ and $n = 1$.

Proof. We have $T_n(0) > 0$ and

$$T_n(\pi) = \sum_{k=1}^{n} (-1)^k \frac{1}{k+1} \geq -\frac{1}{2}$$

with equality if and only if $n = 1$. Thus, it suffices to prove

$$T_n(x) > -\frac{1}{2} \quad (0 < x < \pi;\ n \geq 1). \tag{3.4}$$

Let $x \in (0, \pi)$. If n is odd, then we conclude from (1.3) and (2.2) that (3.4) is valid. Next, we assume that n is even. We consider two cases.

Case 1. $0 < x \leq \pi - \pi/n$.

Applying (1.3) and (2.1) with $b_k = 1/(k+1)$ leads to (3.4).

Case 2. $\pi - \pi/n < x < \pi$.

Let

$$\Theta_n(x) = \frac{1}{2} + T_n(\pi - x) = \frac{1}{2} + C_n^*(x) + S_n^*(x), \tag{3.5}$$

where

$$C_n^*(x) = \sum_{k=1}^{n} (-1)^k \frac{\cos(kx)}{k+1} \quad \text{and} \quad S_n^*(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\sin(kx)}{k+1}.$$

It is enough to prove

$$\Theta_n(x) > 0 \quad \text{for } x \in (0, \pi/n). \tag{3.6}$$

If $0 < x \leq \arccos(3/4) = 0.72\ldots$, then

$$\sqrt{2}\,\Theta_2'''(x) = \frac{16}{3}\sin(\pi/4 + 2x) - \sin(\pi/4 + x) \geq \frac{16}{3}\sin(\pi/4) - 1 = 2.77\ldots.$$

Thus,

$$\Theta_2'(x) \leq \max\{\Theta_2'(0),\ \Theta_2'(\arccos(3/4))\} = -0.03\ldots < 0 \quad \text{and}$$
$$\Theta_2(x) \geq \Theta_2(\arccos(3/4)) = 0.16\ldots.$$

If $\arccos(3/4) \leq x < \pi/2$, then

$$\Theta_2(x) = \frac{2}{3}(\cos(x) - 3/8)^2 + \frac{2}{3}\sin(x)(3/4 - \cos(x)) + \frac{7}{96} > 0.$$

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This settles (3.6) for $n = 2$. For $n = 4, 6, \dots, 18$, we can verify (3.6) by applying the Sturm theory as follows.

We use $n = 4$ for illustration. $C_4^*(x)$ can be expanded into a polynomial of $Y = \cos(x)$, but $S_4^*(x)$ can only be expanded into a polynomial of Y multiplied by $\sin(x)$. We obtain

$$\Theta_4(x) = P_1(Y) + P_2(Y)\sin(x)$$

with

$$P_1(Y) = \frac{11}{30} + \frac{1}{4}Y - \frac{14}{15}Y^2 - Y^3 + \frac{8}{5}Y^4,$$
$$P_2(Y) = \frac{1}{4} + \frac{2}{15}Y + Y^2 - \frac{8}{5}Y^3, \quad \text{and} \quad Y = \cos(x).$$

We assume (for a contradiction) that there exists a number $x_0 \in [0, \pi/4]$ such that $\Theta_4(x_0) = 0$. Then,

$$P_1(Y_0) + P_2(Y_0)\sin(x_0) = 0 \quad \text{with } Y_0 = \cos(x_0).$$

Since P_1 is positive on $[\cos(\pi/4), 1]$, we obtain $P_2(y_0)\sin(x_0) < 0$. Thus, $P_2(y_0) < 0$. Using

$$\sin(x_0) \le \sin(\pi/4) < 0.807$$

gives

$$P_1(Y_0) + (0.807)P_2(Y_0) < P_1(Y_0) + P_2(Y_0)\sin(x_0) = 0. \tag{3.7}$$

The function

$$\Theta_4^*(Y) = P_1(Y) + (0.807)P_2(Y)$$

is a polynomial with $\Theta_4^*(1) = 0.10\dots$. Applying Sturm's theory reveals that Θ_4^* has no zero on $[\cos(\pi/4), 1]$. It follows that Θ_4^* is positive on $[\cos(\pi/4), 1]$. This contradicts (3.7). Therefore, Θ_4 has no zero on $[0, \pi/4]$, so that $\Theta_4(0) = 0.28\dots$ implies that (3.6) holds for $n = 4$. The same procedure can be used to prove (3.6) for n up to 18.

