Weakly weighted sharing and uniqueness of meromorphic functions

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Abstract

With the aid of the notion of weakly weighted sharing, we study the uniqueness of meromorphic functions sharing four pairs of small functions. Our results improve and generalize some results given by T. Czubiak and G. Gundersen, P. Li and C. C. Yang and other authors.

Key words: meromorphic function, value sharing, small function

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1. Introduction and main results

In this article, a meromorphic function means meromorphic in the whole complex plane $\mathbb{C}$. We assume the reader is familiar with the standard notion used in the Nevanlinna value distribution theory such as $T(r,f)$, $N(r,f)$, $m(r,f)$ and $\overline{N}(r,f)$ (see [4, 9]). For any nonconstant meromorphic function f, the term $S(r,f)$ denotes any quantity that satisfies $S(r,f) = o(1)(T(r,f))$ as $r \to \infty$ outside a possible exceptional set of finite linear measure.

Let f be a nonconstant meromorphic function. A meromorphic function a is called a *small function* of f, if $T(r,a) = S(r,f)$. If k is a positive integer, we denote by $\overline{N}_{k)}(r,f)$ the reduced counting function of the poles of f whose multiplicities are less than or equal to k, and denote by $\overline{N}_{(k}(r,f)$ the reduced counting function of the poles of f whose multiplicities are greater than or equal to k.

Let f and g be nonconstant meromorphic functions, and a, b be two values in $\mathbb{C}$. We say that f and g share the value a IM provided that $f(z) - a$ and $g(z) - a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share the value ∞ IM, if $1/f$ and $1/g$ share 0 IM. We say that f and g share the pair of values (a,b) IM provided that $f(z) - a$ and $g(z) - b$ have the same zeros ignoring multiplicities.

The following theorem is is a well known and significant result in the uniqueness theory of meromorphic functions and has been proved by Czubiak and Gundersen.

Theorem A ([2]). *Let f and g be two nonconstant meromorphic functions that share six pairs of values (a_i, b_i), $1 \le i \le 6$ IM, where $a_i \ne a_j$ whenever $i \ne j$ and $b_i \ne b_j$ whenever $i \ne j$. Then f is a Möbius transformation of g.*

The following example was found by Gundersen, shows that the number "six" in Theorem A cannot be replaced with "five".

Example 1 ([3]). Let $f(z) = (e^z+1)/(e^z-1)^2$, $g(z) = (e^z+1)^2/8(e^z-1)$. We see that f, g share $(0,0), (\infty,\infty), (1,1), (-1/8,-1/8)$ and $(-1/2, 1/4)$ IM, and f is not a Möbius transformation of g.

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Let f and g be nonconstant meromorphic functions and a, b be two small meromorphic functions of f and g. We denote by $\overline{N}(r, f = a, g = b)$ the reduced counting function of the common zeros of $f - a$ and $g - b$. We say that f and g share (a, b) IM*, if

$$\overline{N}(r, \tfrac{1}{f-a}) - \overline{N}(r, f = a, g = b) = S(r, f) \quad \text{and} \quad \overline{N}(r, \tfrac{1}{g-b}) - \overline{N}(r, f = a, g = b) = S(r, g).$$

As in Theorem A and throughout this article, when f and g are nonconstant meromorphic functions, we let $S(r)$ denote the term which is both $S(r, f)$ and $S(r, g)$ simultaneously.

We denote by $\overline{N}(r, f = a, g \neq b)$ the reduced counting function of those a-points of f, which are not the b-points of g. We note that f and g share (a, b) IM* if and only if $\overline{N}(r, f = a, g \neq b) = S(r)$ and $\overline{N}(r, g = b, f \neq a) = S(r)$. According to this note, we generalize the definitions of IM and IM* to the *weakly weighted IM sharing* which is given by the following definition:

Defintion 1 ([1]). Let k be a positive integer or infinity, and let a, b be two small functions of nonconstant meromorphic functions f and g. We denote by $\overline{N}_{k)}(r, f = a, g \neq b)$ the reduced counting function of those a-points of f whose multiplicities are less than or equal to k, that are not the b-points of g. If $\overline{N}_{k)}(r, f = a, g \neq b) + \overline{N}_{k)}(r, g = b, f \neq a) = S(r)$, we say that f and g share $(a, b, k)^*$ IM.

We note that, if f and g share $(a, b, k)^*$ IM, then f and g share $(a, b, p)^*$ IM, for all integer $1 \leq p \leq k$. Also, we note that if f and g share (a, b) IM* if and only if f and g share $(a, b, \infty)^*$ IM.