It remains to show that (3.6) holds for all even numbers $n \ge 20$. Let $0 < x < \pi/20$. We define for $k \in \mathbf{N}$:

$$\Delta_k(x) = \sum_{\nu=1}^{k} (-1)^\nu \cos(\nu x).$$

Then,

$$\Delta_k(x) = \frac{-\cos(x/2) + (-1)^k \cos((2k+1)x/2)}{2\cos(x/2)}$$
$$\ge \frac{-2}{2\cos(x/2)} > -\frac{1}{\cos(\pi/40)} = -1.003\dots. \tag{3.8}$$

Let

$$g(x) = \sum_{k=1}^{20} \left(\frac{1}{k+1} - \frac{1}{22}\right)(-1)^k \cos(kx).$$

Using the Sturm theory we can verify that $g(x) + 0.29$ has no root on $[0, \pi/20]$. Since $g(0) + 0.29 = 0.006\dots$, we obtain

$$g(x) > -0.29. \tag{3.9}$$

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We have

$$0.34 + C_n^*(x) = \sum_{k=1}^{n-19} a_k b_k \tag{3.10}$$

with

$$a_1 = 0.34 + C_{20}^*(x), \qquad a_k = (-1)^{k-1} \frac{\cos((k+19)x)}{22} \quad (k = 2, 3, \ldots, n-19),$$

and

$$b_1 = b_2 = 1, \qquad b_k = \frac{22}{k+20} \quad (k = 3, 4, \ldots, n-19).$$

Applying (3.8) and (3.9) leads to

$$A_k = \sum_{\nu=1}^{k} a_\nu = 0.34 + g(x) + \frac{1}{22} \Delta_{k+19}(x) > 0.34 - 0.29 - \frac{1.004}{22} = 0.0043\ldots. \tag{3.11}$$

From (3.10) and (3.11) we obtain by summation by parts

$$0.34 + C_n^*(x) = \sum_{\nu=1}^{n-19} A_\nu (b_\nu - b_{\nu+1}) + A_{n-19} b_{n-18}. \tag{3.12}$$

Since

$$b_1 \geq b_2 \geq \cdots \geq b_{n-19} > b_{n-18} = 0,$$

we conclude from (3.11) and (3.12) that

$$C_n^*(x) \geq -0.34. \tag{3.13}$$

Using (2.2) and (3.13) we get from (3.5):

$$\Theta_n(x) > \frac{1}{2} - 0.34 + S_n^*(x) = 0.16 + S_{n-1}(\pi - x) - \frac{\sin(nx)}{n+1} > 0.16 - \frac{1}{n+1} > 0.$$

This completes the proof of Theorem 2. □

4. Concluding remarks

From (2.2) we conclude that the sums $S_{2m-1}(x)$ and $S_{2m-1}(\pi - x)$ are positive for $x \in (0, \pi)$. This leads to

Corollary 1. *For all real numbers $\mu \in [-1, 1]$, $x \in (0, \pi)$ and all integers $m \geq 1$ we have*

$$\sum_{k=1}^{m} \frac{\sin((2k-1)x)}{2k} + \mu \sum_{k=1}^{m-1} \frac{\sin(2kx)}{2k+1} > 0. \tag{4.1}$$

An inequality closely related to (4.1) is given in [16].

Summation by parts yields

$$\sum_{k=1}^{2m+1} a_k \frac{\sin(kx)}{k+1} = \sum_{k=1}^{2m+1} S_k(x)(a_k - a_{k+1}) + S_{2m+1}(x) a_{2m+2}.$$

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We assume that $a_{2k} = a_{2k+1}$ for $k = 1, \dots, m$ and $a_{2m+2} = 0$. Then we have

$$\sum_{k=1}^{2m+1} a_k \frac{\sin(kx)}{k+1} = a_1 \frac{\sin(x)}{2} + \sum_{k=1}^{m} a_{2k+1}\Big(\frac{\sin(2kx)}{2k+1} + \frac{\sin((2k+1)x)}{2k+2}\Big)$$
$$= \sum_{\substack{k=1 \\ k\, odd}}^{2m+1} S_k(x)(a_k - a_{k+1}).$$

Applying this identity and (2.2) gives

Corollary 2. *If* $b_1 \ge b_2 \ge \cdots \ge b_{m+1} > 0$, *then, for* $x \in (0, \pi)$,

$$b_1 \frac{\sin(x)}{2} + \sum_{k=1}^{m} b_{k+1}\Big(\frac{\sin(2kx)}{2k+1} + \frac{\sin((2k+1)x)}{2k+2}\Big) > 0.$$

Setting $b_k = 1/\binom{2m+1}{k}$ $(k = 1, \dots, m+1)$ we obtain the following special case:

$$\frac{\sin(x)}{2} + \sum_{k=1}^{m} \frac{k+1}{\binom{2m}{k}}\Big(\frac{\sin(2kx)}{2k+1} + \frac{\sin((2k+1)x)}{2k+2}\Big) > 0 \quad (0 < x < \pi;\ m \ge 1).$$

Additional inequalities for trigonometric polynomials involving binomial coefficients are given in [2] and [3].

Inequality (2.2) and the product formula

$$2\sin(x)\cos(y) = \sin(x-y) + \sin(x+y) \tag{4.2}$$

lead to a lower bound for a trigonometric sum in two variables involving sine and cosine.

Corollary 3. *For all real numbers x and y with* $0 \le x - y \le \pi$ *and* $0 \le x + y \le \pi$ *and all odd integers* $n \ge 1$ *we have*

$$\sum_{k=1}^{n} \frac{\sin(kx)\cos(ky)}{k+1} \ge 0. \tag{4.3}$$

We denote the sum given in (4.3) by $U_n(x, y)$. Let $x_0, y_0 \in [0, \pi]$ and let $n \ge 1$ be odd.