Recently, Li and Yang have proved the following:

Theorem B ([6]). *Let f and g be two nonconstant meromorphic functions, and $a_i,\ b_i\ (i = 1, \cdots, 6)$ be small functions of f and g, with $a_i \not\equiv a_j,\ b_i \not\equiv b_j$ whenever $i \neq j$. If f and g share the five pairs (a_i, b_i) IM*, $1 \leq i \leq 5$, and f is not a quasi-Möbius transformation of g, then the following identities or inequalities hold:*

(a) $T(r, f) = T(r, g) + S(r)$;

(b) $3T(r, f) = \sum\limits_{i=1}^{5} \overline{N}(r, \frac{1}{f-a_i}) + S(r)$;

(c) $T(r, f) \leq \overline{N}(r, \frac{1}{f-a_i}) + \overline{N}(r, \frac{1}{f-a_j}) + S(r),\ i \neq j,\ i, j = 1, \cdots, 5$;

(d) $T(r, f) \leq 3\overline{N}(r, \frac{1}{f-a_i}) + S(r),\ i = 1, \cdots, 5$;

(e) $T(r, f) = \overline{N}(r, \frac{1}{f-a_6}) + S(r)$;

(f) $\overline{N}(r, f = a_6, g = b_6) \leq \frac{3}{5}T(r, f) + S(r)$;

(g) $T(r, f) = \overline{N}(r, \frac{1}{f-a_5}) + S(r)$ *and* $T(r, f) = 2\overline{N}(r, \frac{1}{f-a_i}) + S(r)$ *for* $i = 1, \cdots, 4$ *if* $a_i \equiv b_i$, $i = 1, \cdots, 4$.

One may ask the following question: Is it possible to relax the condition "f and g share five pairs of small functions " in Theorem B to the condition "f and g share four pairs of small functions"?

The goal of the present paper is to generalize and improve the Theorems A and B by using the weakly weighted sharing. We now turn to state our results.

Theorem 1. *Let f and g be two nonconstant meromorphic functions, $a_i,\ b_i\ (i = 1, \cdots, 6)$ be small functions of f and g, with $a_i \not\equiv a_j,\ b_i \not\equiv b_j$ whenever $i \neq j$, and let $k_1, \cdots, k_5$ be five positive integers or infinity with $K \equiv \sum\limits_{i=1}^{5} \frac{1}{1+k_i} \leq 1$. Suppose that f and g share $(a_i, b_i, k_i)^*$ IM, $i = 1, \cdots, 4$, and f is not a quasi-Möbius transformation of g. If there exists a number $\lambda \in [0, 1 - K]$ such that*

$$\left.\begin{array}{l} \overline{N}(r, \frac{1}{f-a_6}) + \overline{N}_{k_5)}(r, f = a_5, g \neq b_5) \leq \lambda T(r, f) + S(r), \\ \overline{N}(r, \frac{1}{g-b_6}) + \overline{N}_{k_5)}(r, g = b_5, f \neq a_5) \leq \lambda T(r, g) + S(r), \end{array}\right\} \tag{1.1}$$

then $k_i = \infty$, *for all* $1 \le i \le 5$ *(that means,* f *and* g *share* $(a_i, b_i, \infty)^*$ *IM,* $i = 1, \cdots, 4$*), and the following identities or inequalities hold:*

(a) $T(r, f) = T(r, g) + S(r)$;

(b) $4T(r, f) = \sum_{i=1}^{6} \overline{N}(r, \frac{1}{f-a_i}) + S(r)$;

(c) $T(r, f) \le \overline{N}(r, f = a_i, g = b_i) + \overline{N}(r, f = a_j, g = b_j) + S(r),\ i \ne j,\ i, j = 1, \cdots, 5$;

(d) $T(r, f) \le 3\overline{N}(r, f = a_i, g = b_i) + S(r),\ i = 1, \cdots, 5$*;*

(e) $T(r, f) = \overline{N}(r, \frac{1}{f-a_6}) + \overline{N}(r, f = a_5, g \ne b_5) + S(r)$*;*

(f) $\overline{N}(r, f = a_6, g = b_6) \le \overline{N}(r, f = a_i, g = b_i) + S(r)$ *and*

$\overline{N}(r, f = a_i, g = b_i) \le 3T(r, f) - 4\overline{N}(r, f = a_6, g = b_6) + S(r),\ i = 1, \cdots, 6$*;*

(g) $T(r, f) = \overline{N}(r, f = a_5, g = b_5) + S(r)$ *and* $T(r, f) = 2\overline{N}(r, \frac{1}{f-a_i}) + S(r)$ *for* $i = 1, \cdots, 4$ *if*

$a_i \equiv b_i,\ i = 1, \cdots, 4$ *;*

(h) $T(r, f) = 2\overline{N}(r, f = a_5, g = a_5) + S(r),\ T(r, f) = \overline{N}(r, \frac{1}{f-a_4}) + S(r)$ *and*

$T(r, f) = 2\overline{N}(r, \frac{1}{f-a_i}) + S(r)$ *for* $i = 1, 2, 3$ *if* $a_5 \equiv b_5$ *and* $a_i \equiv b_i,\ i = 1, 2, 3$.

The theorem is true, if $f, a_1, \cdots, a_6$ *are interchanged with* $g, b_1, \cdots, b_6$*, respectively.*

Remark 1. In Theorem 1, if f and g share the five pairs (a_i, b_i) IM*, $1 \le i \le 5$, then from Definition 1 we deduce that the condition (1.1) occurs, and then the properties "(a)-(h)" of Theorem 1 give us the properties "(a)-(g)" of Theorem B. We also see that if f and g share five distinct small functions IM* then, from (g) and (h) of Theorem 1, we deduce that f is a quasi-Möbius transformation of g, and hence, $f \equiv g$. This result was proved in [5] (entire case) and [7] (meromorphic case).

From (e) and (f) of Theorem 1, we can immediately obtain the following corollary.