Case 1. $0 \le x_0 + y_0 \le \pi$.

Let $y_0 \le x_0$. Then we have for $y \in [0, y_0]$:

$$0 \le x_0 - y \le \pi \quad \text{and} \quad 0 \le x_0 + y \le \pi.$$

Applying Corollary 3 gives

$$\int_0^{y_0} U_n(x_0, y)dy = \sum_{k=1}^{n} \frac{\sin(kx_0)\sin(ky_0)}{k(k+1)} \ge 0.$$

This also holds for $y_0 \ge x_0$.

Case 2. $\pi \le x_0 + y_0 \le 2\pi$.

Let $x_0 \le y_0$. Then, for $y \in [y_0, \pi]$:

$$0 \le y - x_0 \le \pi \le x_0 + y \le 2\pi.$$

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If $0 \le a \le \pi \le b \le 2\pi$, then (2.2) leads to $S_n(-a) \le 0$ and $S_n(b) \le 0$. This yields

$$\frac{1}{2}\int_{y_0}^{\pi} [S_n(x_0 - y) + S_n(x_0 + y)]dy = \int_{y_0}^{\pi} U_n(x_0, y)dy = -\sum_{k=1}^{n} \frac{\sin(kx_0)\sin(ky_0)}{k(k+1)} \le 0.$$

The last inequality is also valid if $x_0 \ge y_0$. Hence, we obtain

Corollary 4. *For all real numbers $x \in [0, \pi]$ and $y \in [0, \pi]$ and all odd integers $n \ge 1$ we have*

$$\sum_{k=1}^{n} \frac{\sin(kx)\sin(ky)}{k(k+1)} \ge 0. \tag{4.4}$$

Inequality (4.4) is a striking companion of

$$\sum_{k=1}^{n} \frac{\sin(kx)\sin(ky)}{k^2} \ge 0, \tag{4.5}$$

which holds for all $x, y \in [0, \pi]$ and $n \ge 1$. This result is due to Koschmieder [13]. We note that on contrary to (4.5) inequality (4.4) is not valid if n is even.

Using (4.2) and Theorem 1 yields a sharp lower bound for $U_n(x, y)$, which is valid for all positive integers n. The following extension of Theorem 1 is valid.

Corollary 5. *For all real numbers x and y with $0 \le x - y \le \pi$ and $0 \le x + y \le \pi$ and all integers $n \ge 1$ we have*

$$\sum_{k=1}^{n} \frac{\sin(kx)\cos(ky)}{k+1} \ge \frac{1}{384}(9 - \sqrt{137})\sqrt{110 - 6\sqrt{137}} = -0.044419686\ldots. \tag{4.6}$$

The sign of equality holds in (4.6) if and only if

$$x = \arccos\frac{-3 - \sqrt{137}}{16} = 2.736443821\ldots, \qquad y = 0, \quad \text{and} \quad n = 2.$$

If we apply Theorem 2 and the identities (4.2) and

$$2\cos(x)\cos(y) = \cos(x - y) + \cos(x + y),$$

then we obtain

Corollary 6. *For all real numbers x and y with $0 \le x - y \le \pi$ and $0 \le x + y \le \pi$ and all integers $n \ge 1$ we have*

$$\sum_{k=1}^{n} \frac{\sin(kx)\cos(ky)}{k+1} + \sum_{k=1}^{n} \frac{\cos(kx)\cos(ky)}{k+1} \ge -\frac{1}{2}. \tag{4.7}$$

The sign of equality holds in (4.7) if and only if $x = \pi$, $y = 0$, and $n = 1$.

More inequalities for trigonometric sums in two variables can be found in [1,4,13,20].

We conclude with an inequality for infinite series involving binomial coefficients. A result of Chu [7] states that

$$\sum_{\nu=0}^{\infty} \binom{t}{\nu}\binom{-t}{\nu} = \frac{\sin(\pi t)}{\pi t}.$$

We set $t = kx$ with $x \in (0,1)$, multiply by $k/(k+1)$, and sum from $k = 1$ to $k = n$. Since convergent series may be added term by term, an application of (2.2) leads to

Corollary 7. *For all real numbers $x \in (0,1)$ and all odd integers $n \geq 1$ we have*

$$\sum_{\nu=0}^{\infty}\sum_{k=1}^{n} \frac{k}{k+1}\binom{kx}{\nu}\binom{-kx}{\nu} > 0.$$

Acknowledgments

The authors thank the referees for helpful comments. The research of second author is supported by the Hong Kong Government GRF Grant PolyU 5003/12P and the Hong Kong Polytechnic University Grants G-YK49 and G-U751.

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Please cite this article in press as: H. Alzer, M.K. Kwong, Rogosinski–Szegö type inequalities for trigonometric sums, Journal of Approximation Theory (2014), http://dx.doi.org/10.1016/j.jat.2014.04.007