Corollary 1. *Let* f *and* g *be two meromorphic functions, and* $a_i,\ b_i\ (i = 1, \cdots, 6)$ *be small functions of* f *and* g*, with* $a_i \not\equiv a_j,\ b_i \not\equiv b_j$ *whenever* $i \ne j$*. Suppose that* f *and* g *share the four pairs* (a_i, b_i) *IM*, for* $i = 1, \cdots, 4$ *and*

$$\overline{N}(r, \frac{1}{f-a_6}) + \overline{N}(r, f = a_5, g \ne b_5) \le T(r, f) + S(r),$$

$$\overline{N}(r, \frac{1}{g-b_6}) + \overline{N}(r, g = b_5, f \ne a_5) \le T(r, g) + S(r).$$

If there exists a number $\lambda \in [0, 2/5)$ *such that*

$$\overline{N}(r, f = a_6, g \ne b_6) + \overline{N}(r, f = a_5, g \ne b_5) \le \lambda T(r, f) + S(r),$$

then f *must be a quasi-Möbius transformation of* g*.*

Remark 2. Li and Yang [6, Corollary 2] proved Corollary 1, when f and g share the five pairs (a_i, b_i) IM*, $1 \le i \le 5$, and if there exists a number $\lambda \in [0, 2/5)$ such that $\overline{N}(r, f = a_6, g \ne b_6) \le \lambda T(r, f) + S(r)$. It is evident that Corollary 1 improves Theorem A and the result of Li and Yang [6, Corollary 2].

Obviously, Theorem 1 is a generalization of Theorem B, and Corollary 1 is a generalization of Theorem A and Corollary 2 in [6] .

Example 2. Let f and g be defined as in Example 1, and let $(a_1, b_1) = (0, 0), (a_2, b_2) = (\infty, \infty), (a_3, b_3) = (1, 1), (a_4, b_4) = (-1/8, -1/8), (a_5, b_5) = (-1/2, 1/4)$ and $a_6 = b_6 = 1/4 + \sqrt{3}/4$. It is easy to show that $\overline{N}(r, f = a_6, g \ne b_6) = T(r, f),\ \overline{N}(r, f = a_5, g \ne b_5) = S(r),\ \overline{N}(r, g = b_5, f \ne a_5) = S(r)$ and f is not a quasi-Möbius transformation of g. This shows that the condition "$\lambda \in [0, 2/5)$ in Corollary 1" is necessary.

We have the following more general result.

Theorem 2. *Let f and g be two nonconstant meromorphic functions, a_i, b_i $(i = 1, \cdots, 6)$ be small functions of f and g, with $a_i \not\equiv a_j$, $b_i \not\equiv b_j$ whenever $i \neq j$, and let $k_1, \cdots, k_6$ be six positive integers or infinity with $K \equiv \sum_{i=1}^{6}(1/1+k_i) \leq 1$. Suppose that f and g share $(a_i, b_i, k_i)^*$ IM, for $i = 1, \cdots, 4$. If there are two numbers λ_1 and λ_2 inside the interval $[0, 1-K]$ such that*

$$36 < (8 - 5(\lambda_1 + K))(8 - 5(\lambda_2 + K)) \tag{1.2}$$

and the following two inequalities hold

$$\overline{N}_{k_6)}(r, f = a_6, g \neq b_6) + \overline{N}_{k_5)}(r, f = a_5, g \neq b_5) \leq \lambda_1 T(r, f) + S(r),$$

$$\overline{N}_{k_6)}(r, g = b_6, f \neq a_6) + \overline{N}_{k_5)}(r, g = b_5, f \neq a_5) \leq \lambda_2 T(r, g) + S(r),$$

then f is a quasi-Möbius transformation of g.

We deduce from Theorem 2, the following corollary which is an improvement Theorem A.

Corollary 2. *Let f and g be two nonconstant meromorphic functions, a_i, b_i $(i = 1, \cdots, 6)$ be small functions of f and g, with $a_i \not\equiv a_j$, $b_i \not\equiv b_j$ whenever $i \neq j$. Suppose that f and g share the four pairs (a_i, b_i) IM*, $1 \leq i \leq 4$. If there are two numbers λ_1 and λ_2 inside the interval $[0, 1]$ such that* $36 < (8 - 5\lambda_1)(8 - 5\lambda_2)$ *and the following two inequalities hold*

$$\overline{N}(r, f = a_6, g \neq b_6) + \overline{N}(r, f = a_5, g \neq b_5) \leq \lambda_1 T(r, f) + S(r),$$

$$\overline{N}(r, g = b_6, f \neq a_6) + \overline{N}(r, g = b_5, f \neq a_5) \leq \lambda_2 T(r, g) + S(r),$$

then f is a quasi-Möbius transformation of g.

2. Lemmas

In this section, we introduce some lemmas that will be used to prove the main results in this paper.

Lemma 1. *Let f and g be two nonconstant meromorphic functions such that*

$$F = c_1 f^2 g + c_2 f g + c_3 f^2 + c_4 f + c_5 g + c_6 \equiv 0,$$

where $c_1, \cdots, c_6$ are small functions of f and g and at least one of them is not identically zero.

If f is not a quasi-Möbius transformation of g, then $T(r, g) = 2T(r, f) + S(r)$.

Proof. Since $F \equiv 0$, then $g(c_1 f^2 + c_2 f + c_5) = -(c_3 f^2 + c_4 f + c_6)$. We note that $c_1 f^2 + c_2 f + c_5 \not\equiv 0$, because at least one of $\{c_1, \cdots, c_6\}$ is not zero. Hence,

$$g = -\tfrac{c_3 f^2 + c_4 f + c_6}{c_1 f^2 + c_2 f + c_5}.$$

Since f is not a quasi-Möbius transformation of g, the right-hand side of the above equation is irreducible. Therefore, by applying Valiron-Mokhonko lemma [8] we get $T(r, g) = 2T(r, f) + S(r)$. This completes the proof of Lemma 1.

Lemma 2 ([10]). *Let f be a nonconstant meromorphic function and let $a_1, \cdots, a_n$ be n distinct small functions of f. Then*

$$(n - 2 - \epsilon)T(r, f) \leq \sum_{i=1}^{n} \overline{N}(r, \tfrac{1}{f - a_i}) + S(r, f)$$

holds for any positive number ϵ.

Lemma 3 ([6]). *Suppose that f and g are nonconstant meromorphic functions, $F = F(f,g)$ is a polynomial in f and g with coefficients being small functions of f and g. The degree of F related to f is p, and the degree of F related to g is q. Then we have $T(r,F) \le pT(r,f) + qT(r,g) + S(r)$.*

3. Proofs of Theorems 1 and 2

If $a_6 = \infty$ (or $b_6 = \infty$) then, for $i = 1,2,3,4,5$, we let $F = 1/(f-a)$, $G = 1/(g-a)$, $\overline{a}_i = 1/(a_i - a)$ and $\overline{b}_i = 1/(b_i - a)$, where a is any complex number a such that $a \not\equiv a_i, b_i$ (i=1,2,3,4,5). We can take $F, G, \overline{a}_i$, $\overline{a}_6 = 0$, (or $\overline{b}_6 = 0$, if $b_6 = \infty$) and $\overline{b}_i$ $(i = 1,2,3,4,5)$ in our proofs instead of f, g, a_i and b_i $(i = 1,2,3,4,6)$, respectively. Therefore, we assume that none of a_i and b_i $(i = 1, \cdots, 6)$ is infinity.

3.1. Proof of Theorem 1. It is not difficult to find six small functions c_i $(1 \le i \le 6)$ of f and g that are not all zero, and six other small functions d_i $(1 \le i \le 6)$ of f and g that are not all zero, such that

$$c_1 a_i^2 b_i + c_2 a_i b_i + c_3 a_i^2 + c_4 a_i + c_5 b_i + c_6 \equiv 0, \quad i = 1,2,3,4,5$$

and

$$d_1 a_i b_i^2 + d_2 a_i b_i + d_3 b_i^2 + d_4 a_i + d_5 b_i + d_6 \equiv 0, \quad i = 1,2,3,4,5.$$

Consequently, the following two functions:

$$F := F(f,g) = c_1 f^2 g + c_2 fg + c_3 f^2 + c_4 f + c_5 g + c_6 \tag{3.1}$$

and

$$G := G(f,g) = d_1 f g^2 + d_2 fg + d_3 g^2 + d_4 f + d_5 g + d_6 \tag{3.2}$$

satisfy $F(a_i, b_i) \equiv 0$ and $G(a_i, b_i) \equiv 0$, for $(1 \le i \le 5)$.

Suppose that $F \not\equiv 0$. For all $1 \le i \le 5$, we denote by $\overline{N}_{(k_i+1}(r, f = a_i, g \ne b_i)$ the reduced counting function of those a_i-points of f whose multiplicities are greater than or equal to $k_i + 1$, that are not the b_i-points of g in $\mid z \mid < r$. By using Lemmas 1 and 3, we have

$$\begin{aligned}
4T(r,f) &\le \sum_{i=1}^{6} \overline{N}(r, \tfrac{1}{f-a_i}) + S(r) \\
&\le \sum_{i=1}^{5} \overline{N}(r, f = a_i, g = b_i) + \sum_{i=1}^{5} \overline{N}_{(k_i+1}(r, f = a_i, g \ne b_i) \\
&\quad + \overline{N}_{k_5)}(r, f = a_5, g \ne b_5) + \overline{N}(r, \tfrac{1}{f-a_6}) + S(r) \\
&\le \sum_{i=1}^{5} \overline{N}(r, f = a_i, g = b_i) + \sum_{i=1}^{5} \overline{N}_{(k_i+1}(r, f = a_i, g \ne b_i) + \lambda T(r,f) + S(r) \\
&\le \overline{N}(r, \tfrac{1}{F}) + \lambda T(r,f) + \sum_{i=1}^{5} \tfrac{1}{k_i+1} N(r, \tfrac{1}{f-a_i}) + S(r) \\
&\le 2T(r,f) + T(r,g) + (\lambda + K)T(r,f) + S(r),
\end{aligned} \tag{3.3}$$

this gives us

$$(2 - K - \lambda)T(r,f) \le T(r,g) + S(r). \tag{3.4}$$

In the same method as the above, it can be shown that

$$\begin{aligned}
4T(r,g) &\le \sum_{i=1}^{6} \overline{N}(r, \tfrac{1}{g-b_i}) + S(r) \\
&\le \sum_{i=1}^{5} \overline{N}(r, g = b_i, f = a_i) + \sum_{i=1}^{5} \overline{N}_{(k_i+1}(r, g = b_i, f \ne a_i) + \lambda T(r,g) + S(r)
\end{aligned}$$

$$\leq \overline{N}(r,\tfrac{1}{F})+\lambda T(r,g)+\sum_{i=1}^{5}\tfrac{1}{k_i+1}N(r,\tfrac{1}{g-b_i})+S(r)$$

$$\leq 2T(r,f)+T(r,g)+(\lambda+K)T(r,g)+S(r), \tag{3.5}$$

and this gives us

$$(3-K-\lambda)T(r,g)\leq 2T(r,f)+S(r). \tag{3.6}$$

It follows from (3.4) and (3.6),

$$(3-K-\lambda)T(r,g)\leq 2T(r,f)+S(r)\leq \tfrac{2}{2-K-\lambda}T(r,g)+S(r),$$

which means that $(3-K-\lambda)(2-K-\lambda)\leq 2$, and hence $(K+\lambda-4)(K+\lambda-1)\leq 0$. Since $K+\lambda\leq 1$ (this follows from the assumption of Theorem 1), then, from the last inequality, we deduce that $0\leq K+\lambda-1$, which means that

$$K+\lambda=1. \tag{3.7}$$

Consequently, we deduce (a) from (3.4), (3.6) and (3.7), and then from (3.3), we get (b). Also, (3.3) gives us

$$\overline{N}_{k_5)}(r,f=a_5,g\neq b_5)+\overline{N}(r,\frac{1}{f-a_6})+S(r)=\lambda T(r,f). \tag{3.8}$$

From (a), (3.3) and (3.7), we deduce the following relation

$$\overline{N}_{(k_i+1}(r,f=a_i,g\neq b_i)=\tfrac{1}{k_i+1}N(r,\tfrac{1}{f-a_i})+S(r)$$

$$=\frac{1}{k_i+1}T(r,f)+S(r),\qquad i=1,\cdots,5. \tag{3.9}$$

Let us now prove $K=0$. We first assume that there exits $1\leq i\leq 5$ such that $k_i<\infty$. Then from (3.9), we have

$$N_{k_i)}(r,f=a_i,g\neq b_i)+N_f(r,f=a_i,g=b_i)+N_{k_i}(r,f=a_i)=S(r),$$

where $N_f(r,f=a_i,g=b_i)$ is the counting function of the points $f-a_i=0$ that are zeros of $g-b_i=0$, and $N_{k_i}(r,f=a_i)$ is the counting function of those a_i-points of f whose multiplicities are not equal to k_i+1.

It follows from the preceding equation and (3.9), we have

$$T(r,f)=(k_i+1)\overline{N}(r,f=a_i,g\neq b_i)+S(r)$$

$$=N(r,\frac{1}{f-a_i})+S(r),\qquad \text{if } k_i<\infty \text{ and } 1\leq i\leq 5. \tag{3.10}$$

Let us consider two cases:

Case 1. Suppose that there are two values of $k_i,(i=1,\cdots,5)$ are finite. Suppose that $k_1<\infty$ and $k_2<\infty$. Let Ł be a quasi-Möbius transformation such that $a_j\equiv Ł(b_j)$, $j=3,4,5$. Since f is not a quasi-Möbius transformation of g, then $f\not\equiv Ł(g)$, and then by (a), we have

$$\sum_{i=3}^{4}\overline{N}(r,\tfrac{1}{f-a_i})+\overline{N}(r,f=a_5,g=b_5)\leq \overline{N}(r,\tfrac{1}{f-Ł(g)})+\sum_{i=3}^{4}\tfrac{1}{k_i+1}N(r,\tfrac{1}{f-a_i})+S(r)$$

$$\leq (2+\tfrac{1}{k_3+1}+\tfrac{1}{k_4+1})T(r,f)+S(r).$$

It follows from this, (3,7)-(3,10), (a) and (b)

$$4T(r,f)=\sum_{i=1}^{6}\overline{N}(r,\tfrac{1}{f-a_i})\leq (\tfrac{1}{k_1+1}+\tfrac{1}{k_2+1})T(r,f)+\sum_{i=3}^{6}\overline{N}(r,\tfrac{1}{f-a_i})+S(r)$$

$$\leq (2+\sum_{i=1}^{4}\tfrac{1}{k_i+1})T(r,f)+\overline{N}(r,f=a_5,g\neq b_5)+\overline{N}(r,\tfrac{1}{f-a_6})+S(r)$$
$$\leq (2+K)T(r,f)+\lambda T(r,f)+S(r)=3T(r,f)+S(r),$$

which is impossible.

By using the above method, one can be proved that there are no other two values of $k_i,(i=1,\cdots,5)$ are simultaneously finite. Therefore, this case can not occur.

Case 2. Suppose that there is $1\leq s\leq 5$ such that $k_s<\infty$ and $k_i=\infty$, for all $1\leq i\leq 5$ with $i\neq s$.

Case 2-1. Suppose that $s=4$. That is, $k_i=\infty$, for all $1\leq i\leq 5$ with $i\neq 4$, and hence, $K=1/(1+k_4)$. As the case 1, let Ł be a quasi-Möbius transformation such that $a_j\equiv Ł(b_j)$, $j=1,2,3$. Since f is not a quasi-Möbius transformation of g, then $f\not\equiv Ł(g)$. From (3.10) and (a), we see

$$\sum_{i=1}^{4}\overline{N}(r,\tfrac{1}{f-a_i})\leq \overline{N}(r,\tfrac{1}{f-Ł(g)})+\tfrac{1}{k_4+1}N(r,\tfrac{1}{f-a_4})+S(r)$$
$$\leq (2+\tfrac{1}{k_4+1})T(r,f)+S(r)=(2+K)T(r,f)+S(r).$$

From this, (3,7)-(3.10), (a) and (b)

$$4T(r,f)=\sum_{i=1}^{6}\overline{N}(r,\tfrac{1}{f-a_i})+S(r)$$
$$\leq (2+K)T(r,f)+\overline{N}(r,\tfrac{1}{f-a_5})+\overline{N}(r,\tfrac{1}{f-a_6})+S(r)$$
$$\leq (2+K+\lambda)T(r,f)+\overline{N}(r,f=a_5,g=b_5)+S(r)$$
$$\leq 4T(r,f)+S(r),$$

which gives

$$\overline{N}(r,f=a_5,g=b_5)=T(r,f)+S(r),\ \overline{N}(r,f=a_5,g\neq b_5)+\overline{N}(r,\frac{1}{f-a_6})+S(r)=\lambda T(r,f). \quad (3.11)$$

Thus, from (3.11), we deduce that

$$\overline{N}(r,f=a_5,g\neq b_5)=S(r). \quad (3.12)$$

That means,

$$\overline{N}(r,\frac{1}{f-a_6})+S(r)=\lambda T(r,f). \quad (3.13)$$

On the other hand, by using (3.10)-(3.12) we get

$$\sum_{i=1,2,5}\overline{N}(r,\tfrac{1}{f-a_i})\leq 2T(r,f)+S(r).$$

It follows from the above inequality, (3.7), (3,10), (3.13) and by using (a) and (b), we see

$$4T(r,f)=\sum_{i=1}^{6}\overline{N}(r,\tfrac{1}{f-a_i})+S(r)$$
$$\leq (2+K+\lambda)T(r,f)+\overline{N}(r,\tfrac{1}{f-a_3})+S(r)\leq 4T(r,f)+S(r),$$

thus gives us

$$\overline{N}(r,\tfrac{1}{f-a_3})=T(r,f)+S(r).$$

Similarly, we have

$$\overline{N}(r,\tfrac{1}{f-a_i})=T(r,f)+S(r),\quad i=1,\ 2.$$

From this, (3.11), (3.12), (a) and (b), we get

$$4T(r,f)=\sum_{i=1}^{6}\overline{N}(r,\tfrac{1}{f-a_i})+S(r)=4T(r,f)+\overline{N}(r,\tfrac{1}{f-a_4})+\overline{N}(r,\tfrac{1}{f-a_6})+S(r),$$

which yields

$$\overline{N}(r,\tfrac{1}{f-a_4})+\overline{N}(r,\tfrac{1}{f-a_6})=S(r).$$

From this and (3.9) (remembering $k_4<\infty$), we have $T(r,f)=S(r)$, which is impossible. That means that $k_4=\infty$. In the same method as the above, we can prove that $k_i=\infty,\ i=1,\ 2,\ 3$. This proves that f and g share $(a_i,b_i,\infty)^*$ IM, for all $1\le i\le 4$.

Case 2-2. Suppose that $s=5$. We do as in Case 2-1 to get

$$\sum_{i=1}^{3}\overline{N}(r,\tfrac{1}{f-a_i})+S(r)\le 2T(r,f)+S(r).$$

From this, (3.7)-(3.10), (a) and (b), we get

$$4T(r,f)=\sum_{i=1}^{6}\overline{N}(r,\tfrac{1}{f-a_i})+S(r)\le 2T(r,f)+\overline{N}(r,\tfrac{1}{f-a_4})+KT(r,f)+\lambda T(r,f)+S(r)$$
$$\le 3T(r,f)+KT(r,f)+\lambda T(r,f)+S(r)=4T(r,f)+S(r),$$

from this, we deduce that

$$\overline{N}(r,\tfrac{1}{f-a_4})=T(r,f)+S(r),\quad \overline{N}(r,\tfrac{1}{f-a_6})=\lambda T(r,f)+S(r)$$

Similarly, we have

$$\overline{N}(r,\tfrac{1}{f-a_i})=T(r,f)+S(r),\quad i=1,\ 2,\ 3,$$

from this and (b), it follows that

$$4T(r,f)=\sum_{i=1}^{6}\overline{N}(r,\tfrac{1}{f-a_i})+S(r)=4T(r,f)+\overline{N}(r,\tfrac{1}{f-a_5})+\overline{N}(r,\tfrac{1}{f-a_6})+S(r),$$

which yields

$$\overline{N}(r,\tfrac{1}{f-a_5})+\overline{N}(r,\tfrac{1}{f-a_6})=S(r).$$

From this and (3.9) (remembering $k_5<\infty$), we have $T(r,f)=S(r)$, which is impossible. That means that $k_5=\infty$. Therefore, from (3.7) and (3.8), we get (e). By using the idea which is used in the case 1, then from (a), (b) and (e) and for all $1\le i\le 4$, we observe

$$3T(r,f)=\sum_{i=1}^{4}\overline{N}(r,\tfrac{1}{f-a_i})+\overline{N}(r,f=a_5,g=b_5)+S(r)$$
$$\le 2T(r,f)+\overline{N}(r,\frac{1}{f-a_i})+\overline{N}(r,f=a_5,g=b_5)+S(r). \qquad (3.14)$$

In the same way, for all $1\le i,j\le 4$ and $i\ne j$, we get

$$3T(r,f)=\sum_{i=1}^{4}\overline{N}(r,\tfrac{1}{f-a_i})+\overline{N}(r,f=a_5,g=b_5)+S(r)$$
$$\le 2T(r,f)+\overline{N}(r,\frac{1}{f-a_i})+\overline{N}(r,\frac{1}{f-a_j}). \qquad (3.15)$$

It is obvious that (c) follows from (3.14) and (3.15). It follows from (c) that, for all $1\le j\le 4$,

$$3T(r,f)\le\sum_{\substack{i=1\\ i\ne j}}^{4}\overline{N}(r,\tfrac{1}{f-a_i})+3\overline{N}(r,\tfrac{1}{f-a_j})+S(r)\le 2T(r,f)+3\overline{N}(r,\tfrac{1}{f-a_j})+S(r)$$

and

$$3T(r,f) \leq \sum_{i=1}^{3} \overline{N}(r, \frac{1}{f-a_i}) + 3\overline{N}(r, f = a_5, g = b_5) + S(r) \leq 2T(r,f) + 3\overline{N}(r, f = a_5, g = b_5) + S(r).$$

From the last two inequalities, we deduce (d).

Let $1 \leq j \leq 6$. We select six small functions $A_i (1 \leq i \leq 6)$ of f and g that are not all zero such that the following two function

$$H := H(f,g) = A_1 f^2 g + A_2 fg + A_3 f^2 + A_4 f + A_5 g + A_6$$

satisfy $H(a_i, b_i) \equiv 0$ for $i = 1, \cdots, 6$ with $i \neq j$. Since f is not a quasi-Möbius transformation of g, it follows from Lemma 1 that if $H \equiv 0$, then $T(r,g) = 2T(r,f) + S(r)$, and this contradicts (a). So, $H \not\equiv 0$. It follows from (a) and (b), and by applying Lemma 3 to the function H,

$$\begin{aligned} 4T(r,f) &= \sum_{i=1}^{6} \overline{N}(r, f = a_i, g = b_i) + \overline{N}(r, f = a_5, g \neq b_5) + \overline{N}(r, f = a_6, g \neq b_6) + S(r) \\ &\leq \overline{N}(r \frac{1}{H}) + \overline{N}(r, f = a_j, g = b_j) + \overline{N}(r, f = a_5, g \neq b_5) + \overline{N}(r, f = a_6, g \neq b_6) + S(r) \\ &\leq 3T(r,f) + \overline{N}(r, f = a_j, g = b_j) + \overline{N}(r, f = a_5, g \neq b_5) + \overline{N}(r, f = a_6, g \neq b_6) + S(r), \end{aligned}$$

which yields

$$T(r,f) \leq \overline{N}(r, f = a_j, g = b_j) + \overline{N}(r, f = a_5, g \neq b_5) + \overline{N}(r, f = a_6, g \neq b_6) + S(r),$$

from this and (e), we get

$$\overline{N}(r, f = a_6, g = b_6) \leq \overline{N}(r, f = a_j, g = b_j) + S(r), \qquad j = 1, \cdots, 6, \tag{3.16}$$

it follows from (3.16) and Lemma 2 that for all $1 \leq j \leq 6$, we get

$$\begin{aligned} &4\overline{N}(r, f = a_6, g = b_6) + \overline{N}(r, f = a_j, g = b_j) \\ &\leq \sum_{\substack{i=1 \\ i \neq j}}^{5} \overline{N}(r, f = a_i, g = b_i) + \overline{N}(r, f = a_j, g = b_j) + S(r) \leq 3T(r,f) + S(r). \end{aligned}$$

From this and (3.16), we get (f).

Suppose that $a_i \equiv b_i$, for $i = 1, \cdots, 4$. By using (a), (b) and Lemma 2, we have

$$\begin{aligned} 3T(r,f) &= \sum_{i=1}^{5} \overline{N}(r, f = a_i, g = b_i) + S(r) \leq \overline{N}(r, \frac{1}{f-g}) + \overline{N}(r, f = a_5, g = b_5) + S(r) \\ &\leq 2T(r,f) + \overline{N}(r, f = a_5, g = b_5) + S(r) \leq 3T(r,f) + S(r), \end{aligned}$$

which gives

$$2T(r,f) = \sum_{i=1}^{4} \overline{N}(r, f = a_i, g = b_i) + S(r) \text{ and } T(r,f) = \overline{N}(r, f = a_5, g = b_5) + S(r). \tag{3.17}$$

From (3.17) and (c), we obtain

$$\overline{N}(r, f = a_i, g = b_i) + \overline{N}(r, f = a_j, g = b_j) = T(r,f) + S(r), \quad i, j = 1, \cdots, 4, \ i \neq j,$$

which yields,

$$\overline{N}(r, f = a_i, g = b_i) = \tfrac{1}{2} T(r,f) + S(r), \quad i = 1, \cdots, 4.$$

From this and (3.17), we get (g).

By using the same technique as in the proof of (g), we prove (h). If $f, a_1, \cdots, a_6$ are interchanged with $g, b_1, \cdots, b_6$, respectively, then from (3.3)-(3.7), we deduce Theorem 1 is clear. This completes the proof of Theorem 1, when $F \not\equiv 0$.

Now, assume that $F \equiv 0$. Therefore, by Lemma 1, we deduce that $T(r,g) = 2T(r,f) + S(r)$. If the possibility $G \equiv 0$, we deduce that $T(r,f) = 2T(r,g) + S(r)$, which is impossible. That means, F and G are not simultaneously zero, and hence $G \not\equiv 0$. We use the same way when $F \not\equiv 0$ to show that Theorem 1 is clear. This proves Theorem 1.

3.2. Proof of Theorem 2. Assume that f is not a quasi-Möbius transformation of g. Let F and G be defined as in (3.1) and (3.2). Suppose that $F \not\equiv 0$. Note that

$$\overline{N}(r, \tfrac{1}{f-a_5}) + \overline{N}(r, \tfrac{1}{f-a_6}) \leq \overline{N}(r, f = a_5, g = b_5) + \overline{N}(r, f = a_6, g = b_6) + (\lambda_1 + \tfrac{1}{k_5+1} + \tfrac{1}{k_6+1})T(r,f) + S(r).$$

Since $F(a_i, b_i) \equiv 0$, for all $1 \leq i \leq 5$, it follows from the above inequality and by applying Lemmas 2 and 3,

$$\begin{aligned} 4T(r,f) &\leq \sum_{i=1}^{6} \overline{N}(r, \tfrac{1}{f-a_i}) + S(r) \\ &\leq \sum_{i=1}^{6} \overline{N}(r, f = a_i, g = b_i) + (K + \lambda_1)T(r,f) + S(r) \\ &\leq \overline{N}(r, \tfrac{1}{F}) + \overline{N}(r, f = a_6, g = b_6) + (K + \lambda_1)T(r,f) + S(r) \\ &\leq \overline{N}(r, f = a_6, g = b_6) + (2 + K + \lambda_1)T(r,f) + T(r,g) + S(r), \end{aligned}$$

which implies that

$$2T(r,f) \leq T(r,g) + \overline{N}(r, f = a_6, g = b_6) + (K + \lambda_1)T(r,f) + S(r). \tag{3.18}$$

Of course, if $F \equiv 0$ then according to Lemma 1, we get $T(r,g) = 2T(r,f) + S(r)$, which means that (3.18) remains clear when $F \equiv 0$. Similarly, we have

$$2T(r,f) \leq T(r,g) + \overline{N}(r, f = a_i, g = b_i) + (K + \lambda_1)T(r,f) + S(r), \quad i = 1, \cdots, 5 \tag{3.19}$$

Symmetrically, by using (3.2) we have

$$2T(r,g) \leq T(r,f) + \overline{N}(r, f = a_i, g = b_i) + (K + \lambda_2)T(r,g) + S(r), \quad i = 1, \cdots, 5 \tag{3.20}$$

Suppose that $F \equiv 0$. Then, by Lemma 1, we deduce that $T(r,g) = 2T(r,f) + S(r)$, and $G \not\equiv 0$ (otherwise, we have $T(r,f) = 2T(r,g) + S(r)$). It follows from (3.20) that

$$\begin{aligned} 10T(r,g) &\leq 5T(r,f) + \sum_{i=1}^{5} \overline{N}(r, f = a_i, g = b_i) + 5(K + \lambda_2)T(r,g) + S(r) \\ &\leq 5T(r,f) + \overline{N}(r, \tfrac{1}{G}) + 5(K + \lambda_2)T(r,g) + S(r) \\ &\leq 6T(r,f) + (5(K + \lambda_2) + 2)T(r,g) + S(r) \\ &\leq (5(K + \lambda_2) + 5)T(r,g) + S(r), \end{aligned}$$

this tells us that $1 \leq K + \lambda_2$, that is $K + \lambda_2 = 1$, this together with the inequality (1.2), we get a contradiction. So, $F \not\equiv 0$. In the same way, we prove that $G \not\equiv 0$. From (3.19), we have

$$10T(r,f) \leq 5T(r,g) + \overline{N}(r, \tfrac{1}{F}) + 5(P + \lambda_1)T(r,f) + S(r) \leq 6T(r,g) + (5(K + \lambda_1) + 2)T(r,f) + S(r),$$

which yields

$$(8 - 5(K + \lambda_1))T(r,f) \leq 6T(r,g) + S(r). \tag{3.21}$$

In the same method as the above and by using (3.20), we can show that

$$(8-5(K+\lambda_2))T(r,g) \leq 6T(r,f)+S(r). \quad (3.22)$$

From (3.21) and (3.22), we obtain $(8-5(\lambda_1+K))(8-5(\lambda_2+K)) \leq 36$, which is a contradiction with (1.2). That means, f should be a quasi-Möbius transformation of g. This proves Theorem 2.

